An Introduction to Myerson's Optimal Auction

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- Myerson would go on to win the Nobel Memorial Prize in Economic Sciences in 2007.
- Today, his framework is the underlying foundation used in modern auctions such as online ad auctions and digital marketplace designs.

Economic Terminology

Definition (Direct Mechanism)

A *direct mechanism* is a mechanism in which every message is a report of a valuation. A direct mechanism is a pair (Q, M) of an allocation rule $Q(v) = (Q_1(v), \cdots, Q_n(v))$ and a payment rule $M(v) = (M_1(v), \cdots, M_n(v))$ defined on the report of v.

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Definition (Interim Allocation and Payment)

The expected interim probability of winning and expected payment of bidder i are:

$$q_i(v_i) = \mathbb{E}[Q_i(v_i, v_{-i})|v_i], \quad m_i(v_i) = \mathbb{E}[M_i(v_i, v_{-i})|v_i].$$

Economic Terminology (Cont.)

Definition (Incentive Compatibility)

A direct mechanism is *incentive compatible* (IC) if for every bidder i and all valuations v_i, z:

$$v_i q_i(v_i) - m_i(v_i) \geq v_i q_i(z) - m_i(z) \quad \forall z; \tag{0.1}$$

or in other words, truthful reporting maximizes a bidder's interim expected surplus.

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Definition (Individual Rationality)

A direct mechanism is *individually rational* (IR) if every bidder's interim surplus is nonnegative: $U_i(v_i) := v_i q_i(v_i) - m_i(v_i) \geq 0; \quad \forall i, v_i$.

The Envelope Lemma

The Envelope Lemma

In any incentive-compatible direct mechanism, the interim surplus U_i is non-decreasing and at every point differentiable, $U_i'(v_i) = q_i(v_i)$. Therefore:

$$U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(t)dt. \tag{0.2}$$

The Envelope Lemma (Cont.)

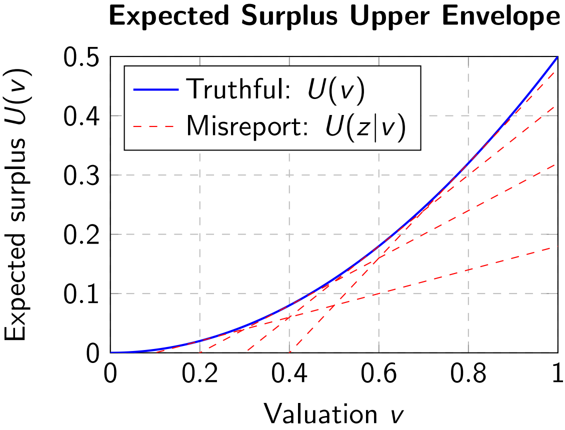


Figure: The truthful expected surplus (solid blue) forms the upper envelope of the payoff functions from misreporting (dashed red).

Myerson's Lemma

Myerson's Lemma

- A direct mechanism is incentive compatible if and only if, for every bidder i, the interim allocation rule $q_i(\cdot)$ is non-decreasing, meaning that as v_i increases $q_i(v_i)$ cannot decrease, and the interim payment rule $m(\cdot)$ obeys the Envelope formula:

$$m_i(v_i) = v_i q_i(v_i) - \int_0^{v_i} q_i(t)dt - U_i(0) \tag{0.3}$$

of the Envelope Lemma.

- Allocations are implementable if and only if $q_i(\cdot)$ is non-decreasing.

Virtual Valuation

Definition (Virtual Valuation)

The *virtual valuation* of bidder i with valuation v_i is:

$$\psi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \tag{0.4}$$

Revenue as a Virtual Surplus

Theorem (Revenue as Virtual Surplus, Myerson 1981)

For any incentive-compatible mechanism, the seller's expected revenue equals the expected virtual surplus.

$$\sum_{i \in N} \mathbb{E}[m_i(v_i)] = \mathbb{E}\left[\sum_{i \in N} \psi_i(v_i) Q_i(v)\right] \quad (0.5)$$

Myerson's Optimal Auction

In a regular environment, a seller's expected revenue is maximized by a mechanism that allocates the object to the bidder with the highest virtual valuation, whenever that maximum is greater than or equal to 0, and withholds the object otherwise:

$$Q_i^*(v) = 1 \quad \textit{iff} \quad \psi_i(v_i) = \max_j \psi_j(v_j) \geq 0.$$

Payments are pinned down by the Envelope Theorem of Myerson's Lemma.

Proof

By the virtual surplus theorem, the revenue of any IC mechanism is given by:

$$\sum_{i \in N} \mathbb{E}[m_i(v_i)] = \mathbb{E}\left[\sum_{i \in N} \psi_i(v_i) Q_i(v)\right].$$

To maximize this expected value, we maximize the integrand pointwise for every value profile v. We let,

$$\psi^*(v) := \max_{i \in N} \psi_i(v_i)$$

represent the highest virtual valuation and

$$\psi^*(v)^+ := \max\{0, \psi^*(v)\}.$$

Proof (Cont.)

Every feasible allocation rule must have $\sum_i Q_i \leq 1$ and $Q_i \geq 0$. Since $\psi_i(v_i) \leq \psi^*(v)$ for every i,

$$\sum_{i \in N} \psi_i(v_i) Q_i(v_i) \leq \psi^*(v) \sum_{i \in N} Q_i(v) \leq \psi^*(v)^+$$

Taking expectations on both sides yields

$$\mathbb{E}[\sum_{i \in N} \psi_i(v_i) Q_i(v)] \leq \mathbb{E}[\psi^*(v)^+].$$

Proof (Cont.)

Next, we define an allocation rule Q^* that strictly allocates the item to the bidder with the highest nonnegative virtual valuation, and withholds it otherwise:

$$Q_i^*(v) = \begin{cases} 1, & \text{if } i \in \arg\max_j \psi_j(v_j) \text{ and } \psi_i(v_i) \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Since F_i is continuous and ψ_i is strictly increasing, ties occur with probability 0.

Proof (Cont.)

By construction, this guarantees for every profile v that:

$$\sum_{i \in N} \psi_i(v_i) Q_i^*(v) = \psi^*(v)^+,$$

meaning that $Q^*(\cdot)$ would attain the upper bound, assuming that it is implementable.

Proof (Cont.)

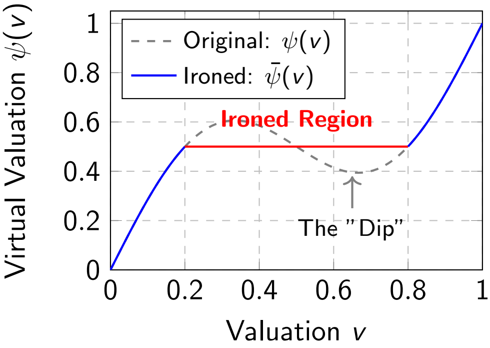
- **Monotonicity via Regularity**: Further, for $Q^*(\cdot)$ to be valid, it must be implementable. By the assumption of regularity, $\psi(\cdot)$ must be strictly increasing, guaranteeing that the induced interim allocation rule $q_i^*(\cdot)$ is non-decreasing with respect to the bid. By Myerson's Lemma, this monotonicity is necessary and sufficient for $Q^*(\cdot)$ to be implementable in any IC mechanism.
- **Individual Rationality**: Since U_i is non-decreasing, its minimum is $U_i(0)$. Setting $U_i(0) = 0$ therefore makes $U_i(v_i) \geq 0$ for all bidders, meaning that individual rationality holds, concluding our proof.

The Irregular Case: Ironing

An Issue in Real-World Data

- **The Problem:** Real-world data is rarely smooth.
- **The Solution:** We can't use broken data, so mathematicians "iron" the data.
- **How It Works:** We calculate a flat average for that entire bumpy region.
- **The Result:** The seller treats every bidder in this flat region exactly the same.

The Ironing Procedure



Optimal Auctions in the Real World

Applying Myerson's Optimal Auction to Professional Sports

- **The Goal:** Maximize competitive balance by awarding the top draft pick to the franchise with the highest "weakness" (virtual valuation).
- **The Irregularity:** If the rule is strictly deterministic (e.g., the NFL), teams have an incentive to *tank* to deliberately losing games to appear weaker.
- **The Ironing Solution:** Leagues like the NBA and NHL mathematically pool the worst-performing teams into a single interval.
- **The Result:** Because the mechanism is forced to treat all teams in this ironed interval identically, it must allocate the top pick using a uniformly random draw, which is called the Draft Lottery.

Thanks for Listening!