# OPTIMAL MECHANISM DESIGN FOR SEQUENTIAL AUCTIONS AND SPORTS DRAFTS

IAN LEUNG

Abstract. This paper first provides an exposition of optimal auction theory, detailing the Revenue Equivalence Theorem, virtual valuations, and ironing. However, classical mechanism design relies on static assumptions where all private information is revealed simultaneously. To model evolving environments, this paper bridges Myerson's Optimal Auction model with sequential mechanism design. By expanding the optimal auction to a multi-period setting, we determine how a seller should optimally adapt their mechanism as bids change. Finally, we apply this framework to professional sports drafts. We derive the concept of dynamic virtual weakness and prove that the draft lottery is mathematically equivalent to an ironed dynamic mechanism.

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## 1. Introduction

For centuries, economists have been studying the behavior of bidders and sellers within fixed markets. However, it was not until 1961, that William Vickrey's foundational work redefined auctions as games with incomplete information. Yet, Vickrey's work only demonstrated how rules governed bidder behavior, leaving a much larger question unanswered. Rather than questioning how bidders behave in auctions, modern auction theory asks the opposite: *If a seller desires to maximize their revenue, what rules must they create to do so?*

The answer to this question gave rise to mechanism design. While many theorists made contributions to the field, it was Roger Myerson that made the most progress in his groundbreaking 1981 paper, *Optimal Auction Design* [Mye81].

By applying the Revelation Principle, Myerson proved that sellers didn't have to be worried about the complexity of auction games. Instead, they could restrict their focus to *direct mechanisms*, where telling the truth about valuations was the optimal strategy. The heart of the framework was Myerson's Lemma [Mye81], and the Revenue Equivalence Theorem [Vic61] [RS81], which established that any two allocation mechanisms resulting in the same probability of a bidder receiving an object will yield the exact same revenue for the seller.

However, Myerson's greatest achievement was solving the seller's revenue-maximization problem through the invention of *virtual values*, which he used to prove that sellers should not allocate the object to the bidder with the highest genuine valuation, but instead to the bidder with the highest virtual valuation. This exact idea is captured in the most iconic equation in mechanism design:

$$\psi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

In this formula, a bidder $i$'s *virtual valuation*: $\psi_i(v_i)$, is defined as their genuine private valuation $v_i$ minus an information rent that incentivizes bidder $i$ to truthfully tell their valuation. The information rent consists of the survival function $1 - F_i(v_i)$, which represents the probability that a randomly selected $v$ is greater than $v_i$, divided by the probability density function $f_i(v_i)$. Though, Myerson's Optimal Auction only functions if the distribution is regular. To solve this, Myerson introduced the mathematical technique *ironing* [Mye81], to ensure that the virtual valuation remains monotone increasing.

Yet, Myerson's framework is fundamentally static, as it assumes all private information is known to all agents at one single point in time. In reality, economic environments aren't static, with information constantly evolving. To capture this idea, this paper bridges Myerson's static foundation with Sequential Mechanism Design [OR68]. In these sequential models, we derive the Dynamic Envelope Lemma and the dynamic virtual valuation, to reveal how a seller should optimally adapt their mechanism to maximize profit when bidders decide to adjust their bids.

Finally, we apply these theoretical ideas to highly visible, multimillion-dollar, sequential mechanisms, professional sports drafts. By viewing the drafts not as a sports event but as a dynamic sequential mechanism, we can expose the structural inefficiencies that lie in the must-watch event. In doing so, we derive the *dynamic virtual weakness* and demonstrate that Myerson's ironing technique is equivalent to draft lotteries utilized across professional sports.

The paper proceeds as follows. Section 2 establishes an economic background. Section 3 presents the standard auction types and the optimal bidding strategy in them. Section 4 introduces the core mechanism design framework, centered heavily on the Envelope Lemma and Myerson's Lemma. Section 5 establishes a main result of the paper, the Revenue Equivalence Theorem. Section 6 explores virtual valuations, the optimal auction, and Myerson's "ironing" technique. Section 7 expands Myerson's optimal auction into a $T-$period environment, adapting the Envelope Lemma, virtual valuations for a dynamic setting. Finally, Section 8 applies this theory to the NFL draft, demonstrating how the draft is a sequential mechanism and providing a mathematical proof that the draft lottery is the ironed solution to deter strategic under-performance.

## 2. Economic Background Knowledge

We begin with a brief review of economic terminology that is required to understand the proof of the theorem.

**Definition 2.1** (Direct Mechanism)**.** A *direct mechanism* is a mechanism in which every message is a report of a valuation. A direct mechanism is a pair $(Q, M)$ of an allocation rule $Q(v) = (Q_1(v), \cdots, Q_n(v))$ and a payment rule $M(v) = (M_1(v), \cdots, M_n(v))$ defined on the report of $v$.

**Definition 2.2** (Interim Allocation and Payment)**.** The expected interim probability of winning and expected payment of bidder $i$ are:

$$q_i(v_i) = \mathbb{E}[Q_i(v_i, v_{-i})|v_i], \quad m_i(v_i) = \mathbb{E}[M_i(v_i, v_{-i})|v_i].$$

**Definition 2.3** (Incentive Compatibility)**.** A direct mechanism is *incentive compatible* (IC) if for every bidder $i$ and all valuations $v_i, z$:

$$v_i q_i(v_i) - m_i(v_i) \geq v_i q_i(z) - m_i(z) \quad \forall z; \tag{2.1}$$

or in other words, truthful reporting maximizes a bidder's interim expected surplus.

**Definition 2.4** (Individual Rationality)**.** A direct mechanism is *individually rational* (IR) if every type's interim surplus is nonnegative: $U_i(v_i) \coloneqq v_i q_i(v_i) - m_i(v_i) \geq 0; \quad \forall i, v_i$.

**Definition 2.5** (Symmetry)**.** Bidders are *symmetric* if their valuations are identically distributed, meaning that there is a common distribution $F$ such that each valuation $v_i$ is distributed according to $F$.

**Definition 2.6** (Expected Payment)**.** The expected payment of bidder $i$ with valuation $v_i$ is $m_i(v_i) = \mathbb{E}[p_i(\mathbf{v}) \mid v_i]$, where $p_i(\mathbf{v})$ is the amount bidder $i$ pays as a function of the full valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$.

**Definition 2.7** (Strategy)**.** A *strategy* for bidder $i$ is a function $\beta_i$ that assigns each value in $[0, \bar{v}]$ a nonnegative bid $\beta_i(v_i)$.

**Definition 2.8** (Risk Neutrality)**.** A bidder is *risk-neutral* if their preferences over uncertain monetary outcomes are represented by the expectations of those outcomes. A risk-neutral bidder $i$'s expected utility is $U_i(v_i) = v_i q_i(v_i) - m_i(v_i)$.

## 3. The Four Standard Auction Types

The four standard auction formats are summarized in Table 1.

| Format | Open or sealed | Price path | Winner pays |
|---|---|---|---|
| English | open | ascending | price at which last rival exits |
| Dutch | open | descending | price at acceptance |
| First-price | sealed | — | own bid |
| Second-price | sealed | — | second-highest bid |

**Table 1.** The four standard auction formats. The Dutch and first-price auctions are strategically equivalent, as are (with private values) the English and second-price auctions.

**Proposition 3.1** (English ≡ second-price under private values)**.** *With private values remaining active in the English auction until the price reaches one's valuation induces the same outcome as bidding one's value in a second-price auction. The equivalence fails when values are interdependent, as bidders may influence the value of other bidders.*

**Proposition 3.2** (Dutch ≡ first-price)**.** *The Dutch auction and the first-price auction are strategically equivalent. A strategy in either auction is a single price chosen with the knowledge of only one's valuation, and identical choices produce identical outcomes. This equivalence does not rely on private values or risk-neutrality.*

### 3.1. Optimal Bidding Strategies in Standard Auction Types.

**Proposition 3.3** (Optimal strategy in second-price auctions)**.** *In an SPA, bidder $i$ maximizes their payoff when he bids their valuation:*

$$\beta(v_i) = v_i. \tag{3.1}$$

*Proof.* We let $b_i$ be bidder $i$'s bid and $B = \max_{j \neq i} b_j$ be the highest bid among all other bidders. If $b_i > B$, bidder $i$ wins and pays $B$, attaining utility $v_i - B$. If $b_i < B$, bidder $i$ loses and receives utility 0. We show that deviating from $b_i = v_i$ can never improve your payoff through the analysis of two cases.

**Case 1: Under-reporting** ($(b_i < v_i)$):

a) If $B > b_i$, bidder $i$ loses with a payoff of 0. Further, if $v_i > B > b_i$, if bidder $i$ had submitted a bid of $v_i$, they would have received a strictly positive payoff of $v_i - B$.
b) If $B < b_i$ bidder $i$ wins regardless of bidding $v_i$ or $b_i$, meaning their payoff remains at $v_i - B$.
c) If $b_i = B$, a tie occurs, yielding an expected payoff of $\frac{1}{2}(v_i - B)$.

**Case 2: Over-reporting** ($(b_i > v_i)$):

a) If $B > b_i$, bidder $i$ loses again with a payoff of 0.
b) If $B < b_i$, bidder $i$ wins, resulting in a payoff of $v_i - B$, which is only positive if $b_i > B$ and is negative if $b_i < B$. However, if $v_i > B$, bidder $i$ bidding $b_i$ would result in a negative expected payoff. Further, negative payoff also arises when $v_i < B < b_i$.
c) If $b_i = B$, a tie occurs, once again yielding an expected payoff of $\frac{1}{2}(v_i - B)$. However, this expected payoff is only positive if $b_i > B$ and is negative otherwise.

□

**Corollary 3.4** (Optimal strategy in English auctions)**.** *We assume a symmetric mechanism with $n$ risk-neutral bidders with private valuations independently and uniformly distributed on $[0, 1]$. It follows immediately from Proposition 3.1 and Proposition 3.3 that a bidder's optimal strategy is to remain active until the price reaches their true valuation:*

$$\beta_{English}(v_i) = v_i \tag{3.2}$$

**Proposition 3.5** (Optimal strategy in first-price auctions)**.** *In an symmetric FPA, with $n$ risk-neutral bidders with valuations that are independently and uniformly distributed on $[0, 1]$ the symmetric Bayesian-Nash equilibrium strategy is given by:*

$$\beta(v) = \left(\frac{n-1}{n}\right) v. \tag{3.3}$$

*Proof.* We let $\beta(\cdot)$ be a symmetric, strictly increasing, and differentiable equilibrium bidding strategy. We suppose that a bidder $i$ has valuation $v_i$, but submits bids as if their valuation were $z$, $(\beta(z) = b_i)$. Bidder $i$ only wins the auction when their bid is higher than all other $n - 1$ bidders. Given that valuations are uniformly distributed on $[0, 1]$, since $\beta(\cdot)$ is strictly increasing, the probability of bidder $i$ winning is $z^{n-1}$, meaning bidder $i$'s expected utility is $U_i(z) = z^{n-1}(v_i - \beta(z))$. To find the optimal bid, we take the derivative of the expected utility with respect to the reported type $z$ and set it to zero to satisfy the First-Order Conditions (FOC):

$$\frac{\partial U_i}{\partial z} = (n-1)z^{n-2}(v_i - \beta(z)) - z^{n-1}\beta'(z) = 0.$$

For $\beta(\cdot)$ to be a Bayesian-Nash equilibrium, it must be incentive compatible, meaning that truth-telling is optimal. Substituting $z = v_i$ into the FOC yields:

$$(n-1)v_i^{n-2}(v_i - \beta(v_i)) - v_i^{n-1}\beta'(v_i) = 0.$$

Rearranging the terms, we expand the left side:

$$(n-1)v_i^{n-1} = (n-1)v_i^{n-2}\beta(v_i) + v_i^{n-1}\beta'(v_i).$$

By the product rule the right side of the equation is the derivative of the product $v_i^{n-1}\beta(v_i)$ with respect to $v_i$. Thus, we can rewrite the equation as:

$$(n-1)v_i^{n-1} = \frac{d}{dv_i}\left[v_i^{n-1}\beta(v_i)\right].$$

Integrating from 0 to $v_i$ on both sides and assuming a boundary condition where a bidder with valuation 0 bids 0:

$$\int_0^{v_i} (n-1)x^{n-1}\, dx = v_i^{n-1}\beta(v_i).$$

Evaluating the integral on the LHS yields:

$$\frac{n-1}{n} v_i^n = v_i^{n-1}\beta(v_i).$$

Finally, dividing both sides by $v_i^{n-1}$ isolates $\beta(v_i)$:

$$\beta(v_i) = \left(\frac{n-1}{n}\right) v_i,$$

completing our proof. □

**Corollary 3.6** (Optimal strategy in Dutch auctions)**.** *We assume a symmetric mechanism with n risk-neutral bidders with private valuations independently and uniformly distributed on* $[0, 1]$*. It follows immediately from Propositions 3.2 and 3.5 that a bidder's optimal strategy is to submit a bid when the price drops to:*

$$\beta_{Dutch}(v_i) = \left(\frac{n-1}{n}\right) v_i. \tag{3.4}$$

## 4. The Mechanism Design Framework

### 4.1. The Revelation Principle.

**Theorem 4.1** (The Revelation Principle)**.** *For any mechanism and any Bayesian-Nash equilibrium of the mechanism, there is an incentive-compatible direct mechanism that yields the same allocation and payments for every valuation profile.*

*Proof.* We let $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ be an equilibrium mechanism of $\Gamma$. We define a direct mechanism that on the profile $v = (v_1, v_2, \cdots, v_n)$ produces the outcome that $\Gamma$ produces under the bids $\beta(v)$. If a bidder $i$ of type $v_i$ could gain by reporting that $z \neq v_i$, then that bidder $i$ could gain by submitting a bid $\beta_i(z)$ instead of $\beta_i(v_i)$, which contradicts the original assumption that $\beta$ is an equilibrium. Thus, truthful reporting is an equilibrium of the direct mechanism, which by construction reproduces the outcomes of $\beta$. □

### 4.2. The Envelope Lemma.

In any incentive-compatible mechanism, a bidder's surplus is determined based on the allocation rule alone, up to an additive constant that represents the expected payoff of the lowest valuation type.

**Lemma 4.2** (Envelope Lemma)**.** *In any incentive-compatible direct mechanism, the interim surplus $U_i$ is non-decreasing and at every point differentiable, $U_i'(v_i) = q_i(v_i)$. Therefore:*

$$U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(t)dt. \tag{4.1}$$

*Proof.* By incentive-compatibility, a type $v_i$ bidder would rather report truthfully rather than reporting a type $z$, so

$$U_i(v_i) = \max_z [v_i q_i(z) - m_i(z)].$$

Thus, the surplus is an upper envelope of the straight lines in type $v_i$ and the line for report $z$ would have slope $q_i(z)$. We fix $v_i < v_i'$ and since $v_i$ could imitate $v_i'$ and vice versa, we have:

$$U_i(v_i') \geq v_i' q_i(v_i) - m_i(v_i) = U_i(v_i) + (v_i' - v_i) q_i(v_i),$$
$$U_i(v_i) \geq v_i q_i(v_i') - m_i(v_i') = U_i(v_i') - (v_i' - v_i) q_i(v_i').$$

Rearranging and dividing by $v_i' - v_i > 0$ gives us:

$$q_i(v_i) \leq \frac{U_i(v_i') - U_i(v_i)}{v_i' - v_i} \leq q_i(v_i')$$

The left inequality shows that $U_i$ is non-decreasing, since $U_i(v_i') \geq U_i(v_i)$. Further, we let $v_i' \to v_i$, which gives us:

$$\frac{U_i(v_i') - U_i(v_i)}{v_i' - v_i} \to U_i'(v_i).$$

and $q_i(v_i') \to q_i(v_i)$, which yields:

$$q_i(v_i) \leq U_i'(v_i) \leq q_i(v_i).$$

Thus, the only possibility is

$$U_i'(v_i) = q_i(v_i).$$

Replacing $v_i$ with $t$ and integrating both sides from 0 to $v_i$:

$$\int_0^{v_i} U_i'(t)dt = \int_0^{v_i} q_i(t)dt.$$

By the Fundamental Theorem of Calculus:

$$U_i(v_i) - U_i(0) = \int_0^{v_i} q_i(t)dt.$$

Thus,

$$U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(t)dt.$$

which completes the proof. □

4.3. **Myerson's Lemma.**

**Lemma 4.3** (Myerson's Lemma)**.** *A direct mechanism is incentive compatible iff, for every bidder i, the interim allocation rule $q_i(\cdot)$ is non-decreasing, meaning that as $v_i$ increases $q_i(v_i)$ cannot decrease, and the interim payment rule $m(\cdot)$ obeys the Envelope formula:*

$$m_i(v_i) = v_i q_i(v_i) - \int_0^{v_i} q_i(t)dt - U_i(0) \tag{4.2}$$

*of Lemma 4.2.*

*Proof.* We define $v_i$ to be bidder $i$'s true valuation, and $z$ to be bidder $i$'s reported valuation. The expected interim utility of bidder $i$ when reporting $z$ is given by $U_i(z|v_i) = v_i q_i(z) - m_i(z)$. Further, we let $U_i(v_i) \equiv U_i(v_i|v_i)$ denote the utility for bidder $i$ telling the truth. By definition, a direct mechanism is IC if $U_i(v_i) \geq U_i(z|v_i) \quad \forall v_i, z$.

*Necessity* : We suppose the direct mechanism is IC, with $v_i' > v_i$. The IC constraints require:

$$v_i q_i(v_i) - m_i(v_i) \geq v_i q_i(v_i') - m_i(v_i'),$$
$$v_i' q_i(v_i') - m_i(v_i') \geq v_i' q_i(v_i) - m_i(v_i).$$

Summing these inequalities yield:

$$(v_i - v_i')(q_i(v_i) - q_i(v_i')) \geq 0.$$

Since $v_i' > v_i$ it follows that $q_i(v_i') \geq q_i(v_i)$, establishing that $q_i(\cdot)$ is non-decreasing, satisfying the first requirement. Then, from Lemma 4.2, substituting $U_i$ gives:

$$v_i q_i(v_i) - m_i(v_i) = U_i(0) + \int_0^{v_i} q_i(t)dt.$$

Solving for $m_i(v_i)$ yields:

$$m_i(v_i) = v_i q_i(v_i) - \int_0^{v_i} q_i(t)dt - U_i(0),$$

satisfying the second requirement.

*Sufficiency*: We suppose that the interim allocation rule $q_i(\cdot)$ is non-decreasing and that the interim payment rule $m(\cdot)$ obeys $m_i(v_i) = v_i q_i(v_i) - \int_0^{v_i} q_i(t)dt - U_i(0)$. To prove IC, we show that truth-telling is a weakly dominant strategy by examining the gain from any unilateral deviation by reporting $z \neq v_i$. By definition, bidder $i$'s utility of truth-telling is $U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(t)dt$, and if bidder $i$ reports $z$, their expected utility becomes $U_i(z|v_i) = v_i q_i(z) - m_i(z)$. Substituting $zq_i(z) - \int_0^z q_i(t)dt - U_i(0)$ for $m_i(z)$ yields:

$$U_i(z|v_i) = q_i(z)(v_i - z) + \int_0^z q_i(t)dt + U_i(0).$$

Substituting the utility definitions, we have:

$$U_i(z|v_i) - U_i(v_i|v_i) = \left[q_i(z)(v_i - z) + \int_0^z q_i(t)dt + U_i(0)\right] - \left[\int_0^{v_i} q_i(t)dt + U_i(0)\right].$$

Simplifying yields:

$$U_i(z|v_i) - U_i(v_i|v_i) = q_i(z)(v_i - z) - \int_z^{v_i} q_i(t)dt.$$

Since $q_i(z)(v_i - z) = \int_z^{v_i} q_i(z)dt$ we can write:

$$U_i(z|v_i) - U_i(v_i|v_i) = \int_z^{v_i} q_i(z)dt - \int_z^{v_i} q_i(t)dt = \int_z^{v_i} [q_i(z) - q_i(t)]dt.$$

We evaluate the integral's sign based on two cases:

- Under-reporting ($z < v_i$): For all $t \in [z, v_i]$ $q_i(t) \geq q_i(z)$ because of $q_i(\cdot)$'s monotonicity. Thus, $q_i(z) - q_i(t) \leq 0$. Integrating a non-positive function with an upper limit greater than lower limit yields a result $\leq 0$.
- Over-reporting ($z > v_i$): By flipping the bounds of the integral we obtain: $\int_z^{v_i}[q_i(z) - q_i(t)]dt = -\int_{v_i}^{z}[q_i(z) - q_i(t)]dt$. For all $t \in [v_i, z]$, the monotonicity of $q_i(\cdot)$ implies that $q_i(t) \leq q_i(z)$, meaning that $q_i(z) - q_i(t) \geq 0$. Integrating a nonnegative function yields a result $\geq 0$. Thus, the leading negative sign ensures that the total expression is $\leq 0$.

In both cases, $U_i(z|v_i) - U_i(v_i|v_i) \leq 0$, confirming that no report $z$ provides a bidder $i$ with higher utility than reporting $v_i$. Thus, the direct mechanism is incentive compatible. □

## 5. The Revenue Equivalence Theorem

**Theorem 5.1** (The Revenue Equivalence Theorem, [Vic61], [Mye81], [RS81])**.** *Among symmetric direct mechanisms that are allocatively efficient and result in the bidder with the lowest valuation receiving zero surplus* ($U_i(0) = 0$)*, every mechanism yields the same interim payment:*

$$m_i(v_i) = v_i F(v_i)^{n-1} - \int_0^{v_i} F(t)^{n-1}dt. \tag{5.1}$$

**Assumption 5.2.** *The direct mechanism is symmetric and allocatively efficient, and bidders' valuations are distributed independently and identically according to a common distribution $F$. Therefore, since a bidder with valuation $v_i$ wins when all other bidders have valuations below $v_i$, it is implied that the mechanism must obey a non-decreasing allocation rule such that $q_i(\cdot) = F(\cdot)^{n-1}$, meaning that the mechanism guarantees the object to be allocated to the bidder with the highest valuation.*

**Assumption 5.3.** *The mechanism must satisfy the normalized condition:* $U_i(\underline{v_i}) = 0, \forall i \in N$. *This specifies that the bidder with the lowest possible valuation has an expected utility of 0.*

*Proof.* By Lemma 4.2, we have that

$$U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(t)dt.$$

Substituting for $F(t)^{n-1}$ yields:

$$U_i(v_i) = \int_0^{v_i} F(t)^{n-1}dt.$$

Rewriting $U_i(v_i)$ as $v_i F(v_i)^{n-1} - m_i(v_i)$ and simplifying gives us:

$$m_i(v_i) = v_i F(v_i)^{n-1} - \int_0^{v_i} F(t)^{n-1}dt,$$

completing the proof. $\square$

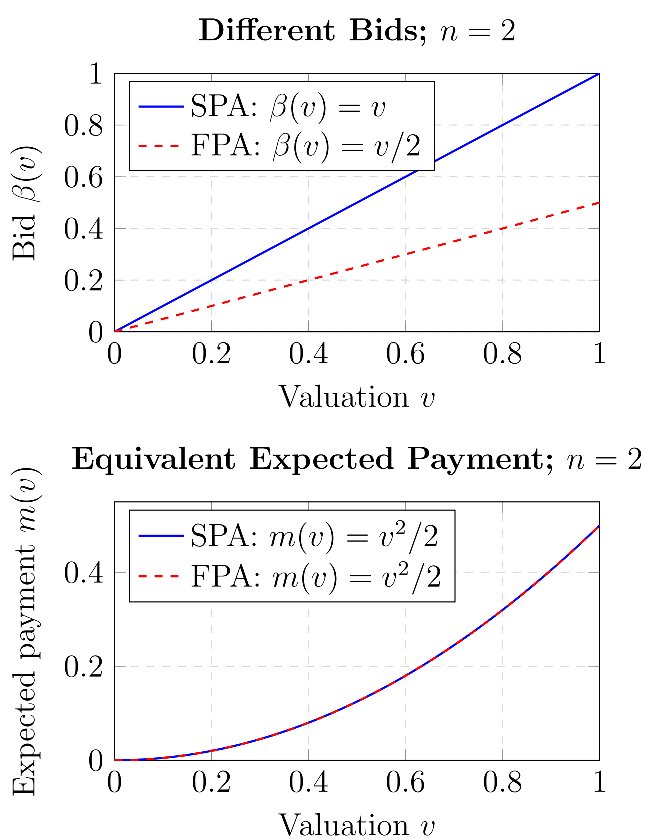


**Figure 1.** Revenue equivalence: the bids and auction formats differ, but the payments do not.

**Example 5.4** (Two bidders $(v_1, v_2)$, uniform values)**.** *In an auction with $n = 2$ and $F(v) = v, v \in [0,1]$, $m(v) = v^2 - \int_0^v t \cdot dt = v^2 - \frac{v^2}{2} = \frac{v^2}{2}$. Thus, the expected revenue is:*

$$2\int_0^1 \frac{v^2}{2} dv = \frac{1}{3}.$$

- *Second-Price Auction (SPA) Immediately from Proposition 3.3 the equilibrium bid is:*

  $$\beta(v) = v.$$

  *Directly computing yields:*

  $$R_{SPA} = \mathbb{E}[\min(v_1, v_2)] = \int_0^1 t \cdot 2(1-t)dt = \frac{1}{3}.$$

- *First-Price Auction (FPA) From Proposition 3.5, the equilibrium bid is:*

  $$\beta(v) = \left(\frac{n-1}{n}\right) v = \frac{1}{2}v.$$

  *Thus we have:*

  $$R_{FPA} = \mathbb{E}[\frac{1}{2}\max(v_1, v_2)] = \frac{1}{2}\mathbb{E}[\max] = \frac{1}{2}\int_0^1 t \cdot 2tdt = \frac{1}{3}$$

**Remark 5.5.** *Theorem 5.1 proves that all efficient, symmetric auctions result in the same expected revenue. Since this renders choice of format irrelevant to revenue, the only way to raise more is to sacrifice efficiency; quantifying this tradeoff is exactly the core of Myerson's problem.*

## 6. Virtual Valuations and the Optimal Auction

**Definition 6.1** (Virtual Valuation)**.** The *virtual valuation* of bidder $i$ with valuation $v_i$ is:

$$\psi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \tag{6.1}$$

*Derivation.* From Lemma 4.3, we have $m_i(v_i) = v_i q(v_i) - \int_0^{v_i} q_i(t)dt$. Taking expectations yields:

$$\mathbb{E}[m_i(v_i)] = \int_0^{\bar{v}_i} \left[ v_i q_i(v_i) - \int_0^{v_i} q_i(t)dt \right] f_i(v_i)dv_i.$$

We first analyze the second term of the integral, which represents the expected information rent:

$$I = \int_0^{\bar{v}_i} \left( \int_0^{v_i} q_i(t)dt \right) f_i(v_i)dv_i.$$

Through integration by parts, we let $u = \int_0^{v_i} q_i(t)dt$ such that $du = q_i(v_i)dv_i$ and let $dv = f_i(v_i)dv_i$ such that $v = F_i(v_i)$. This yields:

$$I = \left[ \left( \int_0^{v_i} q_i(t)dt \right) F_i(v_i) \right]_0^{\bar{v}_i} - \int_0^{\bar{v}_i} q_i(v_i)F_i(v_i)dv_i.$$

Evaluating boundary conditions, we note that $F_i(\bar{v}_i) = 1$ and $F_i(0) = 0$. Consequently, the boundary term simplifies to $\int_0^{\bar{v}_i} q_i(t)dt$. By swapping the variable $t$ to $v_i$, we obtain:

$$I = \int_0^{\bar{v}_i} q_i(v_i)dv_i - \int_0^{\bar{v}_i} q_i(v_i)F_i(v_i)dv_i = \int_0^{\bar{v}_i} q_i(v_i)(1 - F_i(v_i))dv_i.$$

Substituting $I$ back into the original expression gives:

$$\mathbb{E}[m_i(v_i)] = \int_0^{\bar{v}_i} v_i q_i(v_i) f_i(v_i) dv_i - \int^{\bar{v}_i} q_i(v_i)(1 - F_i(v_i)) dv_i.$$

Rearranging and factoring out $q_i(v_i)$ yields:

$$\mathbb{E}[m_i(v_i)] = \int_0^{\bar{v}_i} q_i(v_i)[v_i f_i(v_i) - (1 - F_i(v_i))] dv_i.$$

Moving $f_i(v_i)$ outside the bracket gives us:

$$\mathbb{E}[m_i(v_i)] = \int_0^{\bar{v}_i} q_i(v_i) \underbrace{\left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right]}_{\psi_i(v_i)} f_i(v_i) dv_i, \tag{6.2}$$

which concludes our derivation. □

**Definition 6.2** (Regularity)**.** An environment is *regular* if all $\psi_i$ are monotone strictly increasing.

**Theorem 6.3** (Revenue as Virtual Surplus, Myerson 1981)**.** *For any incentive-compatible mechanism, the seller's expected revenue equals the expected virtual surplus.*

$$\sum_{i \in N} \mathbb{E}[m_i(v_i)] = \mathbb{E}\left[\sum_{i \in N} \psi_i(v_i) Q_i(v)\right] \tag{6.3}$$

*Proof.* Immediately from Equation 6.2 we have:

$$\mathbb{E}[m_i(v_i)] = \int_0^{\bar{v}_i} q_i(v_i) \underbrace{\left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right]}_{\psi_i(v_i)} f_i(v_i) dv_i.$$

Simplifying yields:

$$\mathbb{E}[m_i(v_i)] = \int_0^{\bar{v}_i} q_i(v_i) \psi_i(v_i) f_i(v_i) dv_i = \mathbb{E}[\psi_i(v_i) q_i(v_i)],$$

Summing over all $i$ and writing $q_i(v_i) = \mathbb{E}_{-i}[Q_i(v_i)]$ gives:

$$\sum_{i \in N} \mathbb{E}[m_i(v_i)] = \mathbb{E}\left[\sum_{i \in N} \psi_i(v_i) Q_i(v)\right]$$

completing the proof. □

**Theorem 6.4** (Myerson's Optimal Auction; [Mye81])**.** *In a regular environment, a seller's expected revenue is maximized by a mechanism that allocates the object to the bidder with the highest virtual valuation, whenever that maximum is greater than or equal to 0, and withholds the object otherwise:*

$$Q_i^*(v) = 1 \quad \textit{iff} \quad \psi_i(v_i) = \max_j \psi_j(v_j) \geq 0.$$

*Payments are pinned down by the Envelope Theorem of Lemma 4.3.*

*Proof.* By Theorem 6.3, the revenue of any IC mechanism is given by:

$$\sum_{i \in N} \mathbb{E}[m_i(v_i)] = \mathbb{E}\left[\sum_{i \in N} \psi_i(v_i) Q_i(v)\right].$$

To maximize this expected value, we maximize the integrand pointwise for every value profile $v$. We let $\psi^*(v) := \max_{i \in N} \psi_i(v_i)$ represent the highest virtual valuation and $\psi^*(v)^+ := \max\{0, \psi^*(v)\}$. Every feasible allocation rules must have $\sum_i Q_i \leq 1$ and $Q_i \geq 0$. Since $\psi_i(v_i) \leq \psi^*(v)$ for every $i$,

$$\sum_{i \in N} \psi_i(v_i) Q_i(v_i) \leq \psi^*(v) \sum_{i \in N} Q_i(v) \leq \psi^*(v)^+$$

Taking expectations on both sides yields $\mathbb{E}[\sum_{i \in N} \psi_i(v_i) Q_i(v)] \leq \mathbb{E}[\psi^*(v)^+]$. Next, we define an allocation rule $Q^*$ that strictly allocates the item to the bidder with the highest nonnegative virtual valuation, and withholds it otherwise:

$$Q_i^*(v) = \begin{cases} 1, & \text{if } i \in \arg\max_j \psi_j(v_j) \text{ and } \psi_i(v_i) \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Since $F_i$ is continuous and $\psi_i$ is strictly increasing, ties occur with probability 0. By construction, this guarantees for every profile $v$ that:

$$\sum_{i \in N} \psi_i(v_i) Q_i^*(v) = \psi^*(v)^+,$$

meaning that $Q^*(\cdot)$ would attain the upper bound, assuming that it is implementable. Further, for $Q^*(\cdot)$ to be valid, it must be implementable. By the assumption of regularity, $\psi(\cdot)$ must be strictly increasing, guaranteeing that the induced interim allocation rule $q_i^*(\cdot)$ is non-decreasing with respect to the bid. Since $U_i$ is non-decreasing, its minimum is $U_i(0)$. Setting $U_i(0) = 0$ therefore makes $U_i(v_i) \geq 0$ for all types, meaning that individual rationality holds. By Lemma 4.3, this monotonicity is necessary and sufficient for $Q^*(\cdot)$ to be implementable in any IC mechanism, concluding our proof. □

**Definition 6.5** (Reserve Price)**.** A *reserve price* $r^*$ is a minimum price for the object strategically set by the seller to maximize their payoff. In an efficient auction, any bidder whose valuation exceeds the seller's would win the object, resulting in a sale that would benefit both parties. However, the optimal auction deliberately turns some of these sales away. By witholding the object when all virtual valuations are negative, the seller accepts a positive probability of no sale at all, in exchange for a higher expected price if a sale occurs.

**Corollary 6.6** (Reserve Prices)**.** *With symmetric regular bidders, $\psi_i \equiv \psi$ is increasing, meaning the highest virtual valuation belongs to the bidder with the highest value. Therefore, the optimal auction is a standard efficient auction restricted to bidders with valuations that exceed the reserve price $r^*$ solving $\psi(r^*) = 0$.*

**Example 6.7.** *For $F$ uniform on [0,1], $\psi(v) = 2v - 1$ and $r^* = \frac{1}{2}$, independent to $n$, the number of bidders.*

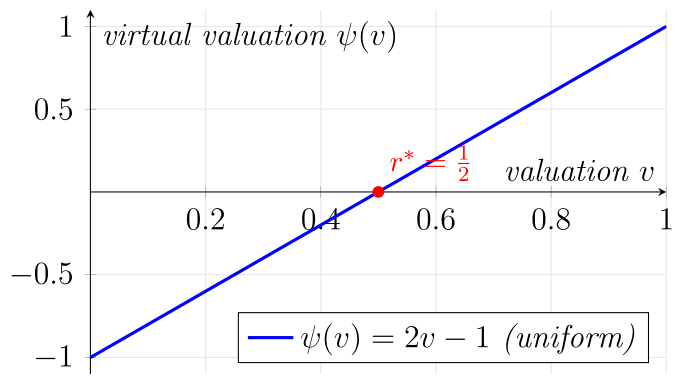


**Figure 2.** The virtual valuation for uniform values. The optimal auction sells only to bidders with $\psi(v) \geq 0$, i.e. $v \geq r^* = \frac{1}{2}$; types below the reserve are excluded, even though their value exceeds the seller's.

6.1. **The Irregular Case: Ironing.** Theorem 6.4 relied on regularity: where each virtual valuation $\psi_i$ is monotone strictly increasing, ranking bidders by their virtual valuation to automatically produce an allocation rule with higher types winning more often. However, not all distributions are friendly. If $\psi_i$ dips downward over a range of values, then maximizing the virtual surplus pointwise would allow an intermediate type to win less than a lower type, forcing a non-monotone, and therefore non-implementable, allocation.

To handle these irregular distributions, Myerson discovered a convexification procedure that was later colloquially named "ironing". This technique was used to replace $\psi_i$ on such intervals by a constant chosen, so that the resulting "ironed" virtual valuation is non-decreasing everywhere, whilst minimizing the distortion of the allocation.

**Definition 6.8** (Ironed Virtual Valuation)**.** For a bidder $i$, we define $h_i(\tau) = \psi_i(F_i^{-1}(\tau))$ for $\tau \in [0, 1]$ and its integral $H_i(\tau) = \int_0^\tau h_i(t)dt$. We let $G_i$ be the convex hull of $H_i$: that is, $G_i$ is the largest convex function such that $G_i(\tau) \leq H_i(\tau)$. We then set $g_i = G_i'$. The *ironed virtual valuation* is $\bar{\psi}_i(v) = g_i(F_i(v))$.

**Theorem 6.9** (A Generalized Optimal Auction, [Mye81])**.** *Without regularity, expected revenue is maximized by the mechanism that allocates the object to the bidder with the highest ironed virtual valuation $\bar{\psi}_i(v_i)$ whenever that maximum is nonnegative. On any interval with $G_i < H_i$, $\bar{\psi}_i$ is constant, meaning all the corresponding types are pooled together and allocated by a tie-break to preserve monotonicity.*

*Proof.* We first establish that the expected revenue of any monotone allocation rule is bounded above by its ironed virtual surplus. We consider a bidder $i$. By Theorem 6.3, the bidder's expected revenue is $\mathbb{E}[\psi_i(v_i)q_i(v_i)]$. Changing variables to the quantile space, we define $\tau = F_i(v_i)$. Next, we let $\hat{q}_i(\tau) = q_i(F_i^{-1}(\tau))$ denote the allocation rule as a function of $\tau$. The expected revenue becomes:

$$\int_0^1 h_i(\tau)\hat{q}_i(\tau)d\tau.$$

By Lemma 4.3, IC forces $q_i$, and consequently, $\hat{q}_i$ are non-decreasing. Since $\hat{q}_i$ is a monotone function bounded on $[0, 1]$, it is of bounded variation. This permits integration through

Lebesgue-Stieltjes, [Rad13], with respect to the measure $d\hat{q}_i$. Integrating by parts yields:

$$\int_0^1 h_i(\tau)\hat{q}_i(\tau)d\tau = [H_i(\tau)\hat{q}_i(\tau)]_0^1 - \int_0^1 H_i(\tau)d\hat{q}_i(\tau).$$

By definition, $H_i(0) = 0$. We let $G_i$ be the convex hull of $H_i$. The convex hull satisfies $G_i(\tau) \leq H_i(\tau)$ for all $\tau \in [0,1]$ and touches the original function at its boundaries, meaning that $G_i(1) = H_i(1)$. Further, the monotonicity of $\hat{q}_i$ ensures that $d\hat{q}_i \geq 0$. Therefore:

$$\int_0^1 H_i(\tau)d\hat{q}_i(\tau) \geq \int_0^1 G_i(\tau)d\hat{q}_i(\tau).$$

Substituting this inequality back into our integration by parts expression yields:

$$\int_0^1 h_i(\tau)\hat{q}_i(\tau)d\tau \leq G_i(1)\hat{q}_i(1) - \int_0^1 G_i(\tau)d\hat{q}_i(\tau)$$

Because $G_i(0) = 0$ and defining $g_i = G_i'$, we reverse the integration by parts to obtain:

$$G_i(1)\hat{q}_i(1) - \int_0^1 G_i(\tau)d\hat{q}_i(\tau) = \int_0^1 g_i(\tau)\hat{q}_i(\tau)d\tau.$$

Thus, replacing the original virtual valuation with the ironed virtual valuation provides an upper bound on the objective for every monotone allocation.

Next, we demonstrate that the allocation rule induced by the ironed virtual valuation is inherently monotonic, and therefore implementable. The ironed allocation awards the object to the bidder with the highest nonnegative virtual valuation $\bar{\psi}_i(v_i) = g_i(F_i(v_i))$ and breaks ties with a fixed rule. Because $G_i$ is a convex function, its derivative is non-decreasing. Further, the cumulative distribution function $F_i$ is non-decreasing. As a result, $\bar{\psi}_i$ must inherently be non-decreasing as well. Consequently, the interim win probability $q_i^*(v_i)$ is also non-decreasing in $v_i$. By Lemma 4.3, monotonicity is both necessary and sufficient for IC, meaning that the ironed allocation is implementable.

Finally, we prove that ironed allocations attain the upper bound. By the first section of the proof, the expected revenue of any implementable allocation under $\psi_i$ is bounded by the virtual surplus of the same allocation under $\bar{\psi}_i$. For the optimal ironed allocation $\hat{q}_i^*$, the allocation strictly increases when $g_i$ increases. This occurs only on non-ironed intervals where $G_i = H_i$ and $g_i = h_i$. Conversely, it is constant on ironed intervals where $G_i < H_i$. Since the function is constant, $d\hat{q}_i^* = 0$ where $G_i < H_i$. Further, because $d\hat{q}_i^* = 0$ on all ironed intervals, the integrals $\int_0^1 H_i(\tau)d\hat{q}_i^*(\tau)$ and $\int_0^1 G_i(\tau)d\hat{q}_i^*(\tau)$ are identical. Thus, the inequality derived in the first part holds, proving that the ironed allocation attains the upper bound. Setting the lowest type's utility $U_i(0) = 0$ and applying the Envelope Theorem of Lemma 4.3 ensures IR, which completes our proof. □

## 7. An Extension: Sequential Mechanisms

Myerson's Optimal Auction solved merely a static problem: one object, with each bidder's type drawn once. However, the same three instruments: the Envelope Theorem, Virtual Valuation, and pointwise maximization of virtual surplus, extend to a *sequence* of allocations in which a bidder's valuation evolves over time. This is the exact theory of *dynamic mechanism design*.

7.1. **A T-Period Environment.**

**Definition 7.1** (Sequential Allocation Environment)**.** There are periods $t \in \{1, \cdots, T\}$ and one object to allocate in each. In period 1, bidder $i$ privately observes a type $v_i \in [0, \bar{v}_i]$, drawn from their own regular distribution $F_i$ with density $f_i$. Bidder $i$'s value from receiving the period-$t$ object is $v_{i,t}$, with $v_{i,t} = \gamma_t v_i + \eta_{i,t}$. $\eta_{i,t}$ is publicly observed and $v_i$ is bidder $i$'s only private information.

**Definition 7.2** (Impulse Responses, [PST14])**.** The *impulse responses*,

$$\gamma_t = \frac{\partial v_{i,t}}{\partial v_i},$$

measures how a marginal change in the initial private type propagates to the value in each later period.

As in the static model, a direct mechanism induces, for each bidder $i$, an interim allocation rule $q_{i,t}(v_i)$, an interim payment of $m_i(v_i)$, and interim surplus $U_i(v_i) = \sum_{t=1}^{T} v_{i,t} q_{i,t} - m_i$. Further, incentive compatibility (IC) requires truth-telling to be optimal.

7.2. **The Dynamic Envelope Lemma.**

**Lemma 7.3** (The Dynamic Envelope Lemma, [PST14])**.** *In any incentive-compatible mechanism, $U_i$ is non-decreasing and differentiable with:*

$$U_i'(v_i) = \sum_{t=1}^{T} \gamma_t q_{i,t}(v_i), \quad \textit{hence} \quad U_i(v_i) = U_i(0) + \int_0^{v_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x)dx. \tag{7.1}$$

*Proof.* We define a function $\phi(z|v_i) = \sum_{t=1}^{T} [v_{i,t} q_{i,t}(z)] - m_i(z)$ By IC, a type $v_i$ bidder prefers truth to any report $z$, so

$$U_i(v_i) = \max_z \phi(z|v_i)$$

Applying the chain rule gives us

$$\frac{\partial \phi(z|v_i)}{\partial v_i} = \sum_{t=1}^{T} \frac{\partial v_{i,t}}{\partial v_i} q_{i,t}(z).$$

Substituting $\gamma$ into our partial derivative yields $\sum_{t=1}^{T} \gamma_t q_{i,t}(z)$. By Lemma 4.2, $U_i(v_i)$ is defined as the upper envelope of a family of affine functions; it is therefore convex and differentiable everywhere. Evaluating the partial derivative at the truth-telling optimum $z = v_i$:

$$U_i'(v_i) = \left.\frac{\partial \phi(z|v_i)}{\partial v_i}\right|_{z=v_i} = \sum_{t=1}^{T} \gamma_t q_{i,t}(v_i).$$

Since each $q_{i,t}, \gamma > 0$, $U_i'(v_i) \geq 0$, meaning that the interim surplus $U_i(v_i)$ is monotonically non-decreasing. Next, we integrate the derivative from 0 to $v_i$ to obtain:

$$U_i(v_i) - U_i(0) = \int_0^{v_i} U_i'(x)dx$$

Substituting and rearranging yields:

$$U_i(v_i) - U_i(0) = \int_0^{v_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x)dx \Longrightarrow U_i(v_i) = U_i(0) + \int_0^{v_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x)dx,$$

completing our proof. □

### 7.3. **The Dynamic Virtual Valuation.**

**Definition 7.4** (Dynamic Virtual Valuation, [PST14])**.** The dynamic virtual valuation of bidder $i$ in period $t$ is:

$$\psi_{i,t}(v_i) = v_{i,t} - \gamma_t \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

*Derivation.* Taking expectations on $m_i(v_i) = \sum_{t=1}^{T} v_{i,t} q_{i,t}(v_i) - U_i(v_i)$ and substituting Lemma 7.3 gives us:

$$\mathbb{E}[m_i] = \int_0^{\bar{v}_i} \sum_{t=1}^{T} v_{i,t} q_{i,t}(v_i) f_i(v_i) dv_i - U_i(0) - \int_0^{\bar{v}_i} \left( \int_0^{v_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x) dx \right) f_i(v_i) dv_i.$$

Analyzing the integral's second term, we let

$$I = \int_0^{\bar{v}_i} \left( \int_0^{v_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x) dx \right) f_i(v_i) dv_i.$$

By integration by parts, we let $u = \int_0^{v_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x) dx$, such that $\sum_{t=1}^{T} \gamma_t q_{i,t}(v_i) dv_i$, and we let $dv = f_i(v_i) dv_i$, such that $v = F_i(v_i) - 1$. This yields:

$$I = \left[ \left( \int_0^{v_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x) dx \right) (F_i(v_i) - 1) \right]_0^{\bar{v}_i} - \int_0^{\bar{v}_i} \left[ (F_i(v_i) - 1) \sum_{t=1}^{T} \gamma_t q_{i,t}(v_i) dv_i \right].$$

Evaluating the boundary conditions: at $v_i = \bar{v}$, $(F_i(\bar{v}) - 1) = 0$, and at $v_i = 0$, the integral term is 0. Thus, the entire boundary term is 0. Distributing the negative sign on the remaining integral gives:

$$I = \int_0^{\bar{v}_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(v_i)(1 - F_i(v_i)) dv_i.$$

Substituting $I$ back into the original equation and writing $1 - F_i(v_i) = \frac{1-F_i(v_i)}{f_i(v_i)} f_i(v_i)$ yields:

$$\mathbb{E}[m_i] = \int_0^{\bar{v}_i} \sum_{t=1}^{T} q_{i,t}(v_i) \underbrace{\left( v_{i,t} - \gamma_t \frac{1 - F_i(v_i)}{f_i(v_i)} \right)}_{\psi_{i,t}(v_i)} f_i(v_i) dv_i - U_i(0),$$

which completes our derivation. □

**Remark 7.5.** *With $\eta_{i,1} = 0$, the period $t = 1$ dynamic virtual valuation is exactly Myerson's static virtual valuation $\psi$.*

### 7.4. **The Optimal Sequential Mechanism.**

**Theorem 7.6** (Optimal Sequential Mechanism, [PST14])**.** *We assume the dynamic environment is regular, such that $\frac{1-F_i(v_i)}{f_i(v_i)}$ is non-increasing, and that the valuation process ensures that $\gamma_t > 0$. Summing the derivation of Definition 7.4 over all bidders, we get that the expected revenue is the expected dynamic virtual surplus:* $\mathbb{E}[\sum_i \sum_{t=1}^{T} \psi_{i,t}(v_i) Q_{i,t}(v)] - \sum_i U_i(0)$.

*It is maximized by the mechanism that allocates the object to the bidder with the highest dynamic virtual valuation, provided that it is non-negative. Payments are pinned down by Lemma 7.3 and $U_i(0)$.*

*Proof.* After setting $U_i(0) = 0$), the seller's objective is to maximize

$$\mathbb{E}[\sum_i \sum_{t=1}^{T} \psi_{i,t}(v_i) Q_{i,t}(v)] - \sum_i U_i(0) = \mathbb{E}[\sum_{i \in N} \sum_{t=1}^{T} \psi_{i,t}(v_i) Q_{i,t}(v)],$$

subject to the constraints $\sum_i Q_{i,t} \leq 1$ and $Q_{i,t} \geq 0$ for all profiles $v$ and time periods $t \in \{1, \cdots, T\}$. To maximize the expected revenue, we maximize the integrand pointwise for every valuation profile $v$. We let $\psi_t^*(v) := \max_{i \in N} \psi_{i,t}(v_i)$ represent the highest dynamic virtual valuation in period $t$, and $\psi_t^*(v)^+ := \max\{0, \psi_t^*(v)\}$. The sum is bounded by: $\sum_{i \in N} \psi_{i,t}(v_i) Q_{i,t}(v) \leq \psi_t^*(v)^+$. Taking expectations yields:

$$\mathbb{E}[\sum_{i \in N} \sum_{t=1}^{T} \psi_{i,t}(v_i) Q_{i,t}(v)] \leq \mathbb{E}[\sum_{t=1}^{T} \psi_t^*(v)^+].$$

Next, we define an allocation rule $Q_{i,t}^*(\cdot)$ that strictly allocates the object to the bidder with the highest dynamic virtual value, and withholds it when that maximum is negative:

$$Q_{i,t}^* = \begin{cases} 1, & \text{if } i \in \arg\max_j \psi_{j,t}(v_j) \text{ and } \psi_{i,t}(v_i) \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

By construction, this guarantees for every profile $v$ and period $t$ that:

$$\sum_{i \in N} \psi_{i,t}(v_i) Q_{i,t}^*(v) = \psi_t^*(v)^+.$$

Finally, $Q_{i,t}^*(\cdot)$ must be implementable. By the assumption of regularity and since $\gamma_t > 0$, each $\psi_{i,t}$ is strictly increasing in $v_i$, guaranteeing that the interim allocation rule $q_{i,t}^*$ is strictly increasing with respect to the type $v_i$. By Lemma 4.3, this monotonicity is sufficient and necessary for $Q^*$ to be implementable into any incentive-compatible mechanism, which concludes our proof. ☐

## 8. Sequential Mechanisms in Sports

### 8.1. **The Normative Sports Draft.**

**Definition 8.1** (Valuation of a Position)**.** The *valuation of a position* $k$, represented as $\hat{V}(k)$, is the expected payoff of the right, but not the obligation, to select the most valuable item still available when position $k$ is reached.

**Definition 8.2** (The Draft as a Sequential Mechanism)**.** Positions $k = 1, \cdots, K$ are draft picks, ordered from worst team first to best team last. The bidders are the NFL teams, with the objects being players. The $K$ players have qualities $S_1, \cdots, S_K$ drawn independently and identically from a cumulative distribution function. Picks are traded on a market whose price $P(k)$ is the league's published pick-value chart.

**Proposition 8.3** (Greedy Selection, [MT05])**.** *The sequential mechanism has a unique subgame-perfect outcome: the holder of each pick selects the best player then available. Consequently, pick $k$ obtains the $k^{th}$ highest quality in the pool, and its value is the order statistic:*

$$\hat{V} = \mathbb{E}[S_{(K-k+1)}], \qquad S_{(1)} \leq \cdots \leq S_{(K)}. \tag{8.1}$$

*Proof.* At every position $k$, the holder of pick $k$ obtains the quality of whichever available player that it selects. The holder faces no subsequent decisions in the continuation of their game, and their game is strictly independent of any decisions made in positions $t > k$. Therefore, since the holder of pick $k$ is unaffected by continued play, they maximize their payoff when selecting the highest-quality available player. We let $S_{(1)} \leq \cdots \leq S_{(K)}$ represent the standard order statistics of the $K$ draws. The maximum quality is $S_{(K)}$, the second highest is $S_{(K-1)}$, and by induction, the $k^{th}$ highest quality is the $(K-k+1)^{th}$ highest order statistic, which is $S_{(K-k+1)}$. Taking expectations yields the expected payoff, Equation 8.1, completing our proof. □

**Lemma 8.4.**

a) *$\hat{V}$ is strictly decreasing in $k$.*
b) *For $F = U[0,1]$, $\hat{V}(k) = \frac{K-k+1}{K+1}$, is linear in $k$.*

*Proof.*

a) Order statistics satisfy $S_{(r)} \leq S_{(r+1)}$ almost surely, meaning that $r \mapsto \mathbb{E}[S_{(r)}]$ is non-decreasing. Further, it is strictly non-decreasing because $F$ is continuous with positive density. Since $r = K - k + 1$ strictly decreases in $k$, $\hat{V}$ strictly decreases in $k$ as well.
b) For uniform samples $S_{(r)} \sim \text{Beta}(r, K-r+1)$, so $\mathbb{E}[S_{(r)}] = r/(K+1)$. Substituting $r = K - k + 1$ gives the claim.

□

8.2. **The Draft Order as a Sequential Mechanism.** In the draft there are $n$ teams and $T$ seasons, with the league designing the draft mechanism. Every team has a *weakness* $w_i \in [0, \bar{w}]$, that is drawn from a continuous distribution function $G_i$ with density $g_i$. A higher weakness is a structurally weaker franchise, that is consequently more deserving of picks. Weakness is carried across seasons through the impulse response defined in Definition 7.2. A team $i$'s pick deservingness in season $t$ is:

$$v_{i,t} = \gamma_t w_i + \eta_{i,t}, \qquad \gamma_1 = 1,$$

where $\gamma_t = \partial v_{i,t}/\partial w_i$ measures how much a franchise's weakness still shows in season $t$, with $\eta_{i,t}$ being a public shock (injuries, practice schedule changes). The league awards a top pick, worth $\bar{V} = \hat{V}(1)$ to its recipient. However, a franchise may *tank*, strategically lowering its standings to appear weaker and receive a top pick.

**Corollary 8.5** (Draft Envelope)**.** *In any incentive-compatible mechanism,*

$$U_i(w_i) = U_i(0) + \int_0^{w_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x) dx.$$

*Proof.* Immediately from Lemma 7.3, we apply type $w_i$ to yield:

$$U_i(w_i) = U_i(0) + \int_0^{w_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x) dx.$$

□

By Corollary 8.5, the rent a franchise earns is its surplus $U_i$. To establish an optimal sequential mechanism for the draft, the league must maximize competitive balance net of these specific rents:

$$\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} v_{i,t}q_{i,t}\right] - \mathbb{E}\left[\sum_{i=1}^{n} U_i\right]. \tag{8.2}$$

The league values competitive balance, which is captured by the first term by rewarding the pick to the franchises with the greatest need. However, the information rent $U_i$ a franchise earns is exactly the surplus it gains from appearing weak, and therefore is the incentive it has to tank. Thus, a league that aims to deter tanking maximizes competitive balance net of these information rents.

**Definition 8.6** (Dynamic Virtual Weakness)**.** The dynamic virtual weakness of $w_i$ is:

$$\psi_{i,t}(w_i) = v_{i,t} - \gamma_t \frac{1 - G_i(w_i)}{g_i(w_i)} = \gamma_t\left(w_i - \frac{1 - G_i(w_i)}{g_i(w_i)}\right) + \eta_{i,t}$$

**Remark 8.7.** *This is exactly the dynamic virtual valuation of Definition 7.4, but with weakness $w_i$ instead of valuation $v_i$.*

**Proposition 8.8.** *Equation 8.2 is equivalent to the expression:* $\mathbb{E}[\sum_t \sum_i \psi_{i,t}(w_i)q_{i,t}] - \sum_i U_i(0)$

*Proof.* By Corollary 8.5, the interim expected surplus of franchise $i$ with weakness $w_i$ is $U_i(w_i) = U_i(0) + \int_0^{w_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x)dx$. Taking expectations over $[0, \bar{w}]$ and with density function $g_i(w)$, we expand the integral:

$$\mathbb{E}\left[\int_0^{w_i} \gamma_t q_{i,t}(x)dx\right] = \int_0^{\bar{w}} \left(\int_0^{w_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x)dx\right) g_i(w_i)dw_i.$$

Applying integration by parts, we let $u = \int_0^{w_i} \sum_{t=1}^{T} \gamma_t q_{i,t}(x)dx$ and $dv = g_i(w_i)dw_i$. Setting $v = -(1 - G_i(w_i))$, and the boundary conditions cause the $du$ term to become 0 at both $\bar{w}$ and 0 leaving:

$$\int_0^{\bar{w}} \left(\sum_{t=1}^{T} \gamma_t q_{i,t}(w_i)\right)(1 - G_i(w_i))dw_i.$$

Dividing and multiplying back the integrand by $g_i(w_i)$ transforms the integral back into an expectation:

$$\int_0^{\bar{w}} \left(\sum_{t=1}^{T} \gamma_t \frac{1 - G_i(w_i)}{g_i(w_i)} q_{i,t}(w_i)\right) g_i(w_i)dw_i = \mathbb{E}\left[\sum_{t=1}^{T} \gamma_t \frac{1 - G_i(w_i)}{g_i(w_i)} q_{i,t}(w_i)\right].$$

Summing across all $n$ franchises gives the total expected rent:

$$\mathbb{E}\left[\sum_{i=1}^{n} U_i\right] = \sum_{i=1}^{n} U_i(0) + \mathbb{E}\left[\sum_{i=1}^{n}\sum_{t=1}^{T} \gamma_t \frac{1 - G_i(w_i)}{g_i(w_i)} q_{i,t}(w_i)\right].$$

Substituting this result back into $\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} v_{i,t}q_{i,t}\right] - \mathbb{E}\left[\sum_{i=1}^{n} U_i\right]$ and factoring out $q_{i,t}$ yields:

$$\mathbb{E}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\left(v_{i,t} - \gamma_t \frac{1 - G_i(w_i)}{g_i(w_i)}\right) q_{i,t}\right] - \sum_{i=1}^{n} U_i(0).$$

By Definition 8.6 the inner parenthesis is exactly the dynamic virtual weakness of $w_i$. The strongest franchise type has zero information rent, fixing $U_i(0) = 0$, reducing the league's goal to maximizing the dynamic virtual surplus: $\mathbb{E}[\sum_{t=1}^{T}\sum_{i=1}^{n}\psi_{i,t}(w_i)q_{i,t}]$, which completes our proof. □

**Theorem 8.9** (Optimal Draft Order)**.** *The competitive-balance optimal sequential mechanism allocates each season's top draft pick to the franchise with the greatest non-negative dynamic virtual weakness $\psi_{i,t}(w_i) \geq 0$. In regions where $w_i \mapsto \psi_{i,t}(w_i)$ is monotone increasing, the allocation is deterministic. Conversely, wherever monotonicity is violated, the mechanism utilizes the ironing procedure, to pool an interval of weakness types and allocate the top pick uniformly at random among the pooled franchises.*

*Proof.* By Proposition 8.8, the league's goal is to maximize $\mathbb{E}[\sum_{t=1}^{T}\sum_{i=1}^{n}\psi_{i,t}(w_i)q_{i,t}]$, with the restraints $\sum_i q_{i,t} \leq 1$ and $q_{i,t} \geq 0$. Since the league's objective is additively separable across different seasons, it is sufficient to maximize $\sum_{i=1}^{n}\psi_{i,t}(w_i)q_{i,t}$ pointwise in each season $t$. The resulting problem is the static optimal auction problem with $\psi_{i,t}$ taking the role of the virtual valuation. Therefore, Theorem 6.9 applies in each season. Pointwise maximization awards the top season $t$ pick to the franchise with the largest non-negative $\psi_{i,t}$. Specifically, when the pointwise rule violates $q_{i,t}$ monotonicity in $w_i$, replacing $\psi_{i,t}$ with its ironed equivalent maintains incentive compatibility. Since $\psi_{i,t}$ is strictly held constant on all ironed intervals, the allocation is identically constant, forcing a uniformly random draw from the pooled franchises, completing our proof. □

**Remark 8.10.** *Unlike a typical revenue-maximizing seller, the league always awards the draft pick, meaning they cannot withhold it. The threshold $\psi_{i,t}(w_i) \geq 0$ in Theorem 8.9 is inherited from the auction analogy, where a seller optimally withholds the object whenever every virtual valuation is negative.*

**Remark 8.11.** *The ironing procedure in Theorem 8.9 is exactly the draft lottery. In professional sports, like the National Basketball Association (NBA) and the National Hockey League (NHL), draft lotteries are implemented for the sole purpose of deterring deliberate tanking. When the distribution of weaknesses $w_i$ creates a dense cluster of low-performing franchises, the franchises must be pooled together to maintain incentive compatibility. Therefore, the draft lottery develops organically when the league can no longer distinguish between the true weaknesses of the worst franchises.*

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