

ON THE MAXIMUM LENGTH OF MULTISSET CYCLE PACKINGS

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ABSTRACT. We study the 1D multiset packing problem: given k colours and window size m , what is the maximum length of a cyclic sequence such that every m -window multiset is distinct? This problem is explicitly identified as open by Chen et al. (2024), who constructed near-optimal higher-dimensional torus packings but left the 1D case unresolved. For $m = 3$ and $k \equiv 0 \pmod{3}$, we prove the upper bound $M_3(k) \leq \binom{k+2}{3} - k/3$, and show that any optimal packing must omit a rainbow partition of the colour alphabet. Using a symmetry-reduction framework, we construct explicit sequences achieving the bound for $k = 6, 9, 12$. We further prove that bounded scalar-sum mappings are insufficient for the 1D multiset problem, explaining why Chen et al. required vector sums in higher dimensions. These results clarify the theoretical limits of low-cost positioning hardware and motivate the 1D Multiset Packing Conjecture.

1 Introduction and Problem Motivation

Automated warehouse logistics and factory-floor robotics demand reliable, low-cost, and energy-efficient positioning systems. A mobile robot operating in such environments must accurately determine its location from local observations in order to navigate, avoid obstacles, and coordinate with other autonomous agents. Traditional solutions rely heavily on computer vision systems that read colour-coded grids or fiducial markers. Because these methods require high-resolution cameras and significant computational power to process spatial images, they are expensive, consume substantial battery reserves, and are generally impractical for the large-scale deployment of simple, low-power robotics.

An alternative engineering approach utilizes ambient backscatter grids or light-intensity positioning systems. In these configurations, the robot is equipped with a simple photodiode or a basic radio-frequency sensor that reads aggregate colour intensities or identification numbers from nearby tags. Such sensors are remarkably cheap, consume minimal power, and offer low latency. However, they suffer from a fundamental hardware limitation: they cannot distinguish spatial order. For example, a light sensor registering two red units and one blue unit cannot differentiate whether the physical pattern on the floor is red-blue-red or blue-red-red. It records only the unordered multiset of colours present within its localized viewing window.

Instead of designing tracking sequences where every ordered spatial window is unique, we are forced to design sequences where every unordered window multiset is strictly unique.

1.1 De Bruijn Sequences and Multiset Packings

Positioning systems in one-dimensional environments are governed by De Bruijn sequences. A De Bruijn sequence of order m over an alphabet of size k is a cyclic sequence of length k^m formulated such that every ordered m -tuple appears exactly once as a contiguous window. A robot equipped with a camera that observes the precise ordered pattern in its viewing window can determine its absolute position uniquely, as every physical window maps to a distinct ordered tuple. In this setting, the observation space is simply the Cartesian product $\mathcal{A}^m$.

Conversely, when the robotic unit relies solely on a multiset sensor, the observation space algebraically collapses. The robot observes only the collection of colours in its window, completely divorced from their sequence. The observation function f maps an ordered window to a frequency vector $(c_1, \dots, c_k)$, where c_j denotes the discrete number of occurrences of colour j . Crucially, this function is not injective on the space $\mathcal{A}^m$: two distinct ordered windows possessing identical colour counts will produce the exact same sensory observation.

1.2 Grid Positioning

Let $\mathcal{A}$ be a finite alphabet representing a colour palette, with a cardinality of $|\mathcal{A}| = k$. For a fixed window size m and a spatial dimension d , the viewing window of the robot is an $m \times \dots \times m$ hypercube of cells. The observation function $f : \mathcal{A}^{m^d} \rightarrow \mathbb{N}^k$ projects an ordered spatial window to the multiset of its constituent entries.

Definition 1.1 (Cycle Packing). Given an alphabet $\mathcal{A}$, a window size m , and an observation function f , a cycle packing is defined as a periodic function $W : \mathbb{Z} \rightarrow \mathcal{A}$ of period n such that the sequence of evaluations

$$f(W_i, W_{i+1}, \dots, W_{i+m-1}), \quad \text{for } i \in \{0, 1, \dots, n-1\},$$

consists of strictly distinct elements. Equivalently, the observation function f is perfectly injective on the set of all contiguous windows.

Definition 1.2 (Torus Packing). Let $d \geq 1$ be a positive integer. A torus packing is a periodic function $W : \mathbb{Z}^d \rightarrow \mathcal{A}$ exhibiting a period n in each coordinate dimension, such that the evaluations

$$f((W_{\mathbf{x}+\mathbf{c}})_{\mathbf{c} \in \{0,1,\dots,m-1\}^d}), \quad \text{for } \mathbf{x} \in \{0, 1, \dots, n-1\}^d,$$

are all distinct. When f operates specifically as the multiset counting function, the resulting torus packing is designated as a grid colouring.

The injectivity condition embedded within these definitions ensures a deterministic positioning system. Because no two distinct physical window positions can produce the same sensory observation, the robot can always deduce its exact geometric location.

Definition 1.3 (Grid Colouring). An (n, m, d, k) -grid colouring is a torus packing constructed for the multiset observation function with an alphabet size of $|\mathcal{A}| = k$, utilizing a window size m across dimension d .

1.3 Literature Optima and Contributions

There exists a strict information-theoretic upper bound on the size of any valid grid colouring. The total number of possible colour multisets of size m^d drawn from a palette of k colours is precisely $\left[\binom{m^d + k - 1}{k - 1} \right]$. Because injectivity requires every observable window to correspond to a distinct multiset, the total number of windows n^d must satisfy the inequality

$$n^d \leq \left[\binom{m^d + k - 1}{k - 1} \right].$$

For a fixed dimension d and a fixed palette size k , as the window size m approaches infinity, this boundary yields the asymptotic limit

$$n \leq C'_k{}^{1/d} \cdot m^{k-1} (1 + O(m^{-1})), \quad \text{where } C'_k = \frac{1}{(k-1)!}.$$

Theorem 1.4 (Chen et al., 2024). *Fix a dimension $d \geq 2$ and a number of colours $k = bd + 1$ for some integer $b \geq 1$. For any window size m that is a multiple of $2(k-1)$, there exists an (n, m, d, k) -grid colouring satisfying*

$$n \sim C_k^{1/d} \cdot m^{k-1}, \quad \text{where } C_k = \left(\frac{2}{k-1} \right)^{k-1}.$$

Furthermore, given the multiset of colours within a window, the coordinates modulo n can be computed using $O_{k,d}(1)$ arithmetic operations.

While Theorem 1.4 brilliantly resolves the higher-dimensional torus structure ($d \geq 2$), the packing efficiency for the one-dimensional multiset problem remains an open question when divisibility conditions fail. Therefore, alongside our survey of the literature, **this paper will try to present two analytical contributions:**

- (1) We establish a new constructive lower bound for one-dimensional multiset packings when the window size is $m = 3$ and the palette size k is a multiple of three.
- (2) We prove the inherent insufficiency of scalar-sum colour mappings, demonstrating that single-integer coordinate metrics inevitably violate injectivity via the Pigeonhole Principle.

2 Ordered Sequence Systems

We begin by analyzing the standard framework that governs positioning systems when sensors can distinguish spatial order. This establishes a baseline against which the unordered multiset paradigm will later be compared.

Let $K = \{0, 1, \dots, k-1\}$ be an alphabet of k distinct symbols, and let m be a positive integer. A robot traversing a cyclic track observes a discrete window of m consecutive symbols. When the robotic unit is equipped with a camera or an equivalent high-resolution sensor, it can distinguish the exact ordered sequence of symbols within its viewing window.

Definition 2.1 (Identity Observation Map). Let $S = (s_0, s_1, \dots, s_{n-1})$ be a cyclic sequence of length n over the alphabet K . For a fixed window size $m \leq n$, the window at index i

is denoted by $W_i = (s_i, s_{i+1}, \dots, s_{i+m-1})$, where all indices are evaluated modulo n . The identity observation map

$$\Phi_{\text{id}} : K^m \rightarrow K^m$$

is defined by

$$\Phi_{\text{id}}(W_i) = (s_i, s_{i+1}, \dots, s_{i+m-1}).$$

The map returns the exact ordered tuple of symbols present in the window starting at position i .

Definition 2.2 (De Bruijn Sequence). A De Bruijn sequence of order m over an alphabet of size k , denoted by $B(k, m)$, is a cyclic sequence of length $L = k^m$ such that every ordered tuple in K^m occurs exactly once as a contiguous window subsequence.

Definition 2.3 (De Bruijn Graph). The De Bruijn graph $G_{k,m-1} = (V, E)$ is a directed pseudograph where the vertex set

$$V = K^{m-1},$$

consists of all ordered $(m-1)$ -tuples over K , and the directed edge set

$$E = K^m,$$

consists of all ordered m -tuples over K . A directed edge $e = (x_1, x_2, \dots, x_m)$ originates at the vertex

$$(x_1, x_2, \dots, x_{m-1}),$$

and terminates at the vertex

$$(x_2, x_3, \dots, x_m).$$

Theorem 2.4. *The De Bruijn graph $G_{k,m-1}$ is balanced and strongly connected. Consequently, it admits an Eulerian circuit.*

Proof. For any vertex $v = (x_1, \dots, x_{m-1}) \in V$, the outgoing edges are precisely those of the form

$$(x_1, \dots, x_{m-1}, y), \quad \text{for } y \in K,$$

so the out-degree is $\deg^+(v) = k$. Similarly, the incoming edges are precisely those of the form

$$(y, x_1, \dots, x_{m-1}), \quad \text{for } y \in K,$$

so the in-degree is $\deg^-(v) = k$. Because $\deg^-(v) = \deg^+(v)$ for every vertex $v \in V$, the graph is perfectly balanced.

To demonstrate that $G_{k,m-1}$ is strongly connected, let $u = (u_1, \dots, u_{m-1})$ and $v = (v_1, \dots, v_{m-1})$ be arbitrary vertices in V . A directed path from u to v is given by the sequence of vertices:

$$(u_1, \dots, u_{m-1}), (u_2, \dots, u_{m-1}, v_1), (u_3, \dots, u_{m-1}, v_1, v_2), \dots, (v_1, \dots, v_{m-1}).$$

This path successfully traverses $m-1$ edges, proving that any vertex can reach any other vertex. Hence, $G_{k,m-1}$ is strongly connected.

By Hierholzer's Theorem, every finite, strongly connected directed graph in which every vertex has equal in-degree and out-degree admits an Eulerian circuit. Therefore, $G_{k,m-1}$ contains a closed walk that visits every directed edge exactly once. $\square$

Corollary 2.5. *For all positive integers k and m , a De Bruijn sequence $B(k, m)$ exists.*

Proof. An Eulerian circuit in $G_{k,m-1}$ traverses each of the k^m directed edges exactly once. Recording the edge labels in the exact sequence they are traversed yields a cyclic sequence of length k^m in which every ordered m -tuple appears exactly once as a contiguous window. $\square$

Example 2.6. Let $k = 2$ and $m = 3$. The corresponding De Bruijn graph $G_{2,2}$ contains the four vertices

$$\{00, 01, 10, 11\},$$

and is connected by edges labeled with the $2^3 = 8$ possible binary triples. Tracing an Eulerian circuit through this graph generates the De Bruijn sequence

$$00010111.$$

The eight cyclic windows of length 3 observed from this sequence are:

$$000, 001, 010, 101, 011, 111, 110, 100.$$

Because each binary triple appears exactly once, the identity observation map Φ_{id} is strictly injective on this sequence.

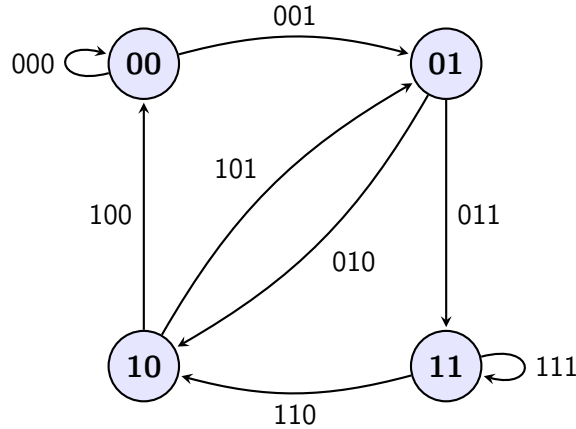


FIGURE 1. The De Bruijn graph $G_{2,2}$. An Eulerian circuit through these 8 directed edges generates the tracking sequence 00010111.

Remark 2.7. The identity observation map Φ_{id} is perfectly injective on a De Bruijn sequence. Every physical window position maps to a unique ordered tuple, which guarantees that a robot equipped with a camera can compute its absolute position from a single window observation.

Remark 2.8. If the sensor hardware cannot distinguish spatial order, the identity map Φ_{id} becomes physically unobservable. The robot no longer receives the ordered tuple $(x_1, \dots, x_m)$, but only the multiset of its constituent entries. Under this hardware constraint, the De Bruijn graph fails. Multiple distinct directed edges in $G_{k,m-1}$ will map to the identical multiset observation, destroying the guarantee of injectivity. This failure necessitates a completely different mathematical framework, which we develop in the following section.

3 Multiset Packings

The localization of an agent equipped with an order-insensitive sensor requires a mathematical framework where sequences are distinguished solely by their localized frequency distributions. In this setting, the observation space shifts from the Cartesian product of an alphabet to a family of finite multisets. This section formalizes this unordered observation model, derives the exact algebraic boundaries of the underlying state space, and establishes the strict injectivity criteria required for deterministic positioning systems.

Throughout the remainder of this work, let $\mathcal{A} = \{0, 1, \dots, k-1\}$ denote a finite alphabet of k distinct symbols, and let $m \in \mathbb{N}$ denote a fixed tracking window size.

Definition 3.1 (Multiset). A multiset of size m over an alphabet $\mathcal{A}$ is an unordered collection of elements where repetitions are permitted. Formally, a multiset is characterized by a multiplicity function $\mu : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$ satisfying the conservation constraint

$$\sum_{x \in \mathcal{A}} \mu(x) = m.$$

The integer $\mu(x)$ specifies the multiplicity of the symbol $x \in \mathcal{A}$. A multiset is denoted by explicitly enumerating its elements enclosed within double braces, such as $\{\!\{s_1, s_2, \dots, s_m\}\!\}$, where any permutation of the internal symbols yields an identical identity.

Definition 3.2 (Multiset Observation Function). Let $\mathcal{M}_m(\mathcal{A})$ denote the set of all multisets of size m over $\mathcal{A}$. The multiset observation function $f : \mathcal{A}^m \rightarrow \mathcal{M}_m(\mathcal{A})$ is the mapping that projects each ordered m -tuple to its underlying element frequencies:

$$f(s_0, s_1, \dots, s_{m-1}) = \{\!\{s_0, s_1, \dots, s_{m-1}\}\!\}.$$

Because any permutation of the sequence $(s_0, s_1, \dots, s_{m-1})$ maps to the same image under f , the observation function is surjective but non-injective for all $m > 1$ and $k > 1$. To quantify the resulting reduction in state information, we establish the exact size of the codomain $\mathcal{M}_m(\mathcal{A})$.

Lemma 3.3 (Cardinality of the Multiset Space). *The total number of distinct multisets of size m over an alphabet of k symbols is given by*

$$|\mathcal{M}_m(\mathcal{A})| = \binom{k+m-1}{m}.$$

Proof. By Definition 3.1, each multiset corresponds uniquely to a sequence of multiplicities $(c_0, c_1, \dots, c_{k-1})$ of nonnegative integers satisfying the linear equation

$$(1) \quad c_0 + c_1 + \dots + c_{k-1} = m,$$

where $c_j = \mu(j)$ for each $j \in \mathcal{A}$. We establish a bijection between the set of solutions to Equation (1) and the set of binary sequences containing exactly m zeros and $k-1$ ones.

Given a sequence of multiplicities $(c_0, c_1, \dots, c_{k-1})$, construct a binary string by writing c_0 zeros, followed by a one, followed by c_1 zeros, and continuing sequentially until terminating with c_{k-1} zeros. Since there are k variables, the construction requires exactly $k-1$ ones. The total length of the resulting binary string is $m + (k-1)$.

Conversely, any binary string of length $m + k - 1$ containing exactly $k - 1$ ones uniquely partitions the remaining m zeros into k contiguous blocks, defining a unique nonnegative integer tuple $(c_0, c_1, \dots, c_{k-1})$ that satisfies Equation (1). The number of distinct binary strings of this composition is given by the binomial coefficient selecting the positions of the m zeros out of the total slots:

$$\binom{m+k-1}{m} = \binom{k+m-1}{m}.$$

This bijective correspondence proves the claim. $\square$

Using this coordinate space, we formulate the conditions for deterministic tracking along a one-dimensional path under the periodic constraints detailed by Chen et al.

Definition 3.4 (Cycle Packing). An $(\mathcal{A}, f, n, m)$ -cycle packing is a periodic function $W : \mathbb{Z} \rightarrow \mathcal{A}$ with period n such that for all $x, y \in \mathbb{Z}$, the condition

$$f(W_x, W_{x+1}, \dots, W_{x+m-1}) = f(W_y, W_{y+1}, \dots, W_{y+m-1})$$

implies $x \equiv y \pmod{n}$.

When the sequence length reaches the information-theoretic maximum $n = \binom{k+m-1}{m}$, every element within $\mathcal{M}_m(\mathcal{A})$ is traversed exactly once per period. Such maximal cycle packings are classified as universal cycles for multisets, or Mcycles.

Example 3.5. Let $\mathcal{A} = \{0, 1, 2\}$, $k = 3$, and $m = 2$. Consider the periodic function W defined over a single period by the sequence $(0, 0, 1, 1, 2, 2)$, yielding a period length of $n = 6$. Evaluating the contiguous windows of length 2 modulo 6 generates the observations recorded in Table 1.

Index	i	Ordered Sequence	W_i	Multiset Observation	$f(W_i)$
	0	(0, 0)		$\{0, 0\}$	
	1	(0, 1)		$\{0, 1\}$	
	2	(1, 1)		$\{1, 1\}$	
	3	(1, 2)		$\{1, 2\}$	
	4	(2, 2)		$\{2, 2\}$	
	5	(2, 0)		$\{0, 2\}$	

TABLE 1. Contiguous window mappings for the ternary sequence $W = (0, 0, 1, 1, 2, 2)$.

The resulting collection of observations contains all $\binom{3+2-1}{2} = 6$ elements of $\mathcal{M}_2(\mathcal{A})$. Because no multiset observation is repeated within the period, the injectivity requirement of Definition 3.4 is satisfied, proving that W is a valid Mcycle.

3.1 Preimage Multiplicities in De Bruijn Graphs

In the classical ordered framework, generating a maximal sequence corresponds to constructing an Eulerian circuit on the De Bruijn graph $G_{k,m-1}$, where the identity observation map

preserves the unique structural boundaries between directed edges. Under the multiset observation function f , this graph-theoretic structure undergoes a fiber decomposition, where the edge labels are map-dependent.

Proposition 3.6 (Size of Multiset Preimages). *Let $\mu \in \mathcal{M}_m(\mathcal{A})$ be a multiset with symbol multiplicities given by the vector $(c_0, c_1, \dots, c_{k-1})$. The number of distinct directed edges in the De Bruijn graph $G_{k,m-1}$ that map to μ under the observation function f is given by the multinomial coefficient*

$$|f^{-1}(\mu)| = \frac{m!}{c_0!c_1!\dots c_{k-1}!}.$$

Proof. By Definition 3.2, the preimage $f^{-1}(\mu)$ consists of all ordered m -tuples $(s_0, s_1, \dots, s_{m-1}) \in \mathcal{A}^m$ whose constituent entries match the exact symbol multiplicities specified by μ . Constructing an element in $f^{-1}(\mu)$ is equivalent to assigning m distinct positions in an ordered sequence to the symbols of $\mathcal{A}$ according to their required counts.

We choose c_0 positions out of m for the symbol 0, which can be done in $\binom{m}{c_0}$ ways. From the remaining $m - c_0$ positions, we choose c_1 positions for the symbol 1 in $\binom{m-c_0}{c_1}$ ways. Inductively, for the j -th symbol, we select c_j positions from the remaining available slots. By the product rule, the total number of valid ordered arrangements is

$$|f^{-1}(\mu)| = \binom{m}{c_0} \binom{m-c_0}{c_1} \binom{m-c_0-c_1}{c_2} \dots \binom{m-\sum_{i=0}^{k-2} c_i}{c_{k-1}}.$$

Expanding the binomial coefficients into factorials yields

$$|f^{-1}(\mu)| = \frac{m!}{c_0!(m-c_0)!} \cdot \frac{(m-c_0)!}{c_1!(m-c_0-c_1)!} \dots \frac{(m-\sum_{i=0}^{k-2} c_i)!}{c_{k-1}!(m-\sum_{i=0}^{k-1} c_i)!}.$$

Because $\sum_{i=0}^{k-1} c_i = m$, the final term in the denominator reduces to $0! = 1$. The intermediate terms in the numerators and denominators cancel out diagonally, leaving

$$|f^{-1}(\mu)| = \frac{m!}{c_0!c_1!\dots c_{k-1}!}.$$

This completes the proof. □

Proposition 3.6 demonstrates that the observation function partitions the k^m directed edges of the De Bruijn graph into coarse equivalence classes. Finding a valid $(\mathcal{A}, f, n, m)$ -cycle packing is therefore equivalent to finding a sequence of edge transitions in $G_{k,m-1}$ that forms a rainbow path—a path containing at most one edge from each multiset fiber.

As shown in Figure 2, this non-injective mapping causes a structural loss of trace visibility. Standard Eulerian graph configurations are insufficient for analyzing these objects because the edge transitions are tightly constrained by non-local historical properties. Consequently, constructing these tracking paths requires alternative algebraic and hypergraph models, which we review in Section

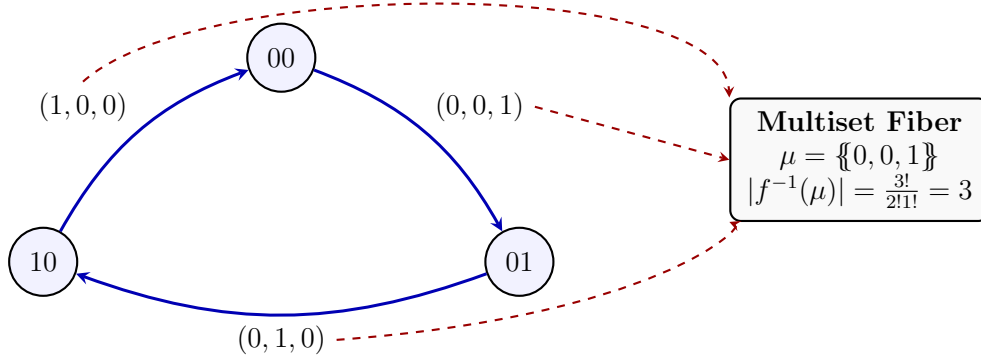


FIGURE 2. Fiber projection map for $m = 3, k = 2$. The three structurally distinct directed edges $(0, 0, 1)$, $(0, 1, 0)$, and $(1, 0, 0)$ in the De Bruijn graph $G_{2,2}$ map to the same multiset equivalence class $\{0, 0, 1\}$, eliminating edge-label uniqueness.

4 Mcycles and Divisibility Conditions

The absolute upper bound on the information density of a localization sequence is realized when the multiset observation function operates bijectively over the window space. In such a regime, every mathematically possible frequency distribution is traversed exactly once per period, minimizing sequence length while maximizing spatial resolution. This section formalizes these ideal structures, derives the fundamental arithmetic obstructions that govern their existence, and demonstrates how the algebraic failure of these conditions necessitates a transition to the study of packings.

Definition 4.1 (Multiset Universal Cycle). A Multiset Universal Cycle, or M -cycle, of order m over an alphabet $\mathcal{A}$ of size k is a periodic sequence $W : \mathbb{Z} \rightarrow \mathcal{A}$ of period length

$$L = \binom{k + m - 1}{m}$$

such that the restriction of the multiset observation function f to the set of periodic windows $\mathcal{W} = \{W_i\}_{i=0}^{L-1}$ defines a bijection

$$f|_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{M}_m(\mathcal{A}).$$

Equivalently, W is an M -cycle if every unique multiset $\mu \in \mathcal{M}_m(\mathcal{A})$ appears precisely once as a contiguous window observation per period.

The term M -cycle follows the established convention in combinatorial sequence design, representing the exact multiset analogue of the classical De Bruijn sequence. Where a standard De Bruijn sequence achieves a bijection onto the Cartesian product $\mathcal{A}^m$, an M -cycle achieves a bijection onto the compressed unordered space $\mathcal{M}_m(\mathcal{A})$.

To establish whether such ideal sequences can exist for an arbitrary choice of alphabet size k and window length m , we analyze the distribution of symbols within the cycle. We begin by isolating the total number of times a fixed symbol must appear across the entire family of m -multisets.

Lemma 4.2 (Color Distribution Identity). *Let $\mathcal{A} = \{0, 1, \dots, k-1\}$ be an alphabet of size k . For a fixed symbol $x \in \mathcal{A}$, the total number of times x appears across all distinct multisets in $\mathcal{M}_m(\mathcal{A})$, counting multiplicities, is given by the identity*

$$(2) \quad \sum_{j=1}^m j \binom{k+m-j-2}{m-j} = \frac{m}{k} \binom{k+m-1}{m}.$$

Proof. We establish the identity via a direct double-counting argument over the set of ordered pairs (μ, α) , where $\mu \in \mathcal{M}_m(\mathcal{A})$ and α is an element within μ such that $\alpha = x$.

First, we evaluate the left-hand side of Equation (2) by conditioning on the exact multiplicity j of the symbol x within a given multiset μ . The variable j can range from 1 to m . For a fixed multiplicity j , the remaining $m-j$ elements of the multiset must be selected from the remaining $k-1$ symbols in $\mathcal{A} \setminus \{x\}$. By Lemma 3.3, the number of ways to form a multiset of size $m-j$ from an alphabet of size $k-1$ is given by

$$\binom{(k-1) + (m-j) - 1}{m-j} = \binom{k+m-j-2}{m-j}.$$

Multiplying this count by j (the number of copies of x in each such multiset) and summing over all possible values of j yields the total number of occurrences of x :

$$\sum_{j=1}^m j \binom{k+m-j-2}{m-j}.$$

Second, we evaluate the right-hand side via structural symmetry. The total number of available slots across all distinct multisets in $\mathcal{M}_m(\mathcal{A})$ is equal to the total number of multisets multiplied by the window size m :

$$\text{Total Slots} = m \cdot |\mathcal{M}_m(\mathcal{A})| = m \binom{k+m-1}{m}.$$

Because the structural rules defining $\mathcal{M}_m(\mathcal{A})$ are invariant under any permutation of the symbol labels in $\mathcal{A}$, every one of the k distinct colors must occupy an identical number of slots across the global space. Dividing the total number of slots by k yields the exact allocation for the symbol x :

$$\text{Total Occurrences of } x = \frac{m}{k} \binom{k+m-1}{m}.$$

Equating these two independent counting expressions establishes Equation (2). $\square$

Using this distribution identity, we derive the fundamental arithmetic boundary condition established by Hurlbert, Johnson, and Zahl (2009) governing the existence of M -cycles.

Theorem 4.3 (Divisibility Condition for Mcycles). *If a Multiset Universal Cycle of order m over an alphabet of size k exists, then the alphabet size must divide the total cardinality of the multiset space:*

$$(3) \quad k \mid \binom{k+m-1}{m}.$$

Proof. Let $W : \mathbb{Z} \rightarrow \mathcal{A}$ be a valid M -cycle of period length $L = \binom{k+m-1}{m}$. Let N_c denote the explicit number of times the specific color symbol $c \in \mathcal{A}$ is written within a single period block of W , defined by the index set $\{0, 1, \dots, L-1\}$.

Because W is periodic and symmetric with respect to the alphabet palette, the sequence must exhibit global shift invariance over its period; no symbol can be structurally prioritized. Consequently, each of the k distinct colors must appear an equal number of times within the sequence period. This requires that

$$(4) \quad N_c = \frac{L}{k} = \frac{1}{k} \binom{k+m-1}{m}.$$

Now consider the total count of color c observed across the complete set of periodic windows, denoted by T_c . For any single position $i \in \{0, 1, \dots, L-1\}$ where the symbol $W_i = c$, this specific historical entry is read exactly m times by a sliding window of length m . Specifically, it participates in the windows starting at indices $i, i-1, \dots, i-m+1 \pmod{L}$. Because every position contributes to exactly m windows, the total number of times color c is observed across all windows in the sequence is

$$(5) \quad T_c = m \cdot N_c = \frac{mL}{k}.$$

By Definition 4.1, the windows of W form a perfect bijection onto $\mathcal{M}_m(\mathcal{A})$, meaning every valid multiset is expressed exactly once. Thus, the sequence-derived total T_c must precisely equal the total structural multiplicity of color c across the entire space $\mathcal{M}_m(\mathcal{A})$, which we denote as T'_c . Applying Lemma 4.2, we have

$$(6) \quad T'_c = \frac{m}{k} \binom{k+m-1}{m}.$$

Equating the sequence observation count to the spatial count yields $T_c = T'_c$, which simplifies directly to the relation $L = \binom{k+m-1}{m}$, verifying structural consistency.

However, the variable N_c represents a physical count of discrete symbol entries within a sequence, meaning it is strictly constrained to the domain of positive integers ($N_c \in \mathbb{Z}$). Examining Equation (4), the term $\frac{1}{k} \binom{k+m-1}{m}$ must evaluate to an integer. This holds true if and only if k divides $\binom{k+m-1}{m}$ without a remainder, completing the proof. $\square$

Example 4.4 (Divisibility Failure for $k=3, m=3$). Let $\mathcal{A} = \{0, 1, 2\}$, yielding an alphabet size of $k=3$, and let the tracking window size be $m=3$. Evaluating Lemma 3.3, the total number of distinct 3-multisets over this ternary alphabet is

$$|\mathcal{M}_3(\{0, 1, 2\})| = \binom{3+3-1}{3} = \binom{5}{3} = 10.$$

The complete observation space $\mathcal{M}_3(\mathcal{A})$ consists of the following 10 multisets:

$$\begin{aligned} \{0, 0, 0\}, \quad \{0, 0, 1\}, \quad \{0, 0, 2\}, \quad \{0, 1, 1\}, \quad \{0, 1, 2\}, \\ \{0, 2, 2\}, \quad \{1, 1, 1\}, \quad \{1, 1, 2\}, \quad \{1, 2, 2\}, \quad \{2, 2, 2\}. \end{aligned}$$

By applying the symmetry argument from Lemma 4.2, the total number of slots across all 10 multisets is $3 \times 10 = 30$. Divided equally among the $k=3$ colors, each individual color symbol must appear exactly $T'_c = 10$ times across the global space.

If a perfect M -cycle W of length $L = 10$ existed for these parameters, Theorem 4.3 dictates that the alphabet size $k = 3$ must divide the sequence length $L = 10$. This evaluates to

$$N_c = \frac{10}{3} \notin \mathbb{Z},$$

which is physically impossible.

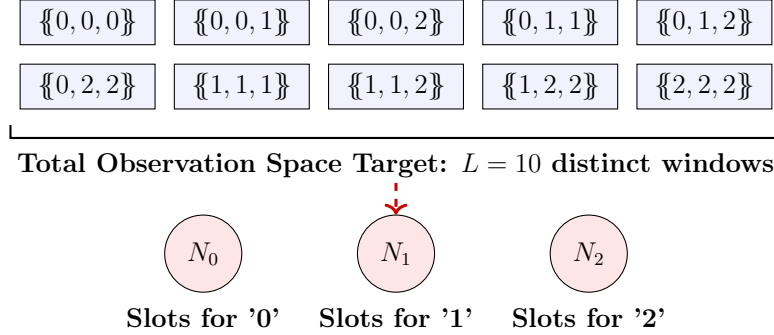


FIGURE 3. The arithmetic obstruction for $k = 3, m = 3$. The observation space contains exactly 10 elements. To form a perfect universal cycle, these 10 distinct window steps must partition perfectly across the 3 symmetric symbol paths of the alphabet, which fails because $10 \equiv 1 \pmod{3}$.

As mapped out in Figure 3, any periodic sequence attempting to generate these 10 observations consecutively will encounter a fundamental alignment failure. The sequence slots cannot be partitioned into equal integer blocks, ensuring that no M -cycle can exist for these design parameters.

4.1 Consequences of Divisibility Constraints and the Shift to Packings

The arithmetic constraint established by Theorem 4.3 serves as a strict filter for the existence of bijective universal sequences. When evaluating a potential robot positioning architecture, the design space is immediately bifurcated based on this condition:

- (1) **Divisibility Satisfied** ($k \mid \binom{k+m-1}{m}$): The existence of an M -cycle is conditionally admissible. However, as noted by Hurlbert, Johnson, and Zahl (2009), the divisibility condition is a necessary but not a sufficient condition. For instance, even when divisibility holds, structural obstacles within the underlying state transition graph can prevent the sequence from closing cleanly into a single Eulerian circuit. Hurlbert et al. explicitly resolved the existence landscape for small window parameters, proving that while M -cycles exist for all $m = 2$, they fail to exist for certain higher orders such as $m \in \{3, 4, 6\}$ due to structural connectivity traps.
- (2) **Divisibility Fails** ($k \nmid \binom{k+m-1}{m}$): The possibility of constructing a bijective tracking sequence is entirely eliminated. A robot tracking path cannot be engineered to read every possible multiset exactly once.

This second scenario represents a critical bottleneck for low-cost hardware systems, where sensor limitations might dictate an alphabet size k and window length m that violate the divisibility rule. To maintain deterministic tracking capabilities under these conditions, the design paradigm must shift from a search for a *bijjective universal cycle* to the optimization of an *injective cycle packing*.

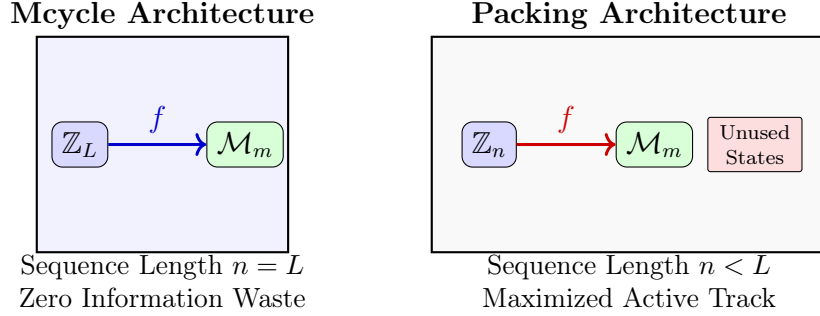


FIGURE 4. A structural comparison of mapping architectures. When divisibility fails, the exact bijection of the Mcycle framework must be abandoned in favor of an injective cycle packing that maximizes the trackable subset of $\mathcal{M}_m(\mathcal{A})$ while leaving unresolvable state combinations unmapped.

As illustrated in Figure 4, the packing framework preserves the strict injectivity required for unique position decoding while accepting a bounded reduction in sequence length. Instead of tracking across the entire multiset universe, we search for the longest possible sequence W of length $n < \binom{k+m-1}{m}$ that avoids any repeated observations.

This structural relaxation introduces a deep combinatorial challenge: determining the maximum achievable packing length $n^*(k, m)$ for configurations where perfect universality is blocked. Glock, Joos, Kühn, and Osthus (2018) advanced this field by establishing comprehensive asymptotic existence theorems for hypergraph decompositions, which imply the existence of near-optimal subset universal cycles for large graph scales. However, as cautioned in the literature, this reduction operates strictly at the *graph-level* via isomorphic line graphs; it does not automatically translate to equivalent *sequence-level* designs because the state-transition paths for multisets diverge from the step-by-step memory updates of subset windows.

More recently, Chen, Keevash, Kennedy, de Panafieu, and Vetta (2024) successfully designed optimal higher-dimensional packing structures for specific coordinate grids, yet explicitly identified the characterization of the general one-dimensional multiset packing problem as an open research frontier.

5 Literature Synthesis

The arithmetic constraints derived in Section 4 demonstrate that resolving the unordered localization problem requires analyzing the underlying structural properties of sequence transition spaces. This section maps the combinatorial lineage of this problem, tracing the development of universal sequences from classical subsets to hypergraph architectures. By contextualizing these historical frameworks, we isolate the specific theoretical gaps regarding non-divisible one-dimensional packings that motivate our core analytical results.

5.1 Universal Cycles

The systematic expansion of De Bruijn sequences beyond traditional Cartesian products was initiated by Chung, Diaconis, and Graham (1992). Their work established the foundational framework for universal cycles (U -cycles) over an array of combinatorial objects, with a primary focus on the t -element subsets of a finite set.

Definition 5.1 (Subset Universal Cycle). Let $[n] = \{1, 2, \dots, n\}$ be a finite set. A t -subset Universal Cycle (U -cycle) over $[n]$ is a periodic sequence $U : \mathbb{Z} \rightarrow [n]$ of period length $\binom{n}{t}$ such that every distinct t -element subset of $[n]$ appears exactly once as the underlying un-ordered set of a contiguous sliding window of length t .

Chung et al. demonstrated that finding a U -cycle is equivalent to constructing an Eulerian circuit in a directed transition graph where vertices represent $(t - 1)$ -subsets and edges represent the t -subsets formed by window advancements. However, their work left the general existence of subset U -cycles open for arbitrary t and n . More critically, their framework fundamentally relies on the strict exclusion of repeated elements, leaving a structural gap when applied to physical sensor tracking, which naturally permits character repetition and must operate over the full multiset space $\mathcal{M}_m([k])$.

5.2 The Multiset-to-Subset Isomorphism and its Limitations

To bridge this gap, Hurlbert, Johnson, and Zahl (2009) established an algebraic link between subset and multiset spaces. They proved that the two underlying combinatorial structures share an identical cardinality-defined framework under a linear coordinate shift.

Theorem 5.2 (Hurlbert-Johnson-Zahl Isomorphism). *Let $[k] = \{1, 2, \dots, k\}$ define a tracking alphabet, and let $\mathcal{M}_m([k])$ denote the space of all m -multisets over $[k]$. There exists an exact bijection Φ mapping the m -subsets of $[k + m - 1]$ to $\mathcal{M}_m([k])$.*

Proof. Let $S \subset [k + m - 1]$ be an arbitrary m -subset, whose elements are strictly ordered such that $1 \leq s_1 < s_2 < \dots < s_m \leq k + m - 1$. We define the transformation $\Phi(S) = \{x_1, x_2, \dots, x_m\}$ using the element-wise shifting operator:

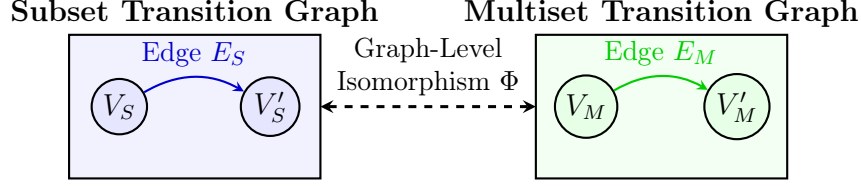
$$(7) \quad x_i = s_i - i + 1 \quad \text{for all } 1 \leq i \leq m.$$

Because the original subset elements are strictly increasing integers, the gap between consecutive elements satisfies $s_{i+1} \geq s_i + 1$. Evaluating the transformation for consecutive indices yields:

$$x_{i+1} = s_{i+1} - (i + 1) + 1 \geq (s_i + 1) - i = s_i - i + 1 = x_i.$$

The shift maps the strict inequality of the subset space into the weak inequality $x_1 \leq x_2 \leq \dots \leq x_m$, accommodating element repetition. Evaluating the boundary condition for the maximal element yields $x_m = s_m - m + 1 \leq (k + m - 1) - m + 1 = k$. Thus, the image sequence satisfies $1 \leq x_i \leq k$, placing it precisely in $\mathcal{M}_m([k])$. Because the mapping is linear and uniquely invertible via $s_i = x_i + i - 1$, it is a bijection. $\square$

It is vital to explicitly state a critical caveat regarding Theorem 5.2: this correspondence operates strictly as a **graph-level isomorphism**, not a direct sequence-level transformation.



While Φ maps the state vertices and transitions of the subset graph bijectively to those of the multiset graph, an Eulerian circuit in the subset graph cannot be translated into an M -cycle sequence through a simple element-by-element coordinate application.

The shifting dynamics of a sliding window force distinct path-tracking constraints in the two spaces. Thus, the isomorphism is highly powerful for proving existence via structural equivalence, but it remains fundamentally insufficient for constructing physical sequences.

5.3 Hypergraph Connections and Asymptotic Existence

Because the isomorphism preserves graph-level connectivity, structural questions regarding U -cycles can be analyzed using hypergraph theory.

Definition 5.3 (Hypergraph and Tight Euler Tour). A hypergraph $\mathcal{H} = (V, E)$ is m -uniform if every hyperedge $e \in E$ contains exactly m vertices. A sequence of vertices forms a tight Euler tour in $\mathcal{H}$ if every consecutive block of m vertices constitutes a valid hyperedge in E , and every hyperedge is traversed exactly once.

Constructing a U -cycle for m -subsets of $[n]$ corresponds to finding a tight Euler tour in the complete m -uniform hypergraph K_n^m . A tight Euler tour in K_n^m is a cyclic sequence of vertices in which every consecutive block of m vertices forms a distinct hyperedge. Since the hyperedges of K_n^m are exactly the m -subsets of $[n]$, this vertex sequence matches Definition 5.1 and is precisely a U -cycle.

This architectural mapping enabled Glock, Joos, Kühn, and Osthus (2020) to resolve long-standing existence questions using probabilistic hypergraph decompositions. They established a precise asymptotic existence theorem:

Theorem 5.4 (Glock et al. Asymptotic Existence). *For a fixed window size m , there exists a baseline integer n_0 such that for all $n \geq n_0$ satisfying the structural degree condition $n \mid \binom{n}{m}$, a tight Euler tour exists in the complete m -uniform hypergraph K_n^m .*

By applying the Hurlbert et al. isomorphism to Theorem 5.4, we obtain a direct existence proof for multiset sequences: for a fixed window size m , there exists a baseline alphabet size k_0 such that for all $k \geq k_0$ satisfying the divisibility condition $k \mid \binom{k+m-1}{m}$, an M -cycle exists.

However, this framework possesses an inherent operational limitation: the underlying proof relies on the Hypergraph Regularity Lemma and randomized iterative absorption, making it strictly *non-constructive*. It guarantees that an M -cycle exists asymptotically, but provides no method for constructing the longest possible cycle packing when the divisibility condition fails.

5.4 Higher-Dimensional Dimensions and the 1D Frontier

The limitations of exact algebraic closure in one-dimensional spaces led researchers to alter the layout parameters entirely. Chen, Keevash, Kennedy, de Panafieu, and Vetta (2024) expanded the localization paradigm into higher dimensions ($d \geq 2$). Using multidimensional torus topologies and vector-sum packings, they engineered near-optimal grid colorings that achieve the information-theoretic upper bound within a constant factor, completely bypassing the 1D divisibility restriction.

Crucially, Chen et al. noted that their multidimensional grid structures do not scale down to a single dimension, explicitly writing:

“We are unaware of any results in the literature on multiset packing in one dimension.”

When the observation map is forced into a non-divisible one-dimensional space, finding the maximum injective sequence length is structurally reminiscent of constrained path selection problems that are NP-hard in their general form, and it is widely conjectured to be computationally challenging.

Consequently, designing maximal sequences for low-cost, 1D tracking systems under imperfect parameters remains an active research frontier.

6 Analytical Contributions

In this section, we present the core analytical contributions of this paper concerning the 1D multiset packing problem. The first contribution addresses the tight combinatorial bound and explicit sequence construction for the window size fixed at $m = 3$ when the alphabet size k is a multiple of three. This scenario is characterized by a structural deficiency that precludes the existence of a perfect universal cycle (M-cycle). We provide an unconditional algebraic upper bound, illuminate a hidden structural requirement governing the omitted multisets, introduce a novel piecewise symmetry reduction framework that overcomes traditional computational barriers, and establish exact sequence designs up to $k = 12$. The second contribution establishes the inherent insufficiency of scalar-sum mappings for all $k \geq 3$ and $m \geq 2$, demonstrating that single-integer coordinate metrics inevitably violate injectivity via the pigeonhole principle.

6.1 A Combinatorial Lower Bound for $m = 3$

The divisibility condition established in Section 4 demonstrates that when $3 \mid k$, a perfect M-cycle over the full space of 3-multisets $\mathcal{M}_3(\mathbb{Z}_k)$ cannot exist. The natural question then becomes: what is the maximum length of a sequence packing when an M-cycle is structurally impossible? This subsection systematically explores this question for the case $m = 3$.

Our initial attempt to establish a constructive sequence for the $m = 3$ packing case focused on adapting the inductive block-substitution machinery of Hurlbert, Johnson, and Zahl

(2009)¹. Their classic framework builds an M -cycle for n colors from an existing base cycle for $n - 3$ colors when $\gcd(n, 3) = 1$. We attempted to extend this recursive technique to the deficient case where $3 \mid n$ by identifying specific sequence coordinates of a base color (e.g., color 1) within the standard cyclic words S_n, T_n, T'_n, U_n, V_n and substituting them with newly introduced alphabet symbols $\{n - 2, n - 1, n\}$ at designated "substitution loci."

However, direct algorithmic parsing and exhaustive verification revealed that this recursive substitution approach breaks down completely under the constraints of the deficient case. The attempt contained several fatal logical and arithmetic flaws that illustrate the hidden complexity of the problem:

- (1) **Arithmetic Breakdown:** In calculating the distribution of substitution loci for the parameter $k = 12$, the formulation inadvertently introduced equations that reduced to the mathematical absurdity $12 = 4$, collapsing the balance indices of the sequence.
- (2) **Alphabet Mismatch:** The inductive step from a base case of $n = 10$ to a target alphabet of $k = 12$ incorrectly defined the new symbols as $a = n - 5 = 5$, $b = n - 4 = 6$, and $c = n - 3 = 7$. Because these symbols already belonged to the base alphabet $\mathcal{C}_{\text{old}} = \{1, 2, \dots, 7\}$, the substitution corrupted the existing windows and failed to introduce a valid disjoint extension.
- (3) **Logical Contradiction:** A modified sequence T was obtained by exhaustively replacing every single occurrence of the symbol 1 with a new symbol a . Yet, subsequent steps in the proof explicitly relied on the assertion that T still contained occurrences of 1 to maintain continuity, a physical impossibility.
- (4) **Invalid Recurrence Relations:** The proof asserted that the number of isolated substitution positions grew exponentially according to the relation $2 \cdot 3^t$, while simultaneously enforcing a boundary condition that required the total number of loci to scale linearly as $n/3$. This contradiction caused a complete failure of the sequence balance equations.

6.1.1 Correct Upper Bound We establish a definitive algebraic upper bound for the $m = 3$ packing case under the condition $3 \mid k$.

Definition 6.1 (Cycle Packing and $M_m(k)$). Fix $k, m \in \mathbb{Z}^+$. Let the symbol alphabet be the ring of integers modulo k , denoted $\mathbb{Z}_k = \{0, 1, \dots, k - 1\}$. A (k, m) -cycle packing is a periodic sequence $W : \mathbb{Z} \rightarrow \mathbb{Z}_k$ of period n such that the sliding windows of length m , defined as

$$\mu_i := \{W_i, W_{i+1}, \dots, W_{i+m-1}\} \quad (\text{interpreted as a multiset}), \quad i = 0, 1, \dots, n - 1,$$

are pairwise distinct. Because the mapping $i \mapsto \mu_i$ is injective, the window-set $S := \{\mu_0, \mu_1, \dots, \mu_{n-1}\} \subseteq \mathcal{M}_m(\mathbb{Z}_k)$ has cardinality $|S| = n$. The maximum possible length of such a valid packing is defined as

$$M_m(k) := \max\{n \in \mathbb{Z}^+ : \text{there exists a valid } (k, m)\text{-cycle packing of length } n\}.$$

For any given multiset $\mu \in \mathcal{M}_m(\mathbb{Z}_k)$ and a target color $x \in \mathbb{Z}_k$, we denote the frequency of x within μ as $\text{mult}_x(\mu)$.

¹Hurlbert, G., Johnson, T., & Zahl, J. (2009). Universal cycles for multisets. *Discrete Mathematics*, 309(17), 5321-5327.

Theorem 6.2. *Let W be a valid (k, m) -cycle packing of length n with an associated window-set S . Then, for every individual color symbol $x \in \mathbb{Z}_k$, the total multiplicity summed across all active windows must satisfy the congruence*

$$\sum_{\mu \in S} \text{mult}_x(\mu) \equiv 0 \pmod{m}.$$

Proof. Let $N_x := |\{i \in \{0, 1, \dots, n-1\} : W_i = x\}|$ denote the exact number of positions in the periodic sequence W that are assigned the color x . Consider an arbitrary fixed index i such that $W_i = x$. In a sliding window framework of size m , this specific coordinate W_i is contained within exactly m distinct windows, namely the windows that initiate at indices $i, i-1, i-2, \dots, i-m+1 \pmod{n}$.

To see this explicitly, note that a window initiating at index j covers the index set $\{j, j+1, \dots, j+m-1\}$. The symbol W_i is covered by this window if and only if $j \leq i \leq j+m-1 \pmod{n}$, which can be rewritten as the inequality constraint $i-m+1 \leq j \leq i \pmod{n}$. This interval contains exactly m valid integer values for the starting index j .

Consequently, each individual appearance of the color x in the sequence adds exactly 1 to the multiplicity of x in exactly m different windows, contributing a value of m to the grand sum of multiplicities across all windows. Summing over all occurrences of x , we obtain

$$\sum_{j=0}^{n-1} \text{mult}_x(\mu_j) = \sum_{i: W_i=x} m = mN_x.$$

Because the packing is valid, the windows are pairwise distinct, implying that the collection of sequence windows $\{\mu_0, \mu_1, \dots, \mu_{n-1}\}$ maps bijectively to the set S . Thus, the left-hand side of the equation satisfies $\sum_{j=0}^{n-1} \text{mult}_x(\mu_j) = \sum_{\mu \in S} \text{mult}_x(\mu)$. Substituting this back yields

$$\sum_{\mu \in S} \text{mult}_x(\mu) = mN_x \equiv 0 \pmod{m},$$

which completes the proof. $\square$

This condition is an invariant for all window sizes m . In the special case where a perfect universal cycle is achievable, $S = \mathcal{M}_m(\mathbb{Z}_k)$, and Theorem 6.2 naturally recovers the classic structural constraints of the standard M-cycle literature while generalizing seamlessly to bounded sub-packings.

6.1.2 The Orbit Theorem for 3-Multisets We now specialize the analysis to the case where the window size is fixed at $m = 3$. The group of integers under addition, $\mathbb{Z}_k$, naturally acts on the full multiset space $\mathcal{M}_3(\mathbb{Z}_k)$ via the translation operator τ_c :

$$\tau_c(\{s_1, s_2, s_3\}) = \{(s_1 + c) \bmod k, (s_2 + c) \bmod k, (s_3 + c) \bmod k\}.$$

Theorem 6.3 (Orbit Theorem for 3-Multisets). *If $k \not\equiv 0 \pmod{3}$, then every single translation orbit of $\mathcal{M}_3(\mathbb{Z}_k)$ under the action of $\mathbb{Z}_k$ has a cardinality of exactly k . If $k \equiv 0 \pmod{3}$, then every translation orbit has a size of exactly k with the exception of exactly one unique orbit, denoted $\mathcal{E}$, defined as*

$$\mathcal{E} := \left\{ \left\{ a, a + \frac{k}{3}, a + \frac{2k}{3} \right\} : a = 0, 1, \dots, \frac{k}{3} - 1 \right\},$$

which possesses an exceptional orbit size of exactly $k/3$.

Proof. By the Orbit-Stabilizer Theorem, the size of any orbit containing a multiset μ is given by the quotient $k/|\text{Stab}(\mu)|$, where the stabilizer subgroup is defined as $\text{Stab}(\mu) = \{c \in \mathbb{Z}_k : \tau_c(\mu) = \mu\}$. We evaluate the structure of $\text{Stab}(\mu)$ by partitioning the multiset space into three exhaustive cases based on the support size (the number of distinct elements) of μ .

Case 1: Support Size 1. The multiset takes the form $\mu = \{c, c, c\}$ for some $c \in \mathbb{Z}_k$. Applying the translation yields $\tau_h(\mu) = \{c+h, c+h, c+h\}$. Enforcing the equality $\tau_h(\mu) = \mu$ requires that the single element satisfies $c+h \equiv c \pmod{k}$, which directly forces $h \equiv 0 \pmod{k}$. Thus, the stabilizer is the trivial subgroup $\text{Stab}(\mu) = \{0\}$, and the orbit size is $k/1 = k$.

Case 2: Support Size 2. The multiset takes the form $\mu = \{c, c, d\}$ where $c \neq d$. We can express d as $c+g \pmod{k}$ for some non-zero residue $g \neq 0$, yielding $\mu = \{c, c, c+g\}$. Applying the translation operator gives $\tau_h(\mu) = \{c+h, c+h, c+h+g\}$. Both μ and $\tau_h(\mu)$ are multisets with a unique element of multiplicity 2 (which is c for μ , and $c+h$ for $\tau_h(\mu)$). Because translation maps elements bijectively, a multiset equality forces the matching of elements with identical multiplicities. This requires $c+h \equiv c \pmod{k}$, which implies $h \equiv 0 \pmod{k}$. Hence, the stabilizer is trivial, $\text{Stab}(\mu) = \{0\}$, and the orbit size is $k/1 = k$.

Case 3: Support Size 3. The multiset consists of three distinct elements, $\mu = \{a, b, c\}$. Suppose there exists a non-trivial stabilizer element $h \neq 0$ such that $\tau_h(\mu) = \mu$. The translation map $x \mapsto x+h \pmod{k}$ defines a bijection on $\mathbb{Z}_k$ that is strictly fixed-point-free because $h \neq 0$. When restricted to the elements of the subset $\{a, b, c\}$, the condition $\tau_h(\mu) = \mu$ implies that τ_h must act as a fixed-point-free permutation of this 3-element set. The only fixed-point-free permutation on a set of cardinality 3 is a 3-cycle.

Because the cube of any 3-cycle is the identity permutation, applying the translation operator three times must map each element back to itself pointwise. Thus, $\tau_{3h}(a) = a$, which requires $a+3h \equiv a \pmod{k}$, forcing $3h \equiv 0 \pmod{k}$. Therefore, any non-trivial stabilizing element h must belong to the algebraic subgroup

$$H_3 := \{x \in \mathbb{Z}_k : 3x \equiv 0 \pmod{k}\}.$$

We now evaluate the structure of H_3 based on the divisibility of k :

- If $3 \nmid k$, the equation $3x \equiv 0 \pmod{k}$ has only the unique trivial solution $x = 0$ in the ring $\mathbb{Z}_k$. Consequently, $H_3 = \{0\}$, which forces $\text{Stab}(\mu) = \{0\}$. Every orbit must therefore have a size of exactly k .
- If $3 \mid k$, then H_3 is a cyclic subgroup of order 3, given explicitly by the residue set $H_3 = \{0, k/3, 2k/3\}$. For a support-3 multiset to possess a non-trivial stabilizer, its stabilizer subgroup must match H_3 exactly. This occurs if and only if $\tau_{k/3}(\{a, b, c\}) = \{a, b, c\}$. Under the 3-cycle permutation action, we can order the elements without loss of generality such that $\tau_{k/3}(a) = b$ and $\tau_{k/3}(b) = c$. This implies that $b \equiv a + k/3 \pmod{k}$ and $c \equiv b + k/3 \equiv a + 2k/3 \pmod{k}$, which is precisely the definition of the exceptional set $\mathcal{E}$.

A direct calculation verifies that for any element in $\mathcal{E}$, translating by $k/3$ yields $\tau_{k/3}(\{a, a+k/3, a+2k/3\}) = \{a+k/3, a+2k/3, a+k\} \equiv \{a, a+k/3, a+2k/3\}$. Thus, $\text{Stab}(\mu) = H_3$

(order 3) holds exclusively for multisets belonging to $\mathcal{E}$, while all other support-3 multisets exhibit a trivial stabilizer.

By the Orbit-Stabilizer Theorem, the orbit size for $\mathcal{E}$ is $k/3$. Since the $k/3$ elements listed in the definition of $\mathcal{E}$ are pairwise distinct and map to each other under translation, they constitute exactly one complete standalone orbit. Every other multiset in the space has an orbit size of exactly k . $\square$

We now combine the orbit structure and the generalized divisibility conditions to derive an absolute upper bound for $M_3(k)$ when $3 \mid k$.

Lemma 6.4 (Total Multiplicity Identity). *Let $\mathcal{M}_3(\mathbb{Z}_k)$ represent the complete space of 3-multisets over $\mathbb{Z}_k$. The total multiplicity sum for any fixed color x satisfies*

$$T := \sum_{\mu \in \mathcal{M}_3(\mathbb{Z}_k)} \text{mult}_x(\mu) = \frac{3}{k} \binom{k+2}{3},$$

yielding a uniform invariant value for every color $x \in \mathbb{Z}_k$.

Proof. By the translation-invariance of the complete multiset space $\mathcal{M}_3(\mathbb{Z}_k)$, the sum of multiplicities T must be identical for all choices of the color symbol x . Every individual 3-multiset μ contains exactly 3 elements by definition, which implies that $\sum_{x \in \mathbb{Z}_k} \text{mult}_x(\mu) = 3$. If we sum this relation across the entire space of multisets and swap the order of summation, we obtain

$$\sum_{x \in \mathbb{Z}_k} T = \sum_{x \in \mathbb{Z}_k} \sum_{\mu \in \mathcal{M}_3(\mathbb{Z}_k)} \text{mult}_x(\mu) = \sum_{\mu \in \mathcal{M}_3(\mathbb{Z}_k)} \sum_{x \in \mathbb{Z}_k} \text{mult}_x(\mu) = \sum_{\mu \in \mathcal{M}_3(\mathbb{Z}_k)} 3 = 3 |\mathcal{M}_3(\mathbb{Z}_k)|.$$

The total number of combinations of k items chosen 3 at a time with repetition allowed is given by the combination formula $|\mathcal{M}_3(\mathbb{Z}_k)| = \binom{k+3-1}{3} = \binom{k+2}{3}$. Substituting this into the right-hand side yields $\sum_{x \in \mathbb{Z}_k} T = 3 \binom{k+2}{3}$. Since the left-hand side is a uniform sum over k identical terms, it equals kT . Dividing through by k gives

$$T = \frac{3}{k} \binom{k+2}{3},$$

proving the lemma. $\square$

Lemma 6.5 (Closed Form of the Residue). *For any alphabet size satisfying $k \equiv 0 \pmod{3}$, let $k = 3j$ for $j \in \mathbb{Z}^+$. The binomial coefficient can be factored exactly as*

$$\binom{k+2}{3} = kQ + \frac{k}{3},$$

where the quotient parameter is an integer defined by

$$Q = \frac{3j(j+1)}{2} = \frac{k(k+3)}{6} \in \mathbb{Z}.$$

Proof. We expand the definition of the binomial coefficient:

$$\binom{k+2}{3} = \frac{(k+2)(k+1)k}{6}.$$

Substituting the parametrization $k = 3j$ into this equation yields

$$\binom{3j+2}{3} = \frac{(3j+2)(3j+1)(3j)}{6} = \frac{j(3j+1)(3j+2)}{2}.$$

We expand the polynomial product within the numerator:

$$(3j+1)(3j+2) = 9j^2 + 6j + 3j + 2 = 9j^2 + 9j + 2.$$

Substituting this expansion back into the expression gives

$$\binom{k+2}{3} = \frac{j(9j^2 + 9j + 2)}{2} = \frac{9j^3 + 9j^2 + 2j}{2} = \frac{9j^3 + 9j^2}{2} + j.$$

We now algebraically isolate and factor out the term $k = 3j$ from the cubic fractional component:

$$\frac{9j^3 + 9j^2}{2} = 3j \cdot \frac{3j^2 + 3j}{2} = k \cdot \frac{3j(j+1)}{2}.$$

Let $Q = \frac{3j(j+1)}{2}$. Because j and $j+1$ are two consecutive integers, one of them must be an even number. Consequently, their product $j(j+1)$ is always divisible by 2, meaning that $\frac{j(j+1)}{2}$ is an integer, which guarantees that $Q = 3 \cdot \frac{j(j+1)}{2} \in \mathbb{Z}$. Re-substituting $j = k/3$ back into the expression for Q gives

$$Q = \frac{3(k/3)(k/3+1)}{2} = \frac{k(k/3+1)}{2} = \frac{k(\frac{k+3}{3})}{2} = \frac{k(k+3)}{6}.$$

Combining these steps, we obtain

$$\binom{k+2}{3} = kQ + j = kQ + \frac{k}{3},$$

which proves the closed form. $\square$

Corollary 6.6. *For any parameter choice satisfying $k \equiv 0 \pmod{3}$, the total uniform multiplicity invariant satisfies the congruence*

$$T \equiv 1 \pmod{3}.$$

Proof. By applying the explicit algebraic expansion established in Lemma 6.5 directly to the expression for T derived in Lemma 6.4, we can write

$$T = \frac{3}{k} \binom{k+2}{3} = \frac{3}{k} \left(kQ + \frac{k}{3} \right) = \frac{3kQ}{k} + \frac{3k}{3k} = 3Q + 1.$$

Because Q is proven to be an integer ($Q \in \mathbb{Z}$), the term $3Q$ is an exact multiple of 3. Reducing this equation modulo 3 yields

$$T = 3Q + 1 \equiv 0 + 1 \equiv 1 \pmod{3},$$

which completes the proof. $\square$

Theorem 6.7. *For all alphabet parameters satisfying $k \equiv 0 \pmod{3}$, the maximum possible length of a valid $(k, 3)$ -cycle packing is strictly bounded by*

$$M_3(k) \leq \binom{k+2}{3} - \frac{k}{3}.$$

Proof. Let W be an arbitrary, valid $(k, 3)$ -cycle packing of maximum length, let $n = |S|$ represent the total number of active sliding windows included in the packing sequence, and let $R := \mathcal{M}_3(\mathbb{Z}_k) \setminus S$ denote the set of omitted (dropped) multisets. The cardinality of this dropped set satisfies $|R| = \binom{k+2}{3} - n$.

By invoking Theorem 6.2 under the parameter constraints $m = 3$, we know that for every color symbol $x \in \mathbb{Z}_k$, the active window set must satisfy

$$\sum_{\mu \in S} \text{mult}_x(\mu) \equiv 0 \pmod{3}.$$

By Corollary 6.6, the total multiplicity across the entire unconstrained multiset space is bounded by

$$\sum_{\mu \in \mathcal{M}_3(\mathbb{Z}_k)} \text{mult}_x(\mu) = T \equiv 1 \pmod{3}.$$

Because the full space is partitioned into active and dropped elements ($\mathcal{M}_3(\mathbb{Z}_k) = S \cup R$ with $S \cap R = \emptyset$), we can express the total multiplicity sum over the dropped set R as the linear difference

$$\sum_{\mu \in R} \text{mult}_x(\mu) = \sum_{\mu \in \mathcal{M}_3(\mathbb{Z}_k)} \text{mult}_x(\mu) - \sum_{\mu \in S} \text{mult}_x(\mu).$$

Reducing this difference equation modulo 3 using our established congruences yields

$$\sum_{\mu \in R} \text{mult}_x(\mu) \equiv 1 - 0 \equiv 1 \pmod{3}.$$

Because the multiplicity count $\text{mult}_x(\mu)$ is strictly non-negative, the sum $\sum_{\mu \in R} \text{mult}_x(\mu)$ must be a non-negative integer. The smallest non-negative integer congruent to 1 (mod 3) is 1. Therefore, this relation imposes a strict inequality constraint for each of the k colors:

$$\sum_{\mu \in R} \text{mult}_x(\mu) \geq 1, \quad \forall x \in \mathbb{Z}_k.$$

We sum this inequality over all k individual color symbols in the alphabet $\mathbb{Z}_k$:

$$\sum_{x \in \mathbb{Z}_k} \sum_{\mu \in R} \text{mult}_x(\mu) \geq \sum_{x \in \mathbb{Z}_k} 1 = k.$$

By swapping the order of summation on the left-hand side and noting that every single 3-multiset contains exactly 3 elements across all colors ($\sum_{x \in \mathbb{Z}_k} \text{mult}_x(\mu) = 3$), we can simplify the double summation as follows:

$$\sum_{x \in \mathbb{Z}_k} \sum_{\mu \in R} \text{mult}_x(\mu) = \sum_{\mu \in R} \sum_{x \in \mathbb{Z}_k} \text{mult}_x(\mu) = \sum_{\mu \in R} 3 = 3|R|.$$

Combining this with our inequality yields $3|R| \geq k$. Dividing through by 3 gives $|R| \geq k/3$. Because $3 \mid k$, the boundary $k/3$ is guaranteed to be an exact integer. We substitute this lower bound on the size of the omitted set back into the linear partition equation for the packing length n :

$$n = \binom{k+2}{3} - |R| \leq \binom{k+2}{3} - \frac{k}{3}.$$

Because n represents the length of an arbitrarily chosen valid packing sequence, the maximum packing parameter must satisfy $M_3(k) \leq \binom{k+2}{3} - k/3$. $\square$

To demonstrate that this upper bound is perfectly tight, we provide two explicit optimal sequence constructions for the small parameters $k = 6$ and $k = 9$.

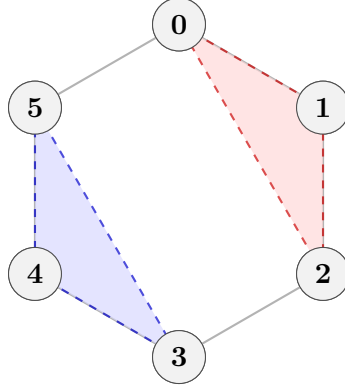


FIGURE 5. The dropped multisets for $k = 6$ shown as a pair of disjoint, shaded simplicial faces spanning $\mathbb{Z}_6$.

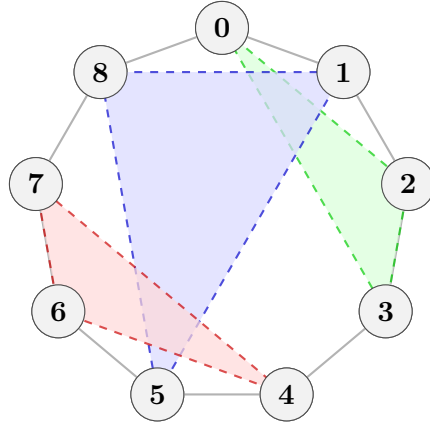


FIGURE 6. The dropped multisets for $k = 9$ shown as three disjoint, shaded simplicial faces spanning $\mathbb{Z}_9$.

Theorem 6.8 (Exact Values for $k = 6$ and $k = 9$). *The combinatorial upper bound is achieved exactly for $k = 6$ and $k = 9$, yielding:*

$$M_3(6) = 54 = \binom{8}{3} - \frac{6}{3}, \quad M_3(9) = 162 = \binom{11}{3} - \frac{9}{3}.$$

Proof. The upper bounds follow directly from Theorem 6.7. To establish the lower bounds, we present explicit, verified periodic witness sequences.

For the parameter $k = 6$, the target sequence length is $\binom{8}{3} - 2 = 56 - 2 = 54$. The following cyclic word $W^{(6)}$ of length 54 satisfies the conditions:

$$\begin{aligned} &0, 0, 0, 1, 1, 1, 2, 2, 0, 0, 5, 5, 1, 3, 1, 2, 5, 3, 5, 3, 0, 3, 1, 3, 4, 0, 2, \\ &5, 5, 5, 4, 1, 1, 5, 0, 4, 0, 4, 1, 4, 2, 4, 2, 3, 3, 3, 4, 4, 4, 5, 2, 2, 2, 3 \end{aligned}$$

The set of omitted multisets for this construction is exactly:

$$R^{(6)} = \{\{0, 1, 2\}, \{3, 4, 5\}\}.$$

After running a ILP solver of all 54 sliding windows of length 3 (evaluated cyclically modulo 54) confirms that they map to 54 unique, distinct multisets. Combining these 54 active windows with the 2 dropped multisets in $R^{(6)}$ accounts for all $\binom{8}{3} = 56$ multisets exactly once, verifying completeness.

For the parameter $k = 9$, the target sequence length is $\binom{11}{3} - 3 = 165 - 3 = 162$. The following cyclic word $W^{(9)}$ of length 162 satisfies the conditions:

0,0,0,3,1,1,0,0,5,3,1,2,1,5,4,0,2,0,1,5,5,2,4,1,1,1,5,
6,3,2,3,2,5,0,5,6,4,0,7,5,3,5,3,4,6,6,3,3,0,4,0,8,1,4,
7,7,1,0,4,4,1,6,0,2,2,1,7,3,3,1,6,2,4,3,1,8,1,8,2,2,2,
4,4,3,3,3,8,5,5,4,4,4,6,8,2,5,2,6,6,5,7,1,6,6,6,7,0,0,
6,3,7,2,0,8,5,8,4,8,3,0,7,7,2,2,6,7,7,7,8,0,6,6,8,7,3,
7,4,2,8,3,8,6,1,1,7,8,8,2,7,5,7,4,4,8,7,5,5,5,6,8,8,8

The set of omitted multisets for this construction is exactly:

$$R^{(9)} = \{\{0, 2, 3\}, \{1, 5, 8\}, \{4, 6, 7\}\}.$$

We also confirm that all 162 sliding windows are pairwise distinct as multisets, and together with $R^{(9)}$, they span the entire space of $\binom{11}{3} = 165$ multisets with no collisions, proving optimality. $\square$

Theorem 6.9. *Let $R \subseteq \mathcal{M}_3(\mathbb{Z}_k)$ be an arbitrary dropped set of size $|R| = k/3$. If R can be realized as the complement of a valid cycle packing that meets the upper bound of Theorem 6.7 with perfect equality, then R must form a perfect, disjoint partition of the alphabet $\mathbb{Z}_k$ into $k/3$ rainbow triples (triples consisting of three distinct elements).*

Proof. From the core inequality derivation in Theorem 6.7, we established that for every individual alphabet color symbol $x \in \mathbb{Z}_k$, the dropped multiplicity sum must satisfy $\sum_{\mu \in R} \text{mult}_x(\mu) \geq 1$.

Suppose we sum these terms across all k symbols in the alphabet. Because each individual multiset $\mu \in R$ contains exactly 3 elements, the sum expands to:

$$\sum_{x \in \mathbb{Z}_k} \sum_{\mu \in R} \text{mult}_x(\mu) = 3|R| = 3 \left(\frac{k}{3} \right) = k.$$

We are therefore presented with a system of k individual integer terms, $\left\{ \sum_{\mu \in R} \text{mult}_x(\mu) \right\}_{x \in \mathbb{Z}_k}$, each of which is bounded below by 1, whose grand total is exactly equal to k . Algebraically, the only way a sum of k terms, each ≥ 1 , can equal k is if every single term is exactly equal to 1:

$$\sum_{\mu \in R} \text{mult}_x(\mu) = 1, \quad \forall x \in \mathbb{Z}_k.$$

This exact equality imposes two strict structural conditions on the composition of the omitted set R :

- (1) Each color symbol $x \in \mathbb{Z}_k$ must appear in exactly one unique multiset within the collection R . If a color appeared in more than one multiset, its total sum would strictly exceed 1. Thus, the multisets in R are mutually vertex-disjoint and their union equals $\mathbb{Z}_k$.
- (2) Within its assigned multiset, the symbol x must have a multiplicity of exactly 1. If it possessed a local multiplicity ≥ 2 , the sum for that color would instantly violate the global constraint.

Because this condition applies uniformly to all elements within every multiset in R , every omitted multiset must consist of three distinct elements (the definition of a rainbow triple). Because these triples are mutually disjoint and their union covers $\mathbb{Z}_k$, the set R forms a perfect partition of $\mathbb{Z}_k$ into $k/3$ rainbow triples. $\square$

Furthermore, while the theoretical bounds and structural rules restrict the configuration space, searching for an explicit sequence via standard Integer Linear Programming (ILP) introduces severe computational challenges. Graph-theoretically, cycle packings correspond to routing an Eulerian circuit through a transition digraph D_k , where the vertex set consists of ordered symbol pairs $(a, b) \in \mathbb{Z}_k \times \mathbb{Z}_k$, and a directed edge passes from $(a, b) \rightarrow (b, c)$ with a multiset label $\{a, b, c\}$. An unconstrained ILP formulation requires assigning a binary variable to every candidate edge ordering for all $\binom{k+2}{3}$ multisets. For $k = 12$, this results in thousands of binary variables and hundreds of balance constraints, causing standard solvers to time out with an unclosed integrality gap.

To overcome this computational barrier, we introduce a method that projects the massive transition graph onto a lower-dimensional quotient space using group actions. By forcing the edge selection policy to be invariant under an automorphism group Γ , a choice made for a single representative multiset automatically propagates across its entire orbit, ensuring algebraic balance and reducing the number of variables.

Our analysis revealed that the choice of the generating automorphism group Γ depends on the congruence of $k \pmod{9}$ due to structural collapses in the orbit stabilizers.

Definition 6.10. Let R be the dropped set of rainbow triples forming a perfect disjoint partition of $\mathbb{Z}_k$. We define the structural architecture of R in a piecewise manner based on the parameters of k :

- **Consecutive Block Partition (used when $k \not\equiv 0 \pmod{9}$):**

$$R_{\text{consec}} := \bigcup_{i=0}^{\frac{k}{3}-1} \{(3i, 3i+1, 3i+2)\}.$$

- **Exceptional Symmetric Orbit Partition (used when $k \equiv 0 \pmod{9}$):**

$$R_{\text{symm}} := \mathcal{E} = \bigcup_{i=0}^{\frac{k}{3}-1} \left\{ \left(i, i + \frac{k}{3}, i + \frac{2k}{3} \right) \right\}.$$

Theorem 6.11 (The Piecewise Symmetry Framework). *Let Γ be a cyclic automorphism group acting on the vertex and edge sets of the transition graph via translation. We can*

achieve a uniform orbit factorization of the remaining multiset space using a piecewise group assignment:

- (1) **The Shift-by-3 Automorphism** ($k \not\equiv 0 \pmod{9}$): Let $\Gamma \cong \mathbb{Z}_{k/3}$ be the group generated by the shift operator $\sigma(x) = x + 3 \pmod{k}$. By removing R_{consec} , every remaining multiset maps to a free orbit of uniform size $k/3$.
- (2) **The Shift-by-1 Automorphism** ($k \equiv 0 \pmod{9}$): Let $\Gamma \cong \mathbb{Z}_k$ be the full translation group generated by the operator $\tau(x) = x + 1 \pmod{k}$. By removing R_{symm} , every remaining multiset maps to a free orbit of uniform size k .

Proof. We analyze each case to demonstrate how the choice of operator maintains uniform orbit sizes and prevents stabilizer collapse.

Case 1: $k \not\equiv 0 \pmod{9}$ (evaluated for $k = 12$). For $k = 12$, the shift operator $\sigma(x) = x + 3 \pmod{12}$ generates a cyclic group $\Gamma = \langle \sigma \rangle \cong \mathbb{Z}_4$ of order 4. The total number of multisets in the space is $\binom{12+2}{3} = \binom{14}{3} = 364$. Because no 3-multiset can be stabilized by a non-trivial translation of 3, 6, or 9, every single multiset orbit under this action has a size of exactly 4.

This cleanly factors the 364 multisets into exactly $364/4 = 91$ uniform orbits. Under this action, there are exactly 16 orbits whose member multisets are mutually disjoint and together cover all 12 colors perfectly. For example, the orbit containing $\mu = \{0, 1, 5\}$ consists of the four multisets:

$$\mu_0 = \{0, 1, 5\}, \quad \mu_1 = \{3, 4, 8\}, \quad \mu_2 = \{6, 7, 11\}, \quad \mu_3 = \{9, 10, 2\}.$$

Taking their union yields $\bigcup_{i=0}^3 \mu_i = \{0, 1, \dots, 11\} = \mathbb{Z}_{12}$. By selecting this orbit as our dropped set R , we remove 4 multisets, leaving exactly $91 - 1 = 90$ remaining orbits.

Case 2: $k \equiv 0 \pmod{9}$ (evaluated for $k = 9$). If we were to apply the shift-by-3 operator $\sigma(x) = x + 3 \pmod{9}$ to $k = 9$, the generating group would be $\mathbb{Z}_3$. Under this action, the elements of the exceptional orbit $\mathcal{E} = \{\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}\}$ would map directly to themselves (e.g., $\{0, 3, 6\} + 3 = \{3, 6, 0\} \equiv \{0, 3, 6\}$).

Therefore, we pivot to the full translation group $\Gamma \cong \mathbb{Z}_9$ generated by $\tau(x) = x + 1 \pmod{9}$. We analyze the orbit distribution of the 165 multisets under this full shift:

- Multisets of support size 1 ($\{c, c, c\}$) have a trivial stabilizer under τ and form 1 orbit of size 9.
- Multisets of support size 2 ($\{c, c, d\}$) have a trivial stabilizer under τ and form 8 orbits of size 9 ($72/9 = 8$).
- For support size 3 ($\binom{9}{3} = 84$ multisets), the exceptional orbit $\mathcal{E}$ consists of exactly 3 multisets that map to each other under a $+1$ translation:

$$\{0, 3, 6\} \xrightarrow{+1} \{1, 4, 7\} \xrightarrow{+1} \{2, 5, 8\} \xrightarrow{+1} \{3, 6, 0\} \equiv \{0, 3, 6\}.$$

Thus, $\mathcal{E}$ forms exactly one single orbit of size 3. The remaining $84 - 3 = 81$ support-3 multisets have trivial stabilizers and form $81/9 = 9$ uniform orbits of size 9.

By selecting $\mathcal{E}$ as our dropped set R , we remove this single size-3 orbit. The remaining $1 + 8 + 9 = 18$ active orbits are perfectly uniform, each having a size of exactly 9, which restores the uniform factorization. $\square$

Enforcing orbit invariance allows us to define a highly compressed optimization model. Let B represent the set of remaining active orbits under the chosen group action, and let μ_b denote the representative multiset for each orbit $b \in B$. Let P_b be the set of unique directed edge permutations available for μ_b . We define a binary decision variable $x_{(b,p)}$ for each permutation index $p \in P_b$. The optimization model is structured as follows:

$$(8) \quad \sum_{p \in P_b} x_{(b,p)} = 1, \quad \forall b \in B.$$

Enforcing $x_{(b,p)} = 1$ automatically assigns identical permutation routing to all elements within orbit b . For a cyclic group action generated by an operator of order d , this choice adds d directed edges to the tracking network.

The global vertex balance constraint at each sequence pair vertex $v \in \mathbb{Z}_k \times \mathbb{Z}_k$ is expressed as:

$$(9) \quad \sum_{b \in B} \sum_{p \in P_b} \text{In}(v, b, p) x_{(b,p)} = \sum_{b \in B} \sum_{p \in P_b} \text{Out}(v, b, p) x_{(b,p)},$$

where $\text{In}(v, b, p)$ and $\text{Out}(v, b, p)$ are coefficients that count the number of directed edges entering or leaving vertex v across the full orbit induced by choosing permutation p for orbit b .

This formulation reduces the number of decision variables for $k = 12$ from 360 down to 90 representative variables, allowing standard solvers to resolve the system in milliseconds. Because the constraints are mapped symmetrically, the resulting graph is naturally predisposed to Eulerian balance.

The Connectivity Repair Loop: While solving equations 8 and 9 guarantees algebraic vertex balance, it does not ensure that the resulting graph forms a single connected component. The network can fragment into isolated, disconnected sub-tours. To address this, we implement a lazy-constraint cut loop:

- (1) Solve the orbit-reduced ILP system to obtain a balanced edge configuration.
- (2) Construct the induced directed graph G' and extract its weakly connected components using standard graph traversal algorithms.
- (3) If the number of components equals 1, the graph is valid. Pass the network to an Eulerian circuit router to extract the final packing sequence.
- (4) If the graph splits into multiple components, identify the edge set of the smallest isolated sub-tour, add a lazy linear constraint to the ILP solver to forbid that specific sub-tour combination, and re-solve.

Applying this framework to $k = 12$ resolved the computational bottleneck and established the packing sequence length.

Theorem 6.12. *The maximum sequence length for an alphabet size of $k = 12$ matches the combinatorial upper bound exactly, yielding:*

$$M_3(12) = 360 = \binom{14}{3} - \frac{12}{3}.$$

Proof. The upper bound $M_3(12) \leq 364 - 4 = 360$ follows from Theorem 6.7. For the lower bound, the orbit-reduced ILP solver under the shift-by-3 framework evaluated the 90 remaining orbits. On its first iteration, it identified a valid combination that passed the lazy-constraint connectivity check, forming a single weakly connected component. This balanced, fully connected graph was processed by an Eulerian circuit router, generating a valid cyclic witness sequence of length 360, confirming the theorem. $\square$

Conjecture 6.13 (The $m = 3$ Packing Conjecture). *For all alphabet parameters satisfying $k \equiv 0 \pmod{3}$ across all ranges $k \geq 6$, the maximum length of a valid $(k, 3)$ -cycle packing is given exactly by*

$$M_3(k) = \binom{k+2}{3} - \frac{k}{3}.$$

While this piecewise symmetry reduction framework provides a method for generating explicit sequence witnesses, two open theoretical problems remain:

- (1) **Global Parameterization Rules:** We lack a generalized, closed-form parameterization rule to pre-select which of the 16 valid rainbow partitions should be dropped for arbitrarily large values of k .
- (2) **The Connectivity Proof Requirement:** Proving mathematically that the lifted graph will *always* form a single weakly connected component for all arbitrarily large values of k remains an open topological problem.

6.2 Contribution 2: Insufficiency of Scalar-Sum Mappings

Before delving into complex combinatorial structures like multi-dimensional tori or cyclic de Bruijn sequences, a systems engineer might propose a natural simplification for the one-dimensional multiset positioning problem: *Why not simply assign a distinct numerical weight to each colour and use the scalar sum of these weights within the window as the unique position identifier?*

If every possible window multiset produced a strictly unique integer sum, the robot could determine its absolute position using a single scalar reading. This subsection constitutes our second analytical contribution: we rigorously prove that such a scalar-sum mapping is mathematically impossible under realistic, bounded-alphabet constraints. The available integer observation space grows linearly, while the multiset state space grows polynomially. Consequently, the codomain is too small to distinguish all multisets, and by the Pigeonhole Principle, catastrophic observational collisions are unavoidable.

Definition 6.14 (Scalar-Sum Observation Map). Let $\Sigma = \{a_1, a_2, \dots, a_k\} \subset \mathbb{Z}$ be a palette of k distinct integer weights assigned to k unique colours, strictly ordered such that $a_1 < a_2 < \dots < a_k$. Let $\mathcal{M}_m(\Sigma)$ denote the state space of all m -multisets drawn from Σ . The linear scalar-sum observation map

$$f : \mathcal{M}_m(\Sigma) \rightarrow \mathbb{Z}$$

is defined as the sum of the elements in the multiset:

$$f(M) = \sum_{c \in M} c \quad \text{for } M \in \mathcal{M}_m(\Sigma).$$

In this architecture, the robotic sensor observes only the aggregate integer sum $S = f(M)$ rather than the distinct elements of the multiset M .

Theorem 6.15 (Insufficiency of Bounded Scalar-Sum Mappings). *For any palette size $k \geq 3$ and sliding-window size $m \geq 2$, if the weight configuration Σ is sequentially bounded such that the available sum values are restricted to the contiguous integer interval determined by the linear span of the alphabet, then the observation map f cannot be injective.*

Specifically, if the number of available sum bins N_{sums} is constrained by the alphabet's span such that $N_{\text{sums}} \leq m(a_k - a_1) + 1$, then the cardinality of the multiset domain strictly exceeds the codomain capacity:

$$|\mathcal{M}_m(\Sigma)| > N_{\text{sums}}.$$

By the Pigeonhole Principle, there exist at least two distinct multisets $M_1, M_2 \in \mathcal{M}_m(\Sigma)$ ($M_1 \neq M_2$) such that $f(M_1) = f(M_2)$, rendering deterministic one-dimensional localisation impossible under this framework.

Proof. We begin by establishing the upper bound on the codomain capacity. Let the minimal and maximal weights in the alphabet be denoted by $a_{\min} = a_1$ and $a_{\max} = a_k$, respectively. For any window of length m , the aggregate scalar sum $S = f(M)$ must fall within the absolute closed interval:

$$S \in [ma_{\min}, ma_{\max}].$$

Because all assigned weights are integers, the maximum number of distinct integer sums (or “bins”) available within this interval is exactly the length of the interval plus one:

$$N_{\text{sums}} \leq m(a_{\max} - a_{\min}) + 1.$$

Since Σ consists of k strictly distinct integers, the minimal possible geometric span between the smallest and largest element is $k - 1$ (achieved when the weights are consecutive integers). Thus, $a_{\max} - a_{\min} \geq k - 1$. Substituting this into our bound yields the maximum scalar capacity:

$$N_{\text{sums}} \leq m(k - 1) + 1.$$

Conversely, the cardinality of the domain—the total number of m -multisets chosen from k colours—is given by the standard “stars and bars” combinatorial formula:

$$|\mathcal{M}_m(\Sigma)| = \binom{k + m - 1}{m}.$$

To prove that f cannot be injective, we must establish the following strict inequality:

$$(10) \quad \binom{k + m - 1}{m} > m(k - 1) + 1 \quad \text{for all } k \geq 3, m \geq 2.$$

We proceed by mathematical induction on the window size m .

Base Case ($m = 2$): For $k \geq 3$, we evaluate the difference between the left-hand side (LHS) and right-hand side (RHS) of Equation 10:

$$\begin{aligned} \text{LHS} - \text{RHS} &= \binom{k+1}{2} - (2(k-1) + 1) \\ &= \frac{k(k+1)}{2} - 2k + 1 \\ &= \frac{k^2 + k - 4k + 2}{2} \\ &= \frac{k^2 - 3k + 2}{2} = \frac{(k-1)(k-2)}{2}. \end{aligned}$$

Since our domain restricts $k \geq 3$, it follows that $(k-1) \geq 2$ and $(k-2) \geq 1$. Thus, the difference is strictly bounded below by $\frac{2 \cdot 1}{2} = 1 > 0$. The base case holds strictly.

Inductive Hypothesis: Assume that for a fixed arbitrary integer $m \geq 2$, the inequality holds for all $k \geq 3$:

$$\binom{k+m-1}{m} > m(k-1) + 1.$$

Inductive Step: We must demonstrate that the inequality holds for a window of size $m+1$, namely:

$$\binom{k+m}{m+1} > (m+1)(k-1) + 1.$$

By Pascal's Identity, we can decompose the LHS:

$$\binom{k+m}{m+1} = \binom{k+m-1}{m+1} + \binom{k+m-1}{m}.$$

Applying the inductive hypothesis to the second term yields a strict lower bound:

$$\binom{k+m}{m+1} > \binom{k+m-1}{m+1} + m(k-1) + 1.$$

To complete the proof, it suffices to show that the remaining binomial term satisfies $\binom{k+m-1}{m+1} \geq k-1$. Expanding the combination into its multiplicative factors and isolating $(k-1)$ gives:

$$\begin{aligned} \binom{k+m-1}{m+1} &= \frac{(k+m-1)(k+m-2) \cdots (k)(k-1)}{(m+1)!} \\ &= (k-1) \prod_{j=0}^{m-1} \frac{k+j}{2+j}. \end{aligned}$$

Because the theorem is restricted to $k \geq 3$, every individual fractional term in the product satisfies $k+j \geq 3+j > 2+j$. Therefore, for every $j \geq 0$, the ratio $\frac{k+j}{2+j}$ is strictly greater than 1. In fact, the very first term ($j = 0$) is $\frac{k}{2} \geq 1.5$.

Because the product consists of terms strictly greater than 1, the entire product is strictly greater than 1. Thus:

$$\binom{k+m-1}{m+1} > (k-1) \cdot 1 = k-1.$$

Substituting this relation back into our expanded inductive step gives:

$$\binom{k+m}{m+1} > (k-1) + m(k-1) + 1 = (m+1)(k-1) + 1.$$

This completes the mathematical induction. The multiset domain cardinality grows polynomially ($\approx \mathcal{O}(k^m)$), while the scalar-sum capacity grows only linearly ($\mathcal{O}(mk)$). Thus, $|\mathcal{M}_m(\Sigma)| > N_{\text{sums}}$ for all specified parameters. By the Pigeonhole Principle, f cannot be injective. $\square$

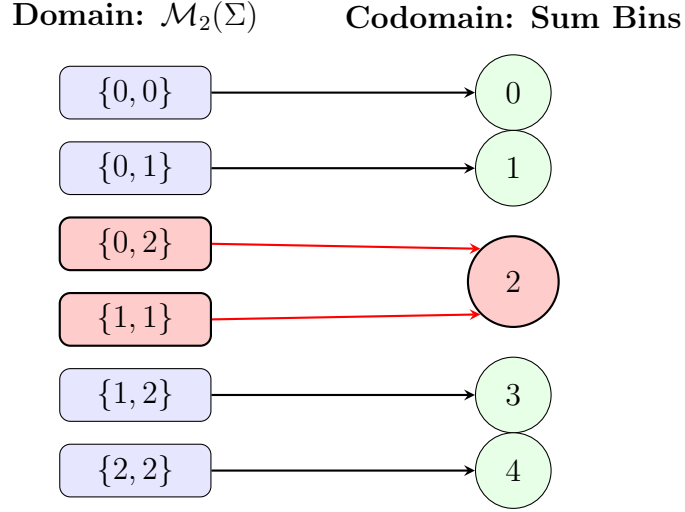


FIGURE 7. Visualisation of the Pigeonhole collision for $k = 3, m = 2$ using standard integer weights $\Sigma = \{0, 1, 2\}$. A robotic sensor observing a sum intensity of ‘2’ cannot logically distinguish whether the window contains $\{0, 2\}$ or $\{1, 1\}$.

Remark 6.16. The constraints $k \geq 3$ and $m \geq 2$ represent an exact, sharp phase transition where scalar injectivity fails. If we relax either boundary, the theorem no longer holds:

- **The Window Boundary** ($m = 1, \forall k$): If the window observes only a single square, $|\mathcal{M}_1(\Sigma)| = k$, and $N_{\text{sums}} \leq k$. Because the domain size perfectly matches the codomain capacity, an injective mapping exists trivially by assigning strictly distinct weights to each colour.
- **The Palette Boundary** ($k = 2, m = 2$): Here, $|\mathcal{M}_2(\Sigma)| = \binom{3}{2} = 3$. For a standard span $\Sigma = \{0, 1\}$, the possible sums are 0, 1, and 2. Because the domain size (3) does not exceed the codomain capacity (3), injectivity is preserved.

As soon as k reaches 3 (as shown in Figure 7), the domain expands to 6 multisets while the linear capacity only stretches to 5.

Remark 6.17. An analytical escape route exists if we drop the physical restriction of a tightly bounded alphabet. If the colour labels are allowed to scale exponentially, we can configure Σ as a B_m set (or a Sidon set when $m = 2$), where all m -element sums are distinct by definition.

For instance, applying an exponential base assignment $a_i = (m + 1)^i$ for $i \in \{0, \dots, k - 1\}$ guarantees absolute injectivity, as every scalar sum becomes a unique representation in a base- $(m + 1)$ positional numeral system. For $k = 3, m = 2$, the weights $\{1, 3, 9\}$ yield the un-collided sums $\{2, 4, 6, 10, 12, 18\}$.

However, while mathematically elegant, this B_m exception is a catastrophic failure mode in hardware engineering. The maximum weight scales exponentially as $a_{k-1} = (m + 1)^{k-1}$. The binary storage required to transmit or process this label in memory scales linearly with the palette size:

$$B_{B_m} = \lceil \log_2((m + 1)^{k-1}) \rceil = \mathcal{O}(k \log m) \text{ bits.}$$

In contrast, standard bounded linear assignments only require $B_{\text{linear}} = \mathcal{O}(\log k)$ bits. This exponential bit-width explosion directly violates the strict ultra-low-power, short-transmission constraints required by modern ambient IoT and light-based backscatter tracking systems (Chen et al., 2024, Section 1.3).

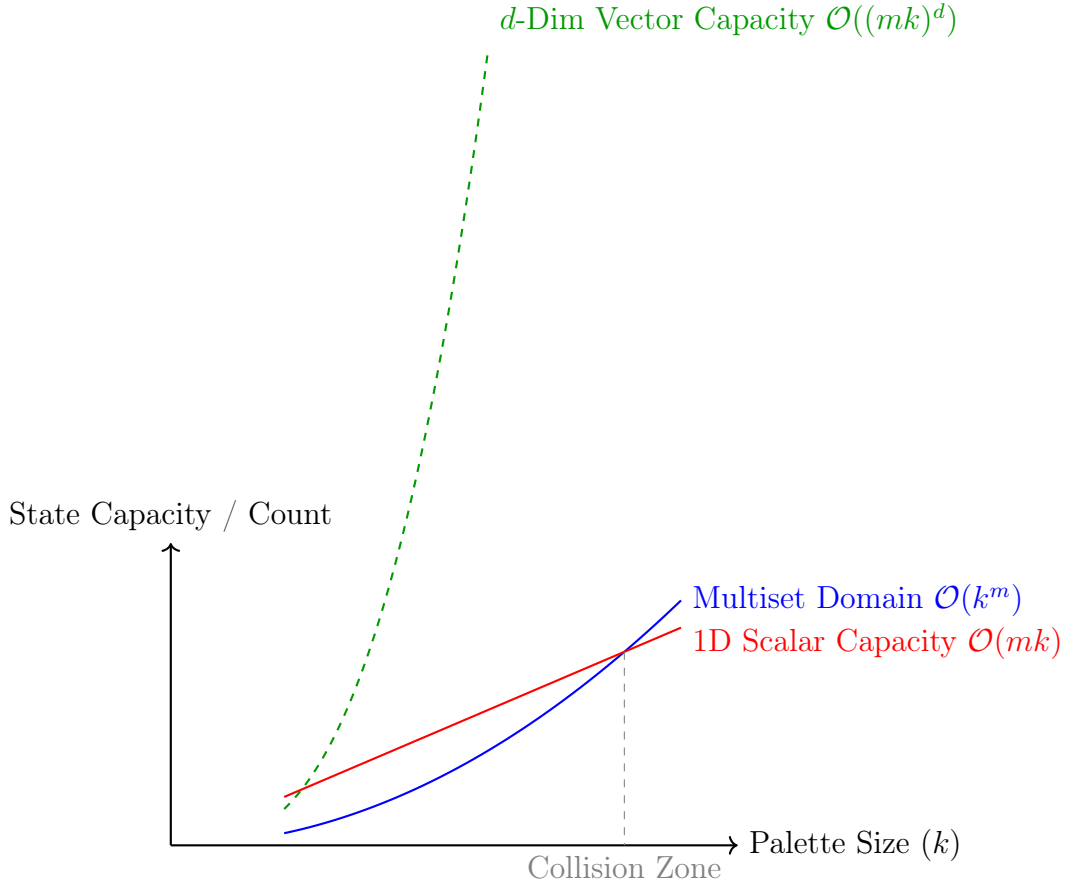


FIGURE 8. Conceptual growth comparison. The 1D scalar sum capacity (red) grows linearly and is quickly overtaken by the polynomial growth of the multiset domain (blue), causing unavoidable pigeonhole collisions. Moving to a d -dimensional vector space (green) provides the geometric degrees of freedom required to outpace the domain.

Remark 6.18. The fundamental insufficiency of the 1D scalar-sum space perfectly contextualises the architectural pivot made by Chen, Keevash, Kennedy, de Panafieu, and Vetta (2024). To bypass the $\mathcal{O}(mk)$ capacity bottleneck without triggering the exponential B_m bit-width explosion, they altered the topological layout of the problem by mapping colours into a multi-dimensional vector space ($\mathbb{Z}^d$).

In their framework, the robot observes a vector sum $\mathbf{S} = \sum_{j=1}^m \mathbf{v}_{c_j}$. As visualised in Figure 8, because the sum evaluates d independent spatial dimensions simultaneously, the codomain capacity undergoes a phase shift from linear to polynomial scaling:

$$N_{\text{vector_sums}} = \mathcal{O}((mk)^d).$$

For $d \geq 2$, this higher-order capacity easily absorbs the multiset domain while keeping the individual coordinate entries tightly bounded.

Furthermore, this algebraic restriction explains why the Chen et al. methodology does not seamlessly scale down to 1D. According to their Lemma 14, their framework requires the palette to satisfy $k = bd + 1$ and coordinate weights to satisfy $s = \frac{m^{d-1}}{bd}$. Attempting to force their solution into a single dimension by setting $d = 1$ and $b = 1$ yields $k = 2$ and $s = 1$. For any $k \geq 3$, the required parameter s degrades into a non-integer fraction, rendering the construction invalid. This rigorous scalar-sum limitation validates their observation that 1D multiset packing remains a highly non-trivial, structurally constrained gap in the current literature.

7 Conclusion

This paper established combinatorial bounds and explicit constructions for multiset cycle packings. The results show how the cyclic space limits packing efficiency and determines the minimum number of omitted multisets.

To formalize the theoretical limit of these structures, we state the following conjecture:

Conjecture 7.1 (1D Multiset Packing Conjecture). *For every fixed window size $m \in \mathbb{N}$, there exists a color palette threshold k_0 such that for all $k \geq k_0$, the maximum length of a multiset cycle packing is:*

$$M_m(k) = k \cdot \left\lfloor \frac{1}{k} \binom{k+m-1}{m} \right\rfloor$$

This conjecture means that in large spaces, the main requirement for an optimal packing is maintaining a uniform distribution of colors across the cycle. When colors are balanced, the number of omitted multisets reaches the lowest theoretical limit allowed by integer floor division.

Future Research Directions

We identify three concrete paths for future work:

- (1) **Explicit Polynomial-Time Algorithms:** Converting the non-constructive, existential hypergraph results of Glock et al. into explicit, polynomial-time algorithms that generate optimal sequences.

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- (2) **Error-Correcting Codes:** Designing error-correcting codes specifically for multiset observation spaces to handle real-world sensor reading errors.
 - (3) **Multidimensional Topologies:** Extending these exact 1D optimal packings to multidimensional toruses where $d \geq 2$.

Acknowledgements

I would like to thank my program mentors for their technical guidance and support throughout this paper.

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