The Black-Scholes Model

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Outline

Introduction

Why This Model Matters

"Before Black-Scholes, option pricing was an art. After Black-Scholes, it became a science."

- Fischer Black, Myron Scholes, Robert Merton — 1973
- Revolutionized how traders think about risk
- Scholes and Merton won the 1997 Nobel Prize (Black had passed away)

Option Basics

Definition (Option)

A contract giving the right, but not the obligation, to buy or sell at a strike price K.

Call Option

- Right to buy
- Profits when price $\uparrow$

Put Option

- Right to sell
- Profits when price $\downarrow$

Exercise Styles

- **European**: exercise only at expiration T
- **American**: exercise anytime before T

Five Inputs

Symbol	Meaning	Source
S_0	Current stock price	Exchange
K	Strike price	Contract
T	Time to expiration	Calendar
r	Risk-free rate	Government bond
σ	Volatility	Estimated

Four are observable. Only σ must be estimated.

Theme 1: Geometric Brownian Motion

From CLT to Random Walk

- Stock returns are influenced by hundreds of small factors like news, policy...
- **Central Limit Theorem**: sum of many small random variables $\rightarrow$ normal distribution

Random walk:

$$S_{t+1} = S_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma^2)$$

Today's price = yesterday's price + a random shock.

From Brownian to Geometric Brownian Motion

- As step size $\to 0$: random walk becomes Brownian motion W_t
- Problem: fixed step size — unrealistic

Solution: **Geometric Brownian Motion**

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

- μ: drift (expected return)
- σ: volatility
- dW_t: random shock
- Both terms multiplied by S_t: percentage changes, not dollar amounts

Log-Normal Distribution

From Itô's Lemma:

$$d(\ln S_t) = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t$$

Therefore:

$$\ln S_T \sim \mathcal{N}\left(\ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)T,\ \sigma^2 T\right)$$

- $S_T > 0$ always — stock price never negative
- Right-skewed — large gains possible, cannot go below zero
- This makes the expected payoff computable

The Black-Scholes Model

The Seven Assumptions

1. Stock price follows **Geometric Brownian Motion**
2. Volatility σ is constant
3. Risk-free rate r is constant
4. **No dividends** during the option's life
5. **No arbitrage** opportunities
6. **Continuous trading**, no transaction costs, no taxes
7. **Fractional shares** can be traded

Idealized assumptions — the model works as a benchmark.

Theme 2: Risk-Neutral Pricing

The Key Insight: μ Disappears!

- **Delta hedging**: hold Δ shares + short 1 option $\Rightarrow$ risk-free portfolio
- Risk-free portfolio must earn exactly r (no arbitrage)

Why must it earn exactly r?

- More than r: arbitrageurs borrow at r, invest, pocket the difference
- Less than r: nobody buys, price adjusts

The Result:

- The stock's expected return μ cancels out entirely

Price as if all assets grow at the risk-free rate r:

$$S_T^{\text{RN}} = S_0 \exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T} \cdot Z\right)$$

The stock's expected return does not matter.

The Black-Scholes Formula

$$C = e^{-rT} \cdot \mathbb{E}^{\mathrm{RN}} [\max(S_T - K, 0)]$$

$$\boxed{C = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Put option:

$$P = K e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)$$

Interpreting the Formula

$$C = \underbrace{S_0\Phi(d_1)}_{\text{Expected stock received}} - \underbrace{Ke^{-rT}\Phi(d_2)}_{\text{Expected strike paid}}$$

- $\Phi(d_2)$: risk-neutral probability of exercise
- $\Phi(d_1)$: weighted probability — always larger than $\Phi(d_2)$
- Ke^{-rT}: strike price discounted to today

Call = expected stock value − expected strike paid. Why $\Phi(d_1) > \Phi(d_2)$?

Because $d_1 = d_2 + \sigma\sqrt{T}$.

Numerical Verification

Monte Carlo Simulation

Method:

1. Set parameters
2. Generate random S_T
3. Compute payoff
4. Discount & average

Parameters:

S_0	100
K	105
T	0.5
r	3%
σ	20%

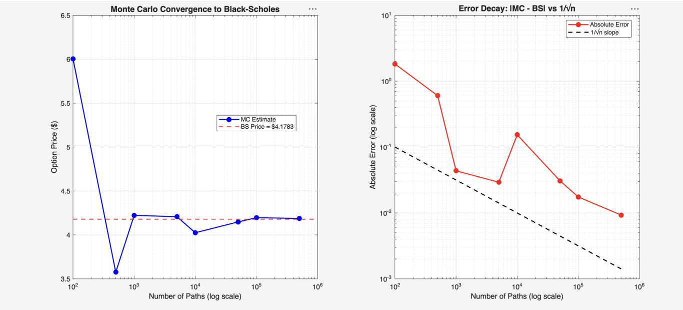
Risk-neutral S_T:

$$S_T = S_0 \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right)T + \sigma\sqrt{T} \cdot Z\right)$$

Call payoff:

$$\max(S_T - K, 0)$$

Monte Carlo Convergence



MC price converges to Black-Scholes theoretical price as paths $\to \infty$.

Theme 3: Binomial Tree

Binomial Tree Method

- Break time into n steps, each $\Delta t = T/n$
- Up factor: $u = e^{\sigma\sqrt{\Delta t}}$, Down factor: $d = 1/u$
- Risk-neutral probability: $p = \dfrac{e^{r\Delta t} - d}{u - d}$

Backward induction:

$$V = e^{-r\Delta t}\left(pV_{\mathrm{up}} + (1-p)V_{\mathrm{down}}\right)$$

- $u \times d = 1$: up then down returns to start
- As $n \to \infty$, binomial tree converges to BSM

American Options & Early Exercise Premium

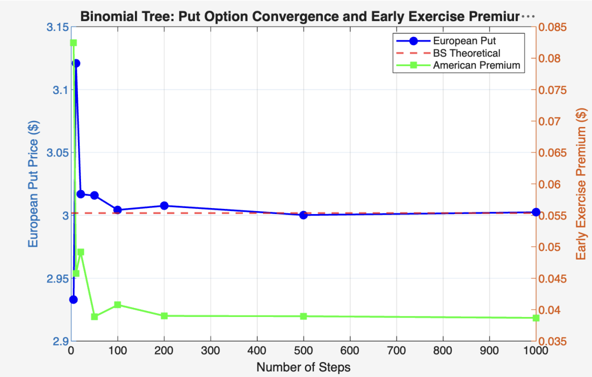
- American put: at each node, choose:

$$V = \max\left(\text{Wait value},\ \max(K - S_{\text{node}}, 0)\right)$$

Steps	**European Put**	**American Put**	**Premium**
5	2.933	3.016	0.082
50	3.016	3.055	0.039
500	3.000	3.039	0.039
1000	3.002	3.041	0.039

American put $>$ European put due to early exercise premium.

American vs European Put



American put (green) consistently higher than European put (blue).

Conclusion

Limitations & Summary

Limitations:

- σ is not constant — volatility smile
- No dividends (standard version)
- Fat tails — extreme events

Summary:

1. GBM: μ drift $+$ σ shock
2. Risk-neutral: μ disappears
3. BSM: closed-form solution
4. Binomial: handles American options

The model is not perfect, but it changed finance forever.

Thank you! Questions?