THE BLACK-SCHOLES MODEL

HAOYU LIU
larry_liuk@126.com

Abstract. Black-Scholes model calculates the European option price and is a fundamental model for option-pricing, which is in some restricted assumptions. In this paper, instead of proving the model and theories completely by Math, we choose to introduce some Math theorem behind it like Brownian motion and mainly focused on validating the model by programming. We collect the price from programming and compare the data with the Monte Carlo simulation. After this, we introduce Binomial Tree's participation in Black-Scholes model a bit to make it applicable in the American stock market. The goal for this paper is to introduce this beautiful and useful model to everyone not necessarily required advanced math knowledge.

Contents

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1. Introduction

Black-Scholes Model is one of the most famous and foundational model for financial derivatives. It was developed by Fischer Black, Myron Scholes, and Robert Merton in the early 1970s and has become a widely used essential tool in finance. So, in this paper, it is possible for us, everyone, to build the actual model and play with it after finishing and have a brief overview to the theorem even if some of us are in lack of some Math knowledge.

The beginning of Black-Scholes Model can be traced back to period of transformation in modern financial markets in the early 1970s. The Black and Scholes method aimed for a unified and simplified model of pricing options. Fischer Black, the University of Chicago economist, and Myron Scholes, the MIT finance professor, worked together to derive an unbiased and isolated value model for stock options. The foundational paper, "The Pricing of Options and Corporate Liabilities", was published in the Journal of Political Economy in 1973 as a revolutionary contribution to the modern option pricing theory. The paper resulted with a closed-form solution for presenting the cost of European call and put options without dividends in non-viable stocks. For applications based on the theory, the most common one is finding the fair value of exchange-traded possibilities globally. Using the underlying asset price, exercise price, the option's period to expiration, and the risk-free rate of return, one can determine the theoretical option price, achievable from the model. So in this paper, we are using advance computing technology as applications to prove the model indirectly by showing its functionality through verification.

This paper aim to help people with limited knowledge but want to know this model. First, we want to introduce an easier and brief understanding of Black-Scholes Model that does not require proficient knowledge. Since its derivation process is bulky and requires a lot of advanced calculus, the model seems hard to prove. However, we choose to program in Matlab and compare the output with the actual data value so that we can have a general proof to the model. In addition, we are able to discover the math behind it and the ideas without derivation so that it is fit and easily understandable for everyone. Second, we want to see some applications and the further use of this model in different stock market about the option pricing in real life. We will introduce the binomial tree to consider the transaction fee and American stock market rules. We also discover the logic and theory behind it while proving it indirectly by actual practices.

In this paper, Section 2 introduces some financial definitions that appears in this paper: the definition of options, the distinction between European and American exercise styles, dividends... Section 3 builds the mathematical background needed for the understanding of the model: random walks, Brownian motion, and the log-normal distribution. Section 4 displays the Black-Scholes model with seven necessary core assumptions, the use of geometric Brownian motion equation for stock prices, the partial differential equation and closed-form formula. Section 5 introduces the way of Monte Carlo simulation, contains our numerical verification, with graphs, and analysis to prove the model indirectly instead of actual option price comparison. Section 6 introduces the binomial tree model as an alternative pricing method and its combinations with the Black-Scholes Model. And then we explore the visualization of it. Section 7 concludes with directions for further exploration as well as the practical adaptations traders employ in real markets.

At last, again, this paper should help anyone interested in the model even someone like me before with little advanced Math background. Therefore, if anything is confusing or cannot be understood fully, just email me. In addition, objectively, this paper is not totally a Math paper while an expository paper surrounding with finance, mathematic and computing technologies.

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2. Basic finance definitions

Before the actual explanation of the model, we need to look over some shared financial vocabulary first. It may sound irrelevant and require some time to go over these boring definitions. Though the main paper is Math related, we still need some financial backgrounds for better understanding.

2.1. **Option.** An option is a financial contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset at a predetermined price on or before a specified date. The buyer pays a fee, called the premium, to acquire this right. The seller, receives the premium and is obligated to fulfill the contract if the buyer chooses to exercise.

A call option gives the holder the right to buy the underlying asset at the strike price.

A put option gives the holder the right to sell the underlying asset at the strike price.

2.2. **Strike Price.** The strike price, denoted by K, is the predetermined price at which the option holder can buy (call) or sell (put) the underlying asset.

2.3. **Expiration Date and Time to Maturity.** The expiration date is when the option contract expires, also called time to maturity, denoted by T, is the time remaining until expiration, measured in years.

2.4. **European and American Options.** European options can only be exercised on the expiration date. American options can be exercised at any time until(including) expiration date. The Black-Scholes model prices European options; the binomial tree method in Section 6 handles American options.

2.5. **Payoff.** The payoff is the amount the option holder receives at exercise, ignoring the premium paid.

2.6. **Risk-Free Interest Rate.** The risk-free interest rate, denoted by r, is the return on a theoretically risk-free investment. Free risk simply means the borrower will one hundred percent sure repay both principal and interest. Most time we use government bonds to estimate as higly possible government would not break the promise. It is one of the assumption about free risk.

2.7. **Volatility.** Volatility, denoted by σ, is the measurement of the annualized standard deviation of the stock's returns. It quantifies how much the stock price fluctuates. A higher volatility means larger price swings and, consequently, a higher option premium. Unlike other inputs, volatility must be estimated from historical data or inferred from market prices.

2.8. **Arbitrage.** Arbitrage is a risk-free profit with zero initial investment. In an efficient market or real world market, arbitrage opportunities will seldom exist, because traders could exploit them until prices adjust. In option pricing, the fair price of an option must preclude arbitrage.

2.9. **Dividends.** A dividend is a payment from a company to its shareholders. The standard Black-Scholes model assumes the stock pays no dividends during the option's life. We maintain this assumption throughout the paper.

2.10. **Hedging and Delta.** Hedging is reducing risk by taking an offsetting position. In option pricing, delta-hedging means holding a combination of the option and the underlying stock such that the portfolio's value is insensitive to small stock price changes. Delta (Δ) measures how much the option price changes per \$1 change in the stock price, and tells the hedger how many shares to hold.

3. Basic Math information

In this section, we just need to know these Math concepts with simple descriptions and examples without proving. Then, we can build each part connected on to understand the real Black-Scholes Model.

3.1. **Normal Distribution.** The normal distribution, also called Gaussian distribution, denoted by $\mathcal{N}(\mu, \sigma^2)$, is the famous bell-shaped curve. Normal distribution is simply the distribution that more values appears in middle and fewer in edges. In this paper we just need to know such shape but should really focus on the most special and important case is the standard normal distribution $\mathcal{N}(0, 1)$, it is symmetric and normal with: mean $= 0$, standard deviation $= 1$. Its cumulative distribution function (CDF) is denoted $\Phi(x)$. This function calculates the probability that a standard normal random variable takes a value less than or equal to x which is basically counting the area under the case of standard normal distribution mentioned above:

$$\Phi(x) = P(Z \leq x), \quad Z \sim \mathcal{N}(0, 1). \tag{3.1}$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt \tag{3.2}$$

The Black-Scholes formula includes $\Phi(d_1)$ and $\Phi(d_2)$ directly wo that is why we should at least know how to calulate those. In our simulations in Matlab, we will generate normal random numbers to model stock price shocks.

3.2. **Central Limit Theorem.** The Central Limit Theorem (CLT) states that the sum of a large number of independent, identically distributed random variables tends toward a normal distribution, regardless of the original distribution's shape. For example, when rolling a 6 face die and taking average from 1- 6, one turn could be random, but ten million turn and those averages will approximately form a normal distribution. Similarly in the Black-Scholes Model, return over a month or a year is the sum of many many small returns over tiny tiny time intervals. The CLT guarantees that this sum is approximately normal.

3.3. **Random Walk.** The Random Walk is the position simulation of a step of walk in random direction in all history while special case is in 1 Dimension it is just negative and positive and the rule is the new position is based on previous position. We can imagine in 2 Dimension it would be the coordinate plane, and 3 Dimension is just walk in the space with randomized direction. Here comes the interest part, over many days the amount of walk increases into a large magnitude. As a result, when we list position of all the tries, it is approximately normal distributed as Central Limit Theorem stated. for a significant distance, it tends to appear in normal distribution. Therefore, we are able to conclude one of its property is after infinite walk the position is approximately normal distributed.

A stock's log-price behaves like a random walk. Each day, the log-price takes a small random step up or down. Because the options' price also follow the random walk as it has many up and rise over the expiration dates, and we can use the randomized directions simulating the stock's movement particularly. The random walk is the simplest model for understanding later Brownian motion comes from.

3.4. **Brownian Motion.** Then, we are thinking something a bit more difficult than random Walk. Now, Consider the time between steps in Random Walk. Instead of one step per second, take one step per millisecond. Instead of ± 1 steps, take ± 0.001 steps. In the limit, as step size and step time both decrement to zero in the right proportion, the random walk becomes Brownian motion, also called a Wiener process, denoted W_t.

Brownian motion has four properties directly given:

(1) $W_0 = 0$. The process starts at zero.
(2) The path is consecutive but cannot be derivated.

(3) Independent increments. What happens between time 0 and time 1 tells you nothing about what happens between time 1 and time 2.
(4) Normal increments. For any $t > s$, the change $W_t - W_s$ follows $\mathcal{N}(0, t-s)$.

The fourth property is the most important one for Black-Scholes Model as we are able to use normal distribution for prediction followed with each increments. The uncertainty of increments scales with the square root of time: standard deviation^2 = time which stated in another way standard deviation = $\sqrt{\text{time}}$.

In the simulations, we multiply random number with $\sqrt{\Delta t}$ instead of Δt. The square root is not arbitrary; it comes from the definition of Brownian motion.

3.5. **Geometric Brownian Motion.** We are making some progress from random walk to Brownian motion and finally attach to the most applicable Geometric Brownian motion. Since Brownian Motion could generate negative number(directions can be negative for a while), it could not be used for stock modeling as there is never a negative amount of money existing for any stocks. So, we introduce the Geometric Brownian Motion as it is always positive. The Black-Scholes model assumes that stock prices follow geometric Brownian motion,. defined by the stochastic differential equation(no need to know how to prove):

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t. \tag{3.3}$$

μ is the expected annual return (drift) and σ is the annual volatility. The term dW_t represents the random shock from Brownian motion.

We can break the equation's right side into two parts. The left part is the trend and expected rate of the stock while the right part of the equation is the noisy part that simulate the arbitrary proportional with σ. The use of S_t in both terms ensures that price changes scale with the current price level. For instance, a \$100 stock experiences larger absolute moves than a \$10 stock even though their percentage change scale is the same.

To simulate later, we just need the discrete-time version of Geometric Brownina Motion. Over a small interval Δt, the stock price then became(equation given directly without proving or deriving):

$$S_{t+\Delta t} = S_t \cdot \exp\left((\mu - \tfrac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\, Z \right), \quad Z \sim \mathcal{N}(0,1). \tag{3.4}$$

We will use this formula later for simulation. And this is the foundation of the Black-Scholes Model's randomized part fo stocks.

3.6. **Itô's Lemma.** Although we claim we would not include some advanced calculus knowledge, it is still necessary to introduce Itô's Lemma while in a short and brief way. So, please do not view this as complicated and annoying, we can have a quick read and understanding instead of deriving the formula. In ordinary calculus, it is known how to differentiate a function from chain rule. But when the underlying variable is a stochastic process like geometric Brownian motion mentioned above, the ordinary chain rule is not really applicable in this case. It is directly because Brownian motion, $(dW_t)^2 \approx dt$, that the second variable is not ignorable.

Then, we required the use of Itô's Lemma for Black-Scholes Model as it extends the chain rule to stochastic processes. For a function $f(S_t, t)$ where S_t follows $dS_t = \mu S_t dt + \sigma S_t dW_t$, the lemma states:

$$df(S_t,t) = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}\right) dt + \sigma S_t \frac{\partial f}{\partial S}\, dW_t. \tag{3.5}$$

The noticeable part $\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}$ is derived from the non-zero quadratic variation of Brownian motion so this is why it is a mathematical bridge connecting Brownian motion with the Black-Scholes Model.

Black-Scholes model uses Itô's Lemma twice. To be more specific, one is applying it to $f(S) = \ln S$ yields the solution for S_T which is the log-normal formula in the next subsection. The other one is applying it to the option price $V(S_t, t)$ leads to the Black-Scholes partial differential equation. We will not perform either derivation, but the lemma is the essential mathematical link that makes both results possible which playing a role in connecting the information above with Black-Scholes model in a computing way.

3.7. **Log-Normal Distribution.** When a variable's log number is normal distributed, then this variable itself is called log normal. The reason for using log instead of original is because we need a positive price and it is known that a negative number is impossible to put in to a log function. In this case, if S_t follows Geometric Brownian motion, then $\ln S_t$ is normally distributed. And a random variable that its logarithm is normal is said to follow a log-normal distribution following by two properties below:

(1) $S_t > 0$ always. The stock price can never become negative which is a basic and realistic requirement.
(2) The distribution is right-skewed. Very high prices are possible, but the price can never fall below zero which indicates the normal shape will have a long tail towards rights.

At time T, under Geometric Brownian motion with drift μ:

$$\ln S_T \sim \mathcal{N}\left(\ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)T,\ \sigma^2 T\right). \tag{3.6}$$

This log-normal property makes it possible and tractable to analyze the Black-Scholes model: the expected payoff $\mathbb{E}[\max(S_T - K, 0)]$ can be computed exactly when S_T is log-normal.

3.8. **Continuous Discounting.** There is a definition of time value inside actual money value. In other words, interestingly, by this idea, the money worth more value in now than in future. Under continuous compounding at rate r, \$1 invested today grows to e^{rT} after time T. Conversely(trace backwards), a payment of \$X to be received at time T has a present value of Xe^{-rT}. In the Black-Scholes model, the term Ke^{-rT} is simply the strike price discounted from expiration date to the buying date.

3.9. **Risk-Neutral Valuation.** The most profound and essential concept is risk-neutral valuation which we will briefly introduce in this section and talk more about it in section 4. In the real world, investors demand higher expected returns for bearing risk. For example, a risky stock might have an expected annual return of $\mu = 10\%$, while the risk-free rate is only $r = 3\%$. The extra 7% is a risk premium that the investors desired.

However, Black-and Scholes showed that when pricing an option, this risk premium is surprisingly irrelevant. By continuously adjusting a portfolio of the option and the underlying stock which also called delta-hedging mentioned before in section 2, an investor can eliminate and reduce all risk over short time intervals by frequent trading. A risk-free portfolio must earn exactly the risk-free rate—otherwise, arbitrage would appear and people can extremely gain profits from it until it is regulated. The consequence is simply that the option's price does not depend on μ at all. We can price the option as if we live in a risk-neutral world where all assets grow at rate r, and then discount expected payoffs at that same rate:

$$C = e^{-rT} \cdot \mathbb{E}^{\mathrm{RN}} \left[\max(S_T - K, 0) \right]. \tag{3.7}$$

Here, $\mathbb{E}^{\mathrm{RN}}$ means the expectation that is taken under the risk-neutral distribution;that is, with the drift μ replaced by the risk-free rate r in equation (3.4) which is amazingly true from Math.

This is the reason that the Black-Scholes model contains r in the formula as later shown but not μ which is one of the most smart part the price is somehow unrelated with μ the stock raising rate by risk neutralization. In our Monte Carlo simulations later in Section 5, we will use r as the drift parameter, directly applying the risk-neutral principle. The simulation will confirm that the average discounted payoff converges to the theoretical Black-Scholes price by applications.

3.10. **Connecting of the Concepts.** We have already talked about these nine mathematical ideas or concepts which mainly covers the thing we need to know for the Black-Scholes Model. Therefore, before studying the Black-Scholes model, we should briefly go over them and take a look at the way they fit together with each other as talking about below as a summarization. A more detailed discussion will follow in Section 4.

(1) **The Central Limit Theorem** demonstrates that a sum of plenty or infinite small independent random terms tends approximately to form a normal distribution as arranging the datas. Consequently, we can use this theorem to approximate daily stock returns as normally distributed because each day's return is one small piece and when calculating its sum over the time interval.
(2) **Random walks and Brownian motion** explain how those random moves of stock accumulate over time independently from each move in a basic way. A stock's log-price does not directly jump to its final value; it drifts and fluctuates through continuous random steps, which Brownian motion simulates mathematically.
(3) **Geometric Brownian motion** applies Brownian motion to percentage returns rather than absolute prices and ameliorate the Brownian motion so that it will not exist negative value so that it is able to anticipate for stocks, options and others. In addition, multiplying by S_t makes the stock price stay positive and that volatility operates on a percentage scale, matching how real stocks behave. Accordingly, Geometric Brownian motion sets up the way Black-Scholes Model's randomizing process.
(4) **Itô's Lemma** is the purely mathematical bridge. It offers the bridge that we can move from the Geometric Brownian motion equation to an elegant formula for S_T, which tells us where the stock ends up. This lemma offers the mathematical for the model. This formula reveals S_T follows a log-normal distribution.
(5) **Risk-neutral valuation** is the economic heart of the model. Through delta-hedging, the stock's true expected return μ becomes irrelevant and can be totally ignored and replaced with r. We can price the option as if all assets grow at the risk-free rate r, and then discount future payoffs at that same rate.
(6) **Continuous discounting** converts the expected future payoff into today's money. The term e^{-rT} appears throughout the Black-Scholes formula because a dollar at expiration is worth less than a dollar today.

Each concept participate in the formation of Black-Scholes model. The Central Limit Theorem and Brownian motion provide the theorem randomness. Geometric Brownian motion and Itô's Lemma turn that randomness into a usable distribution. Risk-neutral valuation and discounting turn that distribution into a price and create an environment without influence of μ the annual return. With these tools in hand, we now are confident to turn to the Black-Scholes model itself.

4. Black-Scholes Model

We understand financial terms from Section 2. Section 3 give us a overview of the mathematics. Now we are able to put them together and present the Black-Scholes model itself without any definition obstacles.

4.1. **The Seven Assumptions.** The Black-Scholes model operates in an idealized situation which can hardly be real as defined by seven core assumptions. Knowing these conditions or assumptions can improve our understanding in how the model works and some real life obstacle for serious and applicable application.

(1) **The stock price follows Geometric Brownian motion.** It assumed the stock's percentage returns over small time intervals are normally distributed, with a constant drift μ and constant volatility σ. This assumption is the same Geometric Brownian motion we introduced in Section 3. The reason for making such assumption is that we cannot know the expected gain without knowing the distribution of the probability for each stock price. And in normal market situation, this assumption of log normal distribution is a realistic assumption. This assumption makes us able to estimate the stock's future price with a known probability distribution which is the log-normal distribution.
(2) **The volatility σ is constant.** The model assumes that the stock's tendency to fluctuate stays constant over the option's lifetime. This is always estimated from market. Which sounds or in reality, it is impossible. In fact, volatility changes daily; sometimes, it sharply changes during market panics. Therefore this assumption is most frequently violated in real markets. But why we still need this assumption? it is because a constant σ is easier for calculation and applying.
(3) **The risk-free interest rate r is constant and known.** It is defined by you can borrow or lend any amount of money at the same rate r, and this rate never changes during the option's life. In practice, we use government bond yields as a approximation for r as we talk about in section 2 government is often viewed as the best risk free background.
(4) **The stock pays no dividends.** No dividends are made to shareholders during the option's life. This helps keep the formula clean as option holders cannot get any dividends from shareholders. Also, if the stock gives out the dividends, the call option will lower its price while the put option would raise.
(5) **There are no arbitrage opportunities.** People cannot make a risk-free profit from nothing which is arbitrage. The no arbitrage is the basis as it sets up baseline for delta hedging, risk assessment. And in reality, almost always in the market, there is no chance or only tiny time interval for the chance of arbitrage so that this assumption is closet to reality.

(6) **Trading is continuous and frictionless.** It states people can buy or sell any amount of the stock or the option at any moment as they want. There are no transaction costs, taxes, and restrictions on short selling. This makes the time continuous and teh relating Math approachable for derivation. While in real life, we will face the transaction fees and taxes obviously...

(7) **Fractional shares can be traded.** It is possible for existence like 0.1 shares. We assume this is possible as a lot of time it produces non-integer result from the model. In reality, people cannot buy fractions of a share easily, but this assumption becomes more reasonable for large institutional traders who deal in thousands of shares.

None of these assumptions are really perfectly fit in the real world even for the third one, the rate does small change in fact as table 1 descript. Yet the model works remarkably well as a benchmark. Traders always adjust the inputs, especially the volatility, to fit market conditions. The model's value is never to estimate the reality in a super perfective way; instead, it provides a solid starting point and begins the era of option pricing.

Assumption	**Definition**	**Real world problem**
Geometric Brownian motion for stock price	Log-returns are normal	Markets have jumps and fat tails
Constant volatility	σ never changes	Volatility changes daily
Constant risk-free rate	r is fixed	Interest rates fluctuate
No dividends	Just no dividends	Many stocks pay dividends
No arbitrage	No free money	Markets are mostly efficient
Continuous trading	Trade anytime, no fees	Fees exist; markets close
Fractional shares	Buy 0.1 shares	Only whole shares for retail trader

Table 1. The seven Black-Scholes assumptions and their real-world limitations.

4.2. **The Stock Price Process.** The Black-Scholes model assumes that the stock price S_t follows Geometric Brownian motion, which we introduced in Section 3:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t. \tag{4.1}$$

We go over this equation again in a simplified and clearer way. Over a tiny time interval, two factors influences the stock price:

- Drift: $\mu S_t dt$. The stock has an underlying tendency to grow. The parameter μ is the expected annual return. For instance, if $\mu = 0.10$, the stock is expected to grow at 10% per year on average. dt is just a short period of time. This term is important as it represents the trend of the stock.
- Diffusion: $\sigma S_t dW_t$. Except the trend, random move affects the price raising or lowering. The parameter σ (sigma) is the volatility which is the annual standard deviation of returns. We can just view the term dW_t, the increment of a Brownian motion, as representing pure, unpredictable randomness. For instance, the volatility that changes according to the news whether good or bad.

Obviously, both terms are multiplied by the current stock price S_t. It again indicates that the model is determined by percentage changes, not absolute dollar changes. This is exactly how real stocks behave that the volatility is only about percentages but not dollar amounts.

4.3. **The Stock Price at Expiration.** Equation (4.1) can be solved to give an elegant and short formula for the stock price at any future time T. We state the solution without the difficult or at least dense process of derivation:

$$S_T = S_0 \cdot \exp\left((\mu - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T} \cdot Z\right), \quad Z \sim \mathcal{N}(0,1). \tag{4.2}$$

This equation is the core of our Monte Carlo simulation and Black-Scholes Model in Section 5. We can still look over simply from words description rather than pure math:

- $(\mu - \frac{1}{2}\sigma^2)T$ is the deterministic drift. The volatility appears here with a negative sign because higher volatility in fact reduces the expected growth rate of the stock price in long term. This is just a mathematical property of Geometric Brownian motion, not a reason of real economic effect. Another part μT is easy to understand as the expected stock price grow with the rate in a given time. As a whole, they together depend where the center would locate, while the drift increases with the time increases.
- $\sigma\sqrt{T} \cdot Z$ is the random component. As Z is standard normal with mean zero and standard deviation one, so this term also has a mean of zero and standard deviation of $\sigma\sqrt{T}$. It makes all possible stock prices into a distribution. And if with a good luck, which Z is positive, stock price is higher than expecting.
- The exponential function $e^{(\cdots)}$ makes sure that $S_T > 0$ always exist. For every value or even negative value of Z, the stock price can never become zero or negative which accords to the real life.

Taking the natural logarithm of both sides gives:

$$\ln S_T = \ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T} \cdot Z. \tag{4.3}$$

Since $Z \sim \mathcal{N}(0,1)$, the log-price $\ln S_T$ is normally distributed. Therefore, S_T itself follows a log-normal distribution as well. We definitely get some insight. So, this becomes the mathematical fact as shown that makes the Black-Scholes formula analytically traceable: when we know the exact distribution of S_T, we are able to calculate the expected option payoff $\mathbb{E}[\max(S_T - K, 0)]$ exactly through some approaches.

4.4. **Risk-Neutral Valuation: The Core Insight.** Equation (4.2) contains the parameter μ, which is the stock's expected annual return. Now we need to shift our thoughts on that nobody knows what μ really is because stock is hardly predictable and there is not a consolidate way to anticipate. One investor could believe stocks for instance Tesla will return 15% per year. Another might expect or predict only 5%. If the option price depended on μ, these two investors would disagree on the fair price, so that it could not be a commending price.

Black-and Scholes found a smart way around this. It circumvents beautifully from the annoying μ. The process works well as following steps.

Step 1: Hedge the risk away. Suppose we sold a call option. Once if the stock price rises, we lose money angrily; meanwhile, the option you sold becomes more valuable to the buyer. In order to protect ourselves, we can then buy Δ shares of the underlying stock, where Δ (delta) measures the approximate sensitivity of the option price to the stock price. In more easy explanation, if the stock rises by \$1, the call option price rises by approximately Δ, so our loss on the option is offset by the gain on our shares from stock. It is the same in other ways that the stock drops its price while option drops then we are still in same situation. So amazingly, if we choose the Δ correctly and adjust it continuously during short time interval, our combined position becomes instantaneously risk-free which meaning we do not lose any money! And our task turns to be find the Δ exactly.

Step 2: As a common sense, a risk-free portfolio must only earn the risk-free rate. Our hedged portfolio contains no risk over a tiny time interval. As a result, we can sadly earn exactly the risk-free interest rate r. We question why. The answer is that it is because if it earned more than r, then other smart investor(like and include us) would borrow money at rate r, invest in this risk-free strategy, and gain the difference quickly. That is arbitrage which is a risk-free profit with no initial investment and basically earn free money. In an efficient and healthy market, arbitrage opportunities cannot persist. By the no-arbitrage principle from the assumption, a risk-free portfolio earns exactly r.

Step 3: Now, it is the magical part. We actually find that the stock's expected return disappears. As the hedged portfolio's return equals r and does not involve μ at all, the option's price cannot depend on μ either. The drift parameter which is the hardest thing to estimate drops from the pricing equation entirely.

Step 4: Price in a risk-neutral world. We simply create the risk-neutral world by hedging delta. The later consequence is risk-neutral valuation. We can price the option by pretending we live in a world where:

- All investors are indifferent to risk (they are "risk-neutral").
- All assets grow at the risk-free rate r instead of their true expected return μ.
- Future cash flows are discounted at the risk-free rate r.

In this risk-neutral world, the call option price is simply just the expected payoff, discounted with rate r:

$$C = e^{-rT} \cdot \mathbb{E}^{\mathrm{RN}}\left[\max(S_T - K, 0)\right], \tag{4.4}$$

where $\mathbb{E}^{\mathrm{RN}}$ means we generate S_T using the risk-free rate r instead of μ:

$$S_T^{\mathrm{RN}} = S_0 \cdot \exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T} \cdot Z\right). \tag{4.5}$$

This is the formula we implement in our matlab simulation. The stock's actual expected return μ can be ignored. Only the risk-free rate r, which anyone can look up from government bond keeps remained.

This is the deepest and most important insight of the Black-Scholes model. The mathematics of solving the equation is difficult, long and impressive, while the economic smartness of the realization that delta hedging makes μ irrelevant is the real key for Black-Scholes Model.

4.5. **The Black-Scholes Formula.** Computing the actual expectation in (4.4) with S_T following (4.5) is totally a calculus problem which we are not to pay too much attention to in this paper. The result is the Black-Scholes formula for a European call option as directly displayed below :

$$\boxed{C = S_0 \cdot \Phi(d_1) - Ke^{-rT} \cdot \Phi(d_2)}, \tag{4.6}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}, \tag{4.7}$$

and $\Phi(x)$ is the cumulative distribution function of the standard normal distribution which is the same Φ we introduced in Section 3. We make the things we learned into a chain and go to the final theorem.

The corresponding formula for a European put option is:

$$\boxed{P = Ke^{-rT} \cdot \Phi(-d_2) - S_0 \cdot \Phi(-d_1)}. \tag{4.8}$$

These formulas take five inputs and return one output: the fair price of the option which we desired for long time:

Symbol	**Meaning**	**How to obtain it**
S_0	Current stock price	Look it up on an exchange
K	Strike price	Written in the option contract
T	Time to expiration (years)	Count days to expiration ÷ 365
r	Risk-free interest rate	Government bond yield
σ	Volatility	Estimate from historical data

Four of the five inputs are directly observable. Only σ needs to be estimated. Also, we do a great job learning that the stock annual growth rate is not related in the formula. This model is perfect as it does not require guess work or data collection for four of the factors except the volatility, the annoying one which estimation from historical data is necessary.

4.6. **Interpreting the Formula.** Formula (4.6) looks abstract and scarily complicated at first glance. We can then take a close look at each element.

We can first take a look over the second term. **The second term:** $Ke^{-rT} \cdot \Phi(d_2)$**.**

- Ke^{-rT} is the strike price discounted to today. We do not pay K now; We pay it at expiration, so from simple Math calculation, the present value would be Ke^{-rT}.
- $\Phi(d_2)$ is the risk-neutral probability that the option earns money in such world which expressed in Math $P^{\text{RN}}(S_T > K)$, the probability that is a number between 0 and 1 of Stock Price at expiration date greater than strike price.
- Put them Together, $Ke^{-rT} \cdot \Phi(d_2)$ is the expected present value of what we will have to pay, which is just the price of the option right now times the probability we actually earn money.

The first term: $S_0 \cdot \Phi(d_1)$**.**

- It is possible to expect the first term to be $S_0 \cdot \Phi(d_2)$, as exact as what the second term does. However, it is not and probability is actually $\Phi(d_1)$ instead, which is always slightly larger than $\Phi(d_2)$ from $d_1 = d_2 + \sigma\sqrt{T}$.
- Then, why is the first probability larger? From common sense, we think the probability of earning would be same. But, in fact, because when the option finishes in the money, the stock price is not just barely above K—it is higher than average. The term $\Phi(d_1)$ is the risk-neutral probability of finishing in the money, which is in fact weighted by how far into the money we expect it to be. It accounts for the fact that conditional on $S_T > K$, the expected stock price is greater than the unconditional expected price as well while just considering more about weighting probability.
- Still, in simple words, $S_0 \cdot \Phi(d_1)$ is the expected present value of the stock we will receive, conditional on exercise.

Put together:

$$C = \underbrace{S_0 \cdot \Phi(d_1)}_{\text{Expected stock received}} - \underbrace{Ke^{-rT} \cdot \Phi(d_2)}_{\text{Expected strike paid}}, \tag{4.9}$$

In one sentence to summarize this formula, is that the call option is equal to the expected stock value minus expected strike paid.

Through delta-hedging and creating the risk-neutral world, the stock's true expected return became irrelevant to the option's price.

Finally, we go over most insights of the model and understand it in some extent. Then, in Section 5, we put this formula to the test and compare the Black-Scholes price to real market data and see where the model succeeds to validate it by programming.

5. Data Analysis

Section 4 gave us the Black-Scholes formula. But a formula derived from these idealized assumptions needs to be tested for sure. Specifically in this paper, we use the testing as a way to prove the validation of the formula. In this section, we verify the model through numerical experiment by Monte Carlo simulation, a method that generates thousands of random future stock prices and averages the results and introducing really soon.

5.1. **Why Simulate with Monte Carlo?** When we desire to test if the model works, we will first think about comparing with actual data in the world by the model. If the model works, its price should be approximate about what traders actually pay. However, this approach has a serious problem we did not realize: we do not know the true volatility σ of the stock. In fact, σ must be estimated from historical data as stated before, and there is not an exact way that future volatility will match the past. If the Black-Scholes price is different from the market price, it is impossible for us to identify whether this inaccuracy blames to formula or our estimate of σ. Testing with real data mixes two unknowns together, so using real-life data makes the problem vague.

Therefore, we are introducing the Monte Carlo simulation that solves this problem by creating a controlled experiment. Instead of using real market data, we are able to obtain a specific fix value for σ and generate thousands of random stock price paths that exactly follow the geometric Brownian motion assumed by the model. Because we know the true parameters that generated the data, we can turn it into a simpler concern. Monte Carlo simulation computes the same expected value by brute force that generate many random paths and averaging their payoffs. Therefore, in a straight way, if the two simulations agree together here, the formula's mathematical derivation is internally consistent. If they disagree, there is an error in the formula.

This is the whole reason of the Monte Carlo experiment in this section. We are not testing whether the Black-Scholes assumptions are realistic that question will be discussed more in Section 7. We are using brute force to check if the model provides an accurate answer.

5.2. **How Monte Carlo Works.** We are listing the steps like pseudocode for the Monte Carlo simulation:

(1) We need to set the parameters at first. Choose S_0, K, T, r, and σ which is the same five inputs the Black-Scholes formula needs as well.

(2) We need to generate a random future stock price with the use the risk-neutral Geometric Brownian motion formula from Section 3:

$$S_T = S_0 \cdot \exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T} \cdot Z\right), \tag{5.1}$$

where Z is the random number from the standard normal distribution. Each Z produces one possible future.

(3) We then calculate the payoff, which for a call option, the payoff is simply the subtraction $\max(S_T - K, 0)$.

(4) Discount to today. Multiply the payoff by e^{-rT}. Money in the future is worth less than money today. This is still the use of Math ideas from section 3.

(5) Repeat these process by brute computing technologies. Go back to step 2 and generate another random S_T. Do this tens of thousands of times.

(6) Finally, the average of all discounted payoffs is the Monte Carlo estimate of the option price that we wanted to for comparison.

Each repetition generates one possible outcome option price. And since the each outcome is equally weighted, the average value of the sum of large amount of repetition would just be converging to the expected value which is what Black-Scholes computes.

5.3. **Simulation Setup.** For our experiment, we use the following parameters as a control group:

Parameter	**Value**	**Meaning**
S_0	100	Current stock price (euros)
K	105	Strike price (euros)
T	0.5	Time to expiration (6 months)
r	0.03	Risk-free rate (2% per year)
σ	0.20	Volatility (20% per year)

Table 2. Parameters used in the Monte Carlo simulation.

These values describe a slightly out of the money call option but still fit for generating the results. First, the S_0, stock price, starts with 100 euros is the standard starting price for option simulation. The K, strike price, is higher than the stock price because it better displays the dependence on volatility. 6 months time to expiration is common in European option market. The r value of 0.03 is the common European government bonds for short term. The volatility of 20% is an average growth rate for big stocks like Euro Stoxx 50. So, we gather all standard and normal data and ready to program it out.

5.4. **Convergence Results.** We then run the Monte Carlo simulation with path counts ranging from 100 to $500,000$. For each path, we record the Monte Carlo price estimate, its standard error, and the absolute difference from the Black-Scholes theoretical price (directly come from the formula) in Table 3.

Paths	MC Price (Euros)	Std Error (Euros)	Abs Error (Euros)
100	6.004987	1.126543	1.826687
500	3.575380	0.317242	0.602920
1,000	4.221890	0.267914	0.043590
5,000	4.207581	0.111504	0.029282
10,000	4.024706	0.076205	0.153594
50,000	4.147536	0.034869	0.030764
100,000	4.195774	0.024797	0.017474
500,000	4.187603	0.011097	0.009304

Table 3. Monte Carlo convergence to the Black-Scholes theoretical price. As the number of simulated paths increases, both of standard Error and absolute error decreases meaning it is closer to the theoritical value .

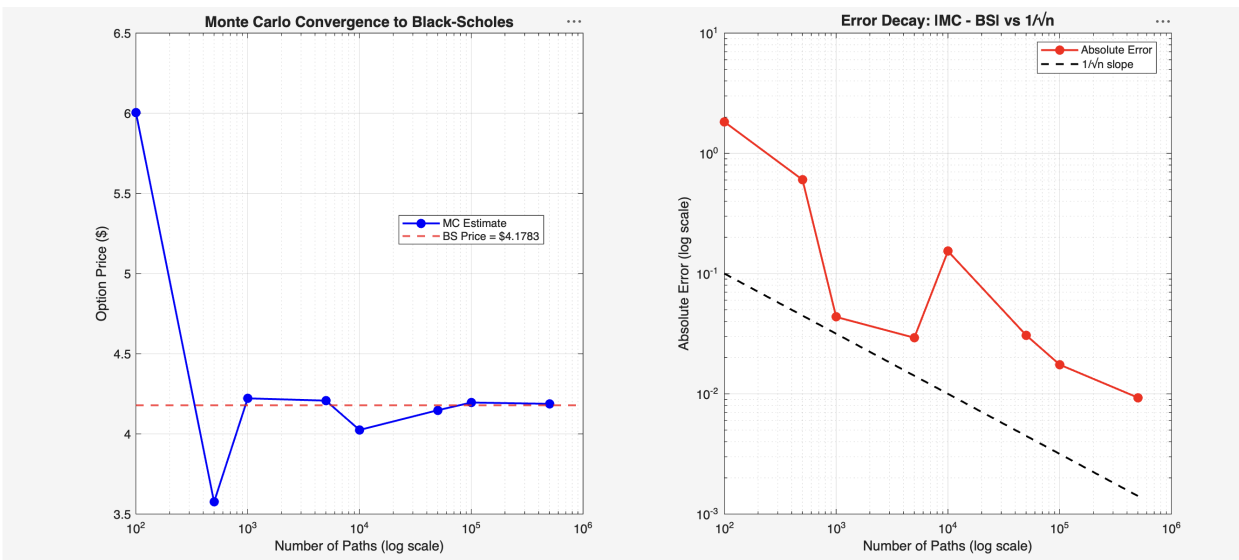

Figure 1. Left: Monte Carlo price estimates converge to the Black-Scholes theoretical price (red dashed line) as the number of paths increases. Right: the absolute error decreases at a rate proportional to $1/\sqrt{n}$ (gray dashed line), consistent with the theoretical convergence rate of Monte Carlo methods.

5.5. **Analysis.** We can then discover or conclude three key observations from the convergence data shown above.

First, notice the Monte Carlo's estimation clearly approach the Black-Scholes price as the number of paths grows. From Table 3: 100 paths, the estimate is away from the theoretical value by roughly 1.8267 euros; with $500,000$ paths, the error drops dramatically to approximately 0.009304 euros. This trend is visible in the left of figure 1: the blue solid line of Monte Carlo estimates oscillates less and moves closer to the red dash line(theoretical value) as we move to larger path counts. Why does it appear a large oscillation at first or in fewer paths? We can state confidently for the reason: most paths finish out of the money with zero payoff, while a small number finish deep in the money with large payoffs as by random. In other words, it does depend on luck at early time. But with adequate samples and paths, the price will converge to the theoretical price.

Second, the rate of convergence matches proved theoretical expectations. Monte Carlo methods are well known to converge at a rate proportional to $1/\sqrt{n}$ (just given without proving), where n is the number of samples. Another way to say is that to half the error, one must quadruple the number of paths. The right of Figure 1 plots the absolute error against the number of paths on a log-log scale. The error points follow the gray dashed line with the $1/\sqrt{n}$ slope in a way as shown. Therefore, confirming that our simulation(Monte Carlo) behaves exactly as statistical theory(proved fact) predicts. This observation increases our confidence for the verification indirectly.

Third, we talk about the standard error column in Table 3. They quantify the precision of each estimate. With 100 paths, the standard error is approximately 1.1265 euros, from the normal distribution mentioned in section 3, we know the true value would be in a large range with decent probability. By $100,000$ paths, the standard error has shrunk to roughly 0.011097 euros—less than one cent. We are in a comparable good range. For any practical purpose, $500,000$ paths deliver a standard deviation of 0.0111, and the remaining uncertainty is negligible compared to the real market.

To conclude, these three observations confirm that the Black-Scholes formula is consistent and actually compelling. Under the model's own assumptions of Geometric Brownian motion with constant volatility σ and constant risk-free rate r, the analytical formula and the brute force Monte Carlo average converge to the same value highly approximately. The experiment does not tell us whether those assumptions hold in real markets; that question is separate totally because real market's volatility is always a problem. However in a mathematical reasoning way with the assumptions to the formula, we prove it or at least validate it with Monte Carlo estimation. The Black-Scholes model, no matter of its practical limitations, is mathematically solid.

6. Intro of Binomial Tree

We grant a a clean, closed-form formula for European option prices from Black-Scholes model. Nevertheless, we also face a significant challenge: it is not able to predict American options directly because American market allows early exercise while European does not. However, with the binomial tree method, introduced by Cox, Ross, and Rubinstein in 1979 [6], solve this obstacle by making time into small steps. We are going to first study and understand the binomial tree model in a simple version and then create some visualization of the use of this model beyond Black-Scholes model specifically in American stock market put option as mentioned later.

6.1. **Binomial Tree's working process.** The binomial tree breaks the continuous time duration it into n small steps, each of length $\Delta t = T/n$. This can be understood by the ability of dealing option any time before expiration date. In each step, the stock price does one of only two things: it goes up, or it goes down. This simplification making the problem into two steps and much more computable.

The focus turns into how much the stock goes up or down. The up factor is $u = e^{\sigma\sqrt{\Delta t}}$, and the down factor is $d = 1/u$. Instead of deriving, we just need to know the expression of these two factors makes the uncertainty amount over one step matches the stock's volatility

σ. Interestingly, $u \times d = 1$. This is because an up move followed by a down move takes the price right back to where it started.

At each step, p is the probability for stock to move up, and down with probability $1 - p$. The formula for p is stated as:

$$p = \frac{e^{r\Delta t} - d}{u - d}. \tag{6.1}$$

We can understand this in a definition way. In the risk-neutral world we discussed in Section 4, all assets must grow at the risk-free rate r on average. The probability p is chosen precisely so that the expected stock price after one step equals $S \cdot e^{r\Delta t}$ which is fit for risk-neutral condition.

Then, after n steps, the tree reaches expiration. At each terminal node that represents each possible final stock price, the option payoff is easy to compute. For a put, it is $\max(K - S_T, 0)$: the amount by which the strike exceeds the stock price, or zero if the option is out of the money which is actually similar logic with Black-Scholes logic.

Here comes the smartest and different part: we work backward from the tree. One step before expiration, at any given node of the tree, we know the two possible option values one step ahead which is the branches before this node. The value at the current node is simply the average of those two future values, weighted by p and $1 - p$, and then discounted by $e^{-r\Delta t}$ to account for the time value of money:

$$V = e^{-r\Delta t} \left(p \cdot V_{\text{up}} + (1 - p) \cdot V_{\text{down}} \right). \tag{6.2}$$

We later repeat this calculation at every node which is moving one step back at a time until today. The number at the starting node is then amazingly option price. And we know a way different from Black-Scholes model to calculate the option price.

6.2. **Binomial Tree and American Options.** For a European option with a fixed exipration date, the binomial tree is just another different way to compute the same value of option that Black-Scholes gives us. We can use it to check but not helpful. Where the binomial tree makes a difference is in American options, specifically of a put option as talking about later.

At each single node in the tree, we as option holder have two decisions as describing below:

- Wait. The option is worth the discounted expected value of what it might be worth next period. This is the number we computed in the European case.
- Exercise now which happens when the option is worth its immediate payoff—for a put, $\max(K - S_{\text{node}}, 0)$.

We as smart investors pick whichever is larger:

$$V = \max \left(\text{Wait value, Exercise value} \right). \tag{6.3}$$

This single comparison repeated every node help with our decision. It is somewhat similar with greedy algorithm but anyway it gives us two interesting facts between call option and put option

Waiting is always better than exercising early for a call option without dividends. Because exercising means paying the strike price K now, whereas waiting means paying K later. From the continuous discounting in section 3, future money is cheaper than money now. Also, in even easier examples, we hold a call option with strike \$100, if the stock grows to \$ 105, we definitely wait as the price could grow higher, while if the stock drops to \$ 99, we are waiting because it is possible to climb up. For the percentage of dropping significant amount like \$100 is unpredictable and least likely happen while we will talk about this in section 7. Therefore, in this situation of options on a stock without any dividends, American call is equivalent to European call.

However, the game changes for a put option. Imagine we now hold a put with strike \$95, and the stock has crashed to \$40. What a lose! But it good for us because we buy the put option. Exercising now gives us \$55 immediately, and we could put that money in the bank and earn the interest. Waiting might bring a slightly larger payoff if the stock falls further, but unlike rising without any restriction of upper bond, from a Math way, we know the minium it could drop is only 0. And we then give up the interest we could have earned by that. At some nodes of the tree, the promised interest outweighs the low chance of a further decline which is the signal for us to exercise rather than waiting. The American put is worth more than its European counterpart as the exisitng chance describing above, and the binomial tree captures exactly how much more

6.3. **Analysis.** We finally test the binomial method on a put option with $S = 100$, $K = 95$, $T = 0.5$, $r = 0.02$, and $\sigma = 0.20$ which is the similar data explaining above previously in section 5 better for comparing. Table 4 shows the results for step counts from 5 to 1000.

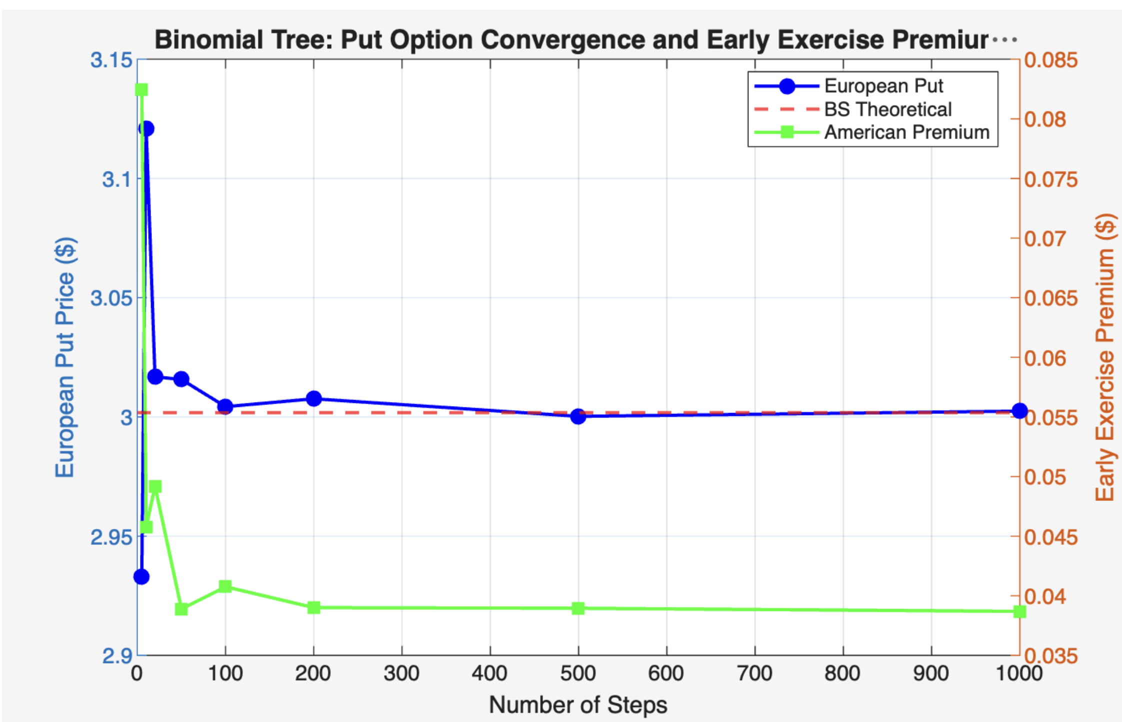

Figure 2. Binomial tree for a put option. European put price (blue) converges to the Black-Scholes value (red dashed). Early American exercise premium (green) which is the difference between American put and European put remains positive across all step counts.

Steps	European Put ($)	American Put ($)	Premium ($)
5	2.933109	3.015532	0.082423
10	3.121065	3.166819	0.045755
20	3.016824	3.066009	0.049185
50	3.015695	3.054579	0.038884
100	3.004291	3.045046	0.040755
200	3.007607	3.046611	0.039004
500	3.000231	3.039181	0.038950
1000	3.002437	3.041127	0.038690

Table 4. Binomial tree prices for a put option ($S = 100$, $K = 95$). The European put converges to the Black-Scholes price. The American put is consistently higher due to the value of early exercise.

We can notice two observations from the graph. First of all, the European binomial price converges to the Black-Scholes value as n increases, confirming the method's accuracy that Binomial tree can actually estimate the option price like Black-Scholes model. As $n \to \infty$, the European binomial price converges to Black-Scholes, we then have confidence that binomial tree model is a right method Second, the American put is consistently more valuable than the European put, with an early exercise premium of approximately $[0.0824]. This difference exists because when the stock price drops sufficiently, exercising immediately and earning interest on the proceeds beats waiting in most conditions. The binomial tree captures this by comparing hold and exercise values at every node which is something the Black-Scholes formula cannot do.

In conclusion, we can imagine Black-Scholes model values the process in an overall view while the Binomial Tree views each step individually. The binomial tree actually extends option pricing beyond the Black-Scholes as it covers. what Black-Scholes model can do and cannot do.

7. Conclusion

In this section, we are concluding with some further questions.

First, the Black-Scholes model's most not realistic assumption is the constant volatility. In fact, volatility changes over time and across strike prices which is a phenomenon called the volatility smile. When traders invert the Black-Scholes formula to back out implied volatility from market prices, they find that out-of-the-money options carry higher implied volatility than at-the-money options. From this, we can say the market prices stays in a greater probability of extreme moves than the log-normal distribution allows.

This results in the second question: the model assumes stock returns are normally distributed while real markets always represents in fat tails that the large price moves occur far more frequently than the normal distribution predicts. We can imagine the bell curve's right side being extended a lot. Extensions like the Heston model, allows volatility itself to follow a random process, and jump-diffusion models, incorporate sudden price discontinuities, address these shortcomings. However, the 1987 stock market crash, the 2008 financial crisis, and the 2020 COVID-19 sell-off are all well-known "black swan" events that the price lie many standard deviations from the mean under the normal assumption. Black swan is a story that thousands years of believe that swan cannot be black in Europe while people find the black swan in Africa. It basically indicates some irregular events that are totally not predictable from history data. No model built on geometric Brownian motion can capture these risks. And no specific model can solve this right now because they are higly abrupt.

Excluding these limitations, traders still do not use it as a perfect model in reality. Every option price on a screen can be translated into an implied volatility number via Black-Scholes, allowing traders to compare options across strikes and maturities in a standardized way. So the key here in market in nowadays is the estimation of volatility. The model's real importance is not the formula itself, but the smart creation of risk-neutral valuation that it introduced modern derivative pricings..

For the interested reader wants to grind more Math behind the theorem and beyond this paper, further topics include stochastic volatility models (Heston, 1993), numerical methods beyond binomial trees (finite difference methods, least-squares Monte Carlo for American options) are recommended for reading. The Black-Scholes formula serves as the solid start point of the option pricing.

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Appendix: MATLAB Code

The complete MATLAB code is available at: `https://www.overleaf.com/read/txhdsgxxsfqb#9404c4`