Counting Domino Tilings

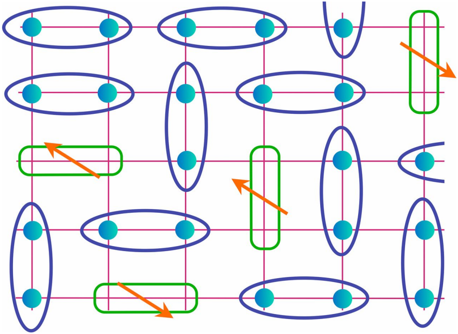
Derek Hu

Euler Circle IRPW

07 09, 2026

History

Domino tilings have applications in quantum lattice spin models and the Ising model. The number of domino tilings of an $m \times n$ grid was independently discovered by three physicists Pieter Kasteleyn, Harold Temperley, and Michael Fisher in 1961.



A quantum dimer model

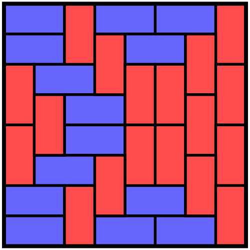
What is a domino tiling?

Definition

A *domino* is a set of two directly adjacent unit squares.

Definition

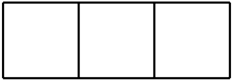
A *domino tiling* of a grid is a set of disjoint dominoes such that the entire grid is covered with dominoes and that no dominoes are sticking out of the grid.



A domino tiling of an 8×8 grid

Does a tiling exist?

Not all grids have domino tilings. For example, this 1×3 grid doesn't have any domino tilings.

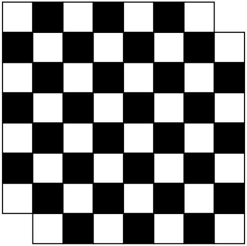


A 1×3 grid

Theorem

A domino tiling exists for an $m \times n$ grid if and only if mn is even.

The Mutilated Chessboard



Mutilated chessboard with opposite corners missing

Theorem

The mutilated chessboard does not have any valid domino tilings.

Using Recursion

Definition

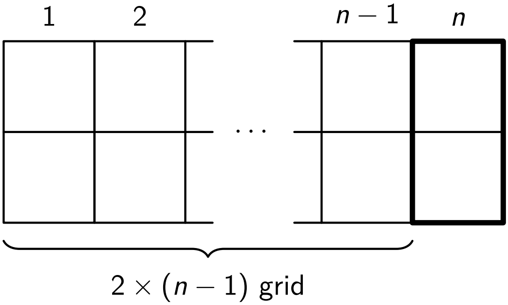
The n^{th} Fibonacci number, F_n, is defined as $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n > 2$.

The first few Fibonacci numbers: $1, 1, 2, 3, 5, 8, \ldots$

Theorem

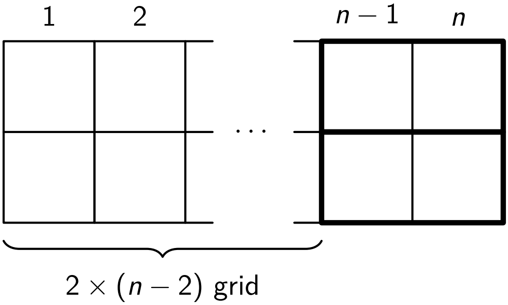
A $2 \times n$ grid has F_{n+1} domino tilings.

Proof



Vertical domino on the end of $2 \times n$ grid

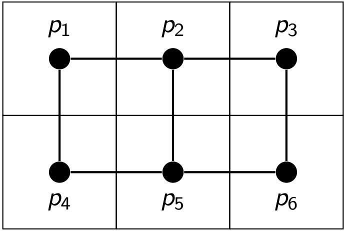
Proof



Horizontal dominoes on the end of a $2 \times n$ grid

Graphs

We can represent each square in a grid by a vertex in a graph and draw edges based on adjacent squares.



Example with a 2×3 grid

Adjacency Matrices

From the graph, we can create an adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

With this adjacency matrix, it is simple to see if a given tiling is valid, since a pair of squares $\{p_i, p_j\}$ is a valid domino if and only if $a_{ij} = 1$. Thus, one way we could count the total number of domino tilings is by iterating through all the possible partitions of the squares into pairs and counting the partitions where all the pairs are dominoes.

Skew Symmetric Matrices

Definition

For a square matrix A, if a_{ij} denotes the entry in the i-th row and j-th column, A is considered *skew-symmetric* if $a_{ij} = -a_{ji}$ for all entries in the matrix.

$$A = \begin{bmatrix} 0 & 1 & -2 & 3 & 0 & -4 \\ -1 & 0 & 5 & 0 & -1 & 2 \\ 2 & -5 & 0 & -3 & 4 & 0 \\ -3 & 0 & 3 & 0 & 1 & -5 \\ 0 & 1 & -4 & -1 & 0 & 6 \\ 4 & -2 & 0 & 5 & -6 & 0 \end{bmatrix}$$

The Pfaffian

Definition

The *Pfaffian* of a $2N \times 2N$ skew-symmetric matrix A with entries a_{ij} is defined as

$$\mathrm{Pf}(A) = \sum_{\sigma \in \Pi} \mathrm{sgn}(\sigma) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_N j_N} = \sqrt{|\det(A)|}$$

where Π is the set of all the ways to partition the set $\{1, 2, ..., 2N\}$ into N pairs ordered such that $i_k < j_k$ and $i_1 < i_2 < ... < i_N$, σ is a permutation of $\{1, 2, ..., 2N\}$ given by the sequence $\{i_1, j_1, i_2, j_2, ..., i_N, j_N\}$, and $\mathrm{sgn}(\sigma) = (-1)^M$ where M is the number of inversions for σ. An inversion is a pair of indices x, y such that $x < y$ and $\sigma(x) > \sigma(y)$.

The Pfaffian

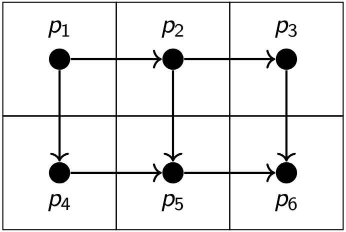
We can turn our regular adjacency matrices into skew-symmetric matrices by allowing $a_{ij} = -a_{ji} = 1$ for $i < j$ if p_i is adjacent to p_j. The Pfaffian of A would perfectly count the total number of domino tilings if it weren't for the $\operatorname{sgn}(\sigma)$ term.

$$\{\{p_1, p_2\}, \{p_3, p_6\}, \{p_4, p_5\}\}$$

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & \mathbf{1} \\ -1 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

Kasteleyn Signings

Since we are using skew-symmetric matrices, our graph looks a little different.



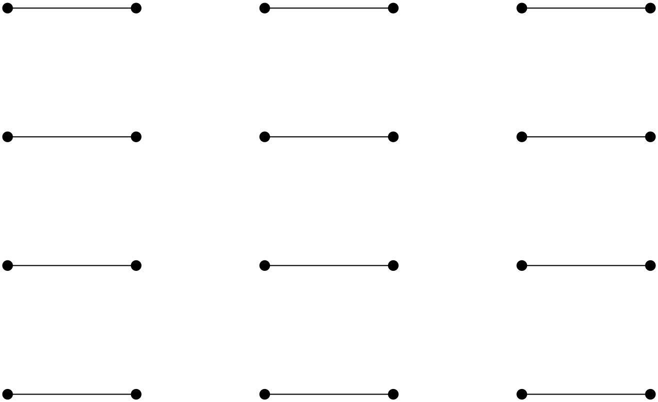
Example with a 2×3 grid

It is possible to change the directions of some of the edges of this graph such that the Pfaffian of its skew-symmetric adjacency matrix is equal to the number of domino tilings.

The Standard Configuration

Definition

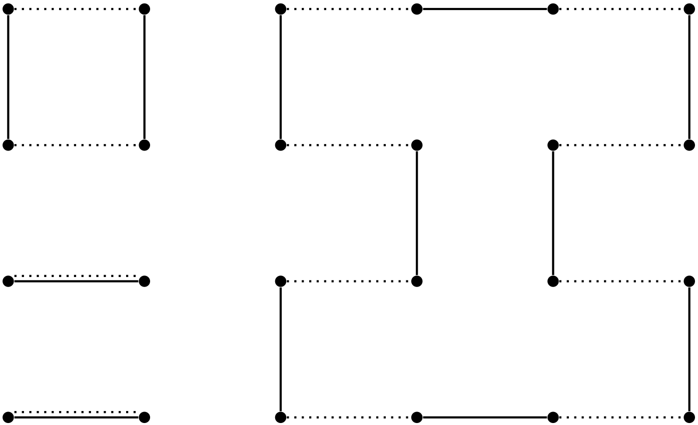
The standard configuration of an $m \times n$ grid, n even, denoted as C_0, is the domino tiling $\{\{p_1, p_2\}, \{p_3, p_4\}, \ldots \{p_{mn-1}, p_{mn}\}\}$.



C_0 for a 4×6 grid

C_0 and C

We can overlay C_0 and an arbitrary domino tiling C:



Both C and C_0

Alternating Polygons

This diagram consists of polygons with edges alternating between those of C_0 and C. It can be proved that each alternating polygon contributes a factor of -1 to the $\text{sgn}(\sigma)$ term, which is a byproduct of the conversion from C_0 to C.

Clockwise Edges

We can find that

$$\sum_{i=1}^{F} \omega(f_i) \equiv \omega(P) + E_{\text{int}} \pmod{2}.$$

Euler's formula for planar graphs states that $V - E + F = 2$. We can manipulate this:

$$V - E + F = 2 \implies E_{\text{int}} = V_{\text{int}} + F - 1.$$

We can then substitute this into our earlier equation:

$$\omega(P) \equiv \left(\sum_{i=1}^{F} (\omega(f_i) + 1) \right) + V_{\text{int}} + 1 \pmod{2}.$$

If $\omega(f_i) \equiv 1 \pmod{2}$, then $\omega(P) \equiv 1 \pmod{2}$.

Clockwise Edges

Let P be a polygon of C, where M_e represents the matrix entries along the edges of the polygon. The total sign contribution of P to the Pfaffian expansion is

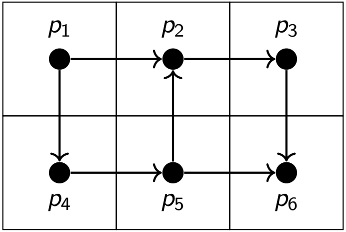
$$\operatorname{sgn}(P) \prod_{e \in P} M_e.$$

If we trace P clockwise, the product of its edge weights will be

$$(1)^{\omega(P)}(-1)^{E_{\text{boundary}}-\omega(P)} = (-1)^{E_{\text{boundary}}-\omega(P)}.$$

Creating a Kasteleyn Signing

If we find a way to orient the edges in the graph such that each face has an odd number of clockwise edges, it will be a Kasteleyn signing and will produce a Kasteleyn matrix.



2×3 grid with every other vertical column of edges reversed

Pfaffian Evaluation

The total number of domino tilings of an $m \times n$ grid will be equal to the Pfaffian of the $mn \times mn$ Kastelyn matrix K. Since $\text{Pf}(K) = \sqrt{|\det(K)|}$, we can find the Pfaffian by finding the determinant. The determinant of K is equal to the product of the eigenvalues of K. We can then use spectral graph theory and linear algebra to find these eigenvalues.

Pfaffian Evaluation

$$\mathrm{Pf}(K) = \sqrt{|\det(K)|} = \sqrt{\left|\prod_{k=1}^{mn} \lambda_k\right|}$$

$$\mathrm{Pf}(K) = \sqrt{\left|\prod_{j=1}^{m}\prod_{k=1}^{n} 2i\sqrt{\cos^2\left(\frac{\pi j}{m+1}\right) + \cos^2\left(\frac{\pi k}{n+1}\right)}\right|}$$

$$\mathrm{Pf}(K) = \prod_{j=1}^{\lceil \frac{m}{2} \rceil}\prod_{k=1}^{\lceil \frac{n}{2} \rceil}\left(4\cos^2\left(\frac{\pi j}{m+1}\right) + 4\cos^2\left(\frac{\pi k}{n+1}\right)\right)$$

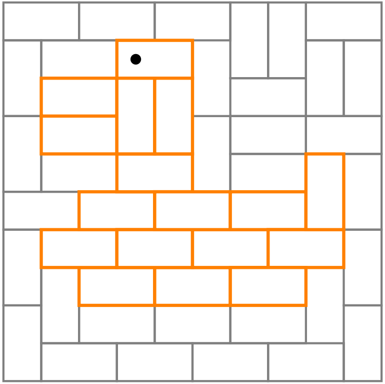
Theorem

The number of domino tilings of an $m \times n$ grid is

$$\prod_{j=1}^{\lceil \frac{m}{2} \rceil}\prod_{k=1}^{\lceil \frac{n}{2} \rceil}\left(4\cos^2\left(\frac{\pi j}{m+1}\right) + 4\cos^2\left(\frac{\pi k}{n+1}\right)\right)$$

Thank You

Any questions?



A domino tiling of a 10×10 grid