# Counting the Domino Tilings of a Rectangular Grid

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**Abstract**

We present a derivation of a formula to count the number of domino tilings of rectangular grids. We shall first see if a tiling exists, and then count them for $1 \times n$ and $2 \times n$ grids with recursion. We represent the grid as a graph with each square corresponding to a vertex. We can then construct adjacency matrices to represent the relationships between the squares. We shall create a skew-symmetric adjacency matrix and utilize Kasteleyn's orientation theorem to solve the sign problems of the determinant, reducing the problem to finding the Pfaffian of a Kasteleyn matrix. Finally, we arrive at the formula to count the number of domino tilings of rectangular grids.

## 1 Introduction

A *domino* is a set of two directly adjacent unit squares. A *domino tiling* of a square grid is a configuration such that every square is covered by exactly one domino (also known as a dimer). Scientists have long been interested in domino tiling, especially due to its applications in physics. It is often used in quantum mechanics to model the physics of a lattice spin system and in the Ising model for magnetism. Many grids may be tiled by dominoes in a multitude of ways. For example, there are three ways to tile a $2 \times 3$ grid. To find the exact number of tilings of a given grid is a curious and often difficult problem in combinatorics. Not all grids have domino tilings, with a famous example being the mutilated chessboard, which we will discuss in Section 3. For certain cases such as a $2 \times n$ grid, the solution can be found through recursion. However, recursion becomes impractical for $m \times n$ grids if $m$ and $n$ are large. As it turns out, the number of domino tilings of an $m \times n$ grid can be expressed as the Pfaffian of a certain matrix $K$, yielding

$$\prod_{j=1}^{\lceil \frac{m}{2} \rceil} \prod_{k=1}^{\lceil \frac{n}{2} \rceil} \left( 4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1} \right).$$

This formula was independently discovered by three physicists Pieter Kasteleyn, Harold Temperley, and Michael Fisher in 1961. Where did all of those cosines come from? Isn't this a discrete combinatorics problem? In this paper, we will primarily focus on the derivation of this formula. We can represent the grid with a bipartite graph. We can create a biadjacency matrix representing this graph, and the permanent of this matrix is equal to the number of domino tilings. However, computing the permanent is impractical for large grids. Finding the Pfaffian is very similar to finding the permanent, as their structures are nearly identical, and is much easier to do for large grids, but there are some sign issues that lead to undercounting. A grid can be seen as a graph by designating the squares as vertices and creating edges when two squares are adjacent. We can then create a Kasteleyn signing of this graph and create a new adjacency matrix to fix the sign issues. The Pfaffian of this matrix is then equal to the number of domino tilings. The Pfaffian squared is equal to the absolute value of the determinant, which allows us to streamline its evaluation. We can find the determinant of the matrix by finding the product of its eigenvalues. We may then use spectral graph theory, which is the study of graphs in relation to linear algebra. The matrix can be broken down into a sum of the independent vertical and horizontal adjacency relations through Kronecker products, and the eigenvalues of each of these can be found individually. The matrix may then be squared to find its eigenvalues. Finally, the Pfaffian can be calculated from these eigenvalues.

## 2 Preliminaries

**Definition 2.1.** An $m \times n$ *square grid*, where $m$ and $n$ are positive integers, is a rectangular arrangement containing $m$ horizontal rows and $n$ vertical columns.

Each square may be denoted by $(i, j)$ or $p_k$ for $i = 1, 2, ..., m, j = 1, 2, ..., n, k = 1, 2, ..., mn$. We may use both these forms interchangably, obtaining the corresponding value of $k$ from $(i, j)$ by $k = (j - 1)m + i$.

**Definition 2.2.** A *domino* is a set of two directly adjacent unit squares.

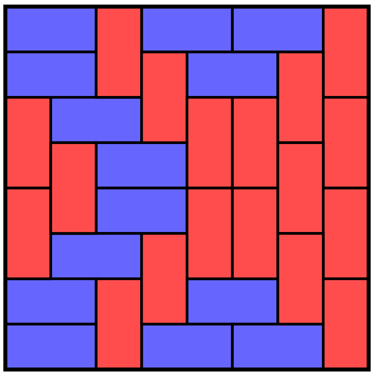

Figure 1: A domino tiling of an $8 \times 8$ grid

**Definition 2.3.** A *domino tiling* of a grid is a set of disjoint dominoes such that the every unit square is covered with by a domino and that no dominoes are sticking out of the grid.

Of course, many grids have lots of different domino tilings. Let $T(m, n)$ be equal to the number of domino tilings of an $m \times n$ grid. Note that $T(m, n) = T(n, m)$.

**Definition 2.4.** A *permutation* is an ordered arrangement of the items in a set.

For example, a permutation of $\{1, 2, 3\}$ is $(2, 1, 3)$. We will often use $\sigma$ to represent a permutation.

# 3 Does a tiling exist?

Not all grids have a domino tiling. For example, it is easy to see that a $1 \times 1$ grid does not have any domino tilings. For which rectangular grids does a domino tiling exist?

**Theorem 3.1.** *A domino tiling exists for an $m \times n$ grid if and only if $mn$ is even.*

*Proof.* Let a given domino tiling consist of $N$ dominoes. Since each domino consists of 2 unit squares, and since every unit square is covered by a domino, we have $2N = mn$. Thus, $mn$ must be even. Conversely, suppose $m$ and $n$ are both positive integers such that $mn$ is even. This implies that $m$ or $n$ is also even. Without loss of generality, let $m$ be even with $m = 2k$ for $k \in \mathbb{Z}_{>0}$. This means that each column will have $2k$ squares. We can then fill up each column with exactly $k$ dominoes $n$ times. This will give us a valid domino tiling. Thus, any rectangular grid with even area must have at least one domino tiling. $\square$

Note that having even area is not enough to ensure the existence of a domino tiling for non-rectangular grids. A famous example of this is the mutilated chessboard, which is an $8 \times 8$ chessboard that has had 2 of its opposite corners removed, as seen in Figure 2. The mutilated chessboard does not have any domino tilings despite containing an even number of squares.

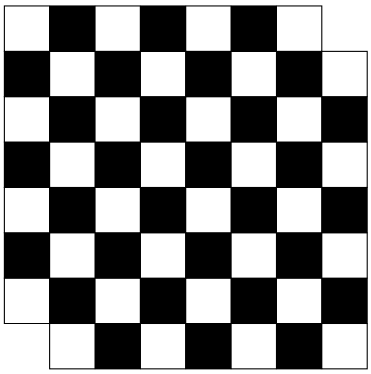

Figure 2: Mutilated chessboard with opposite corners missing

**Theorem 3.2.** *(Black, 1946) The mutilated chessboard does not have any valid domino tilings.*

*Proof.* We can color each square $(i, j)$ in the mutilated chessboard white if $i + j$ is even and black if $i + j$ is odd. As we can see in Figure 2, the mutilated chessboard contains a total of 30 black squares and 32 white squares. Each black square is only adjacent to white squares and each white square is only adjacent to black squares. This is because each square is only adjacent to squares with different parities of $i + j$. Since a domino occupies two adjacent squares, any domino on the mutilated chessboard would cover both a white square and a black square. Thus, a domino tiling of the mutilated chessboard would cover the same number of white and black squares. However, the mutilated chessboard contains an unequal number of black and white squares. Thus, the mutilated chessboard does not have any valid domino tilings. □

In general, coloring arguments are very useful for determining the existence of tilings for all sorts of figures, including grids that aren't composed of squares. Now that we have found types of rectangular grids that have domino tilings, we can get to counting them.

# 4 Counting with simpler rectangular grids

Before we look at $m \times n$ grids in general, we shall first look at an example where $m = 1$.

**Theorem 4.1.** *A $1 \times n$ grid has 1 domino tiling if $n$ is even and 0 domino tilings if $n$ is odd.*

*Proof.* If $n$ is odd, our $1 \times n$ grid will have an odd number of unit squares. By Theorem 3.1, no domino tilings will exist. If $n$ is even, the opposite will be true, and by Theorem 3.1 at least 1 domino tiling must exist. Consider the last two squares in the $1 \times n$ grid for even $n$. In total, there is one way to tile these two squares, which is with one horizontal domino. If we look at the remaining $1 \times (n-2)$ grid, the number of ways to tile it is $T(1, n-2)$. Thus, we have $T(1, n) = T(1, n-2)$. We can do the same thing to $T(1, n-2)$ and realize that it is equal to $T(1, n-4)$. We can continue doing this until we reach $T(1, n) = T(1, 2)$. A $1 \times 2$ grid has one domino tiling. Thus, if $n$ is even, there is one domino tiling. □

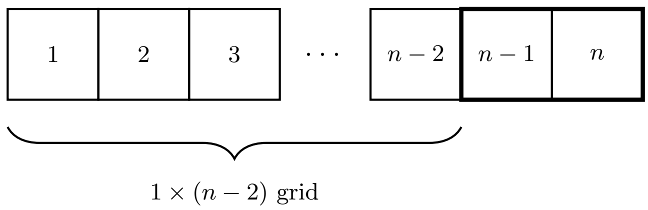


Figure 3: A $1 \times n$ grid

We may wonder if such a recursive approach would also work for higher $m$. Indeed, a result for $m = 2$ can also be found.

**Definition 4.1.** The $n^{th}$ Fibonacci number, $F_n$, is defined as $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all integers $n > 2$.

The Fibonacci numbers are helpful for finding the number of domino tilings of some grids. Besides this, they also have uses in architectural design, modeling natural growth patterns, analyzing financial markets, and optomizing computer algorithms.

**Theorem 4.2.** *A $2 \times n$ grid has $F_{n+1}$ domino tilings.*

*Proof.* Consider all of the domino tilings of a $2 \times n$ grid for $n > 2$. By Theorem 3.1, we must have at least one domino tiling. Some of these domino tilings will have a vertical domino on the end covering the squares $(1, n)$ and $(2, n)$, and others will have two horizontal dominoes on the end covering $(1, n-1), (1, n)$ and $(2, n-1), (2, n)$ respectively. If we remove the vertical domino, we are left with a $2 \times (n-1)$ grid, which has $T(2, n-1)$ domino tilings. If we remove the two horizontal dominoes, we are left with a $2 \times (n-2)$ grid, which has $T(2, n-2)$ domino tilings. Thus, $T(2, n) = T(2, n-1) + T(2, n-2)$. Since $T(2, 1) = 1$ and $T(2, 2) = 2$, we see that this is very similar to the definition of the Fibonacci numbers, except shifted by 1. Thus, a $2 \times n$ grid has $F_{n+1}$ domino tilings. □

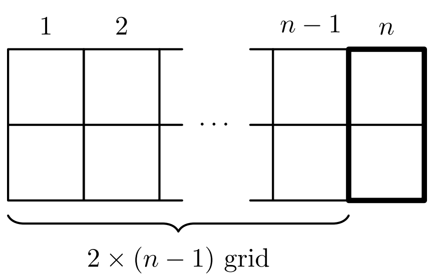


Figure 4: Vertical domino on the end of $2 \times n$ grid

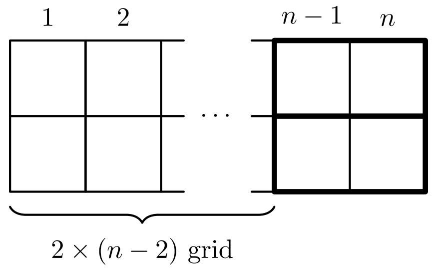


Figure 5: Horizontal dominoes on the end of a $2 \times n$ grid

Would recursion work for $m = 3$? If we had a lot of time to spare, perhaps it could, but the task becomes very complicated very quickly. Even with a small $3 \times 4$ grid, it is rather difficult. It is hard to determine which grids we should try to reduce to. There are also plenty of non-rectangular subtilings, which didn't appear for $m = 1, 2$. The problem only becomes harder with larger grids. We must use a different approach to try to tackle this problem for $m > 2$.

# 5 Adjacency matrices and the permanent

**Definition 5.1.** A *graph* $G = (V, E)$ is a set of vertices $V$ and a set of edges $v_i, v_j \in E$.

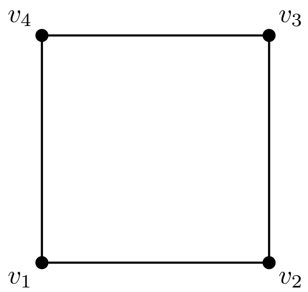


Figure 6: A graph

**Definition 5.2.** The *adjacency matrix* of a graph $G = (V, E)$ is a square matrix $A$ such that $a_{ij} = 1$ if there is an edge from $v_i$ to $v_j$, and $a_{ij} = 0$ otherwise.

The adjacency matrix of the graph in Figure 6 is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

**Definition 5.3.** A *bipartite graph* is a graph whose vertices can be partitioned into two disjoint sets such that each edge connects an element of the first set with an element of the second set.

The graph in Figure 6 is also a bipartite graph. The edges of the graph only go between the vertices of the sets $\{v_1, v_3\}$ and $\{v_2, v_4\}$. These two sets are placed on opposite ends of Figure 7 to show this:

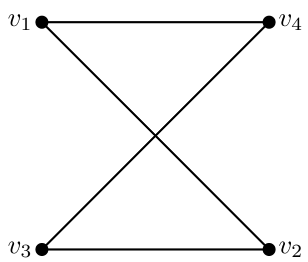


Figure 7: Graph redrawn to show that it is bipartite

**Definition 5.4.** A *biadjacency matrix* $B$ is a rectangular $r \times s$ matrix representing a bipartite graph whose vertices are partitioned into two disjoint sets $U$ of size $r$ and $V$ of size $s$. $b_{ij} = 1$ if there is an edge from $u_i$ to $v_j$, and $b_{ij} = 0$ otherwise.

The biadjacency matrix of the graph in Figure 7 is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

**Definition 5.5.** Given a graph $G = (V, E)$, a *matching* of $G$ is a subset of edges where no two edges share a common vertex.

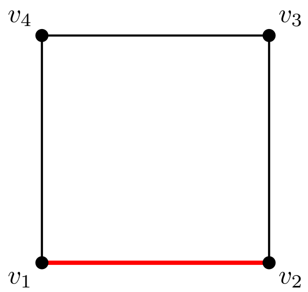


Figure 8: A matching of the graph which contains a single edge $(v_1, v_2)$

**Definition 5.6.** Given a graph $G = (V, E)$, a *perfect matching* of $G$ is a matching of $G$ where each vertex of $G$ is adjacent to an edge.

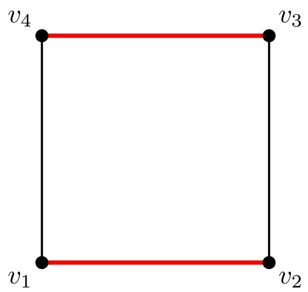


Figure 9: A perfect matching of the graph

**Definition 5.7.** The *permanent* of an $N \times N$ matrix $A$ with entries $a_{ij}$ is defined as

$$\text{per}(A) = \sum_{\sigma \in S_N} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{N\sigma(N)}$$

where $S_n$ is the set of all the permutations of $\{1, 2, ..., N\}$, and $\sigma$ is a permutation.

We can construct an undirected graph from a grid by considering each square as a vertex and drawing edges between vertices corresponding to adjacent squares on the grid. If we color the squares and vertices black and white like in a chessboard, we see that each domino is an edge between a white and black vertex. Then, a domino tiling corresponds to a perfect matching between white and black vertices, and vice versa. We shall use a $2 \times 3$ grid as an example:

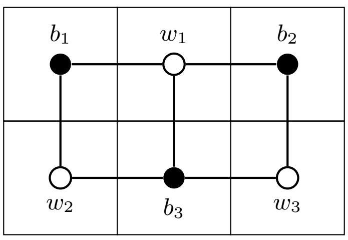


Figure 10: Example with a $2 \times 3$ grid

This is a bipartite graph since each edge connects a black vertex to a white vertex. From here, we can construct a biadjacency matrix for this graph, with rows for black vertices and columns for white vertices:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

A way for us to count the perfect matchings of this graph would be to take all of the permutations of the set $\{w_1, w_2, w_3\}$ and map them onto the black vertices and check if it is a valid perfect matching (no non-existent edges). For example, $(w_1, w_3, w_2)$ is a valid perfect matching, but $(w_1, w_2, w_3)$ is not since $b_2$ is not adjacent to $w_2$. Our biadjacency matrix can help us count these perfect matchings, as it encodes the edges of the graph. A permutation $\sigma$ is a valid perfect matching if and only if $a_{i\sigma(i)} = 1$ for all $i$, since if $a_{i\sigma(i)} = 0$ for some $i$ then $b_i$ and $w_{\sigma(i)}$ wouldn't be covered by a domino. Then, we have

$$T(2,3) = \sum_{\sigma \in S_3} a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$$

where $S_3$ is the set of all the permutations of $\{1, 2, 3\}$. This works because any invalid permutation would contain a 0 as established earlier, and thus would have negligible contribution.

We can extend this to all $m \times n$ grids where the total number of vertices $mn$ is even. Let $N = \frac{mn}{2}$ denote the number of dominoes required to tile the grid, which is equal to the number of black/white squares. The total number of tilings is given by:

$$T(m,n) = \sum_{\sigma \in S_N} a_{1\sigma(1)} a_{2\sigma(2)} ... a_{N\sigma(N)}$$

where $S_N$ is the set of all the permutations of $\{1, 2, ..., N\}$. This is exactly the same as the formula for the permanent! Thus, the number of domino tilings of an $m \times n$ grid is equal to the permanent of its biadjacency matrix.

Unfortunately, permanents for most matrices cannot be computed in a reasonable amount of time. In fact, finding the permanent is a #P problem, which means that it is one of the most difficult problems in terms of computational complexity. Computational complexity theory is a branch of computer science that classifies problems by the amount of resources (time and memory) required to solve them. #P problems are as difficult as counting the number of solutions to NP problems, which are already computationally difficult to solve. It is impossible for a computer or anything for that matter to compute the permanent of most matrices in a reasonable amount of time. Annoyingly, we must look for another approach to this problem.

# 6 Using the Pfaffian

**Definition 6.1.** A permutation $\sigma$ of squares in a grid is considered *canonical* if it satisfies the following conditions:

$$\sigma(p_1) < \sigma(p_2); \sigma(p_3) < \sigma(p_4); ...; \sigma(p_{mn-1}) < \sigma(p_{mn}),$$
$$\sigma(p_1) < \sigma(p_3) < ... < \sigma(p_{mn-1}).$$

The notion of canonical permutations is used to prevent overcounting. The first condition ensures that each domino is unique. We wouldn't say $(p_1, p_2)$ is different from $(p_2, p_1)$. (Note that these parenthesis denote tuples rather than points.) The second condition ensures that the ordering of the dominoes is unique.

**Definition 6.2.** For a square matrix $A$, if $a_{ij}$ denotes the entry in the $i$-th row and $j$-th column, $A$ is considered *skew-symmetric* if $a_{ij} = -a_{ji}$ for all entries in the matrix.

**Definition 6.3.** The *Pfaffian* of a $2N \times 2N$ skew-symmetric matrix $A$ with entries $a_{ij}$ is defined as

$$\mathrm{Pf}(A) = \sum_{\sigma \in \Pi} \mathrm{sgn}(\sigma) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_N j_N} = \sqrt{|\det(A)|}$$

where $\Pi$ is the set of all the ways to partition the set $\{1, 2, ..., 2N\}$ into $N$ pairs ordered such that $i_k < j_k$ and $i_1 < i_2 < ... < i_N$, $\sigma$ is a permutation of $\{1, 2, ..., 2N\}$ given by the sequence $\{i_1, j_1, i_2, j_2, ..., i_N, j_N\}$, and $\mathrm{sgn}(\sigma) = (-1)^M$ where $M$ is the number of inversions for $\sigma$. An inversion is a pair of indices $x, y$ such that $x < y$ and $\sigma(x) > \sigma(y)$.

Instead of constructing a biadjacency matrix, we can try using a regular adjacency matrix. Here is the adjacency matrix for our earlier example of a $2 \times 3$ grid:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

A way for us to count the perfect matchings of this graph would be to take all of the canonical partitions of the vertex set $\{p_1, p_2, p_3, p_4, p_5, p_6\}$ into disjoint pairs and determining which ones correspond to valid edge configurations. The adjacency matrix we constructed us earlier can greatly aid us in counting these partitions. A partition is valid if and only if each of the chosen pairs $\{p_i, p_j\}$ satisfies $a_{ij} = 1$, since if $a_{ij} = 0$, $p_i$ and $p_j$ wouldn't be adjacent and thus an edge wouldn't be possible. For example, $\{p_1, p_2\}, \{p_3, p_6\}, \{p_4, p_5\}$ is a valid partition, but $\{p_1, p_2\}, \{p_3, p_4\}, \{p_5, p_6\}$ is not a valid partition since $a_{34} = 0$.

Then, we have

$$T(2,3) = \sum_{\sigma \in \Pi} a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3}$$

where $\Pi = \{\{(\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_3\}\} \mid \{i_1, j_1, i_2, j_2, i_3, j_3\} = \{1, 2, 3, 4, 5, 6\},\ i_k < j_k,\ i_1 < i_2 < i_3\}$. This works because any invalid vertex partition includes pairs of squares that are not adjacent which will contain a 0 entry from the adjacency matrix, and thus won't contribute anything to the final sum. The restrictions placed on the ordering of the indices $i_k$ and $j_k$ ensure that only canonical partitions are considered.

We can extend this to all $m \times n$ grids where the total number of vertices $mn$ is even. Let $N = \frac{mn}{2}$ denote the number of dominoes required to tile the grid. The total number of tilings is given by:

$$T(m,n) = \sum_{\sigma \in \Pi} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_N j_N}$$

where $\Pi$ is the set of all canonical vertex matchings:

$$\Pi = \left\{ \{(i_1, j_1), \ldots, (i_N, j_N)\} \;\middle|\; \bigcup_{k=1}^{N} \{i_k, j_k\} = \{1, \ldots, 2N\},\ i_k < j_k,\ i_1 < \cdots < i_N \right\}.$$

This looks a lot like the formula for the Pfaffian:

$$\mathrm{Pf}(A) = \sum_{\sigma \in \Pi} \mathrm{sgn}(\sigma) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_N j_N}.$$

We can create a skew-symmetric adjacency matrix where $a_{ij} = 1 = -a_{ji}$ for $i < j$ that are adjacent without affecting the calculation due to the fact that $i_k < j_k$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

If it weren't for the sign, each valid configuration would contribute $+1$ to the sum, so we could directly find the number of domino tilings by computing the Pfaffian of the skew-symmetric adjacency matrix of the grid. However, since the sign is sometimes equal to $+1$ and other times $-1$, some configurations will contribute $+1$ while others will contribute $-1$ to the final sum and thus undercount the actual number of domino tilings. For example, $\mathrm{Pf}(A) = 1$, while there are actually 3 valid domino tilings of the $2 \times 3$ grid. Can we somehow make it so that each domino tiling will always contribute $+1$ to the sum?

## 7 Kasteleyn matrices

Perhaps we could change the signs of some of the elements in our skew-symmetric adjacency matrix to get a correct result. Let us use the $2 \times 3$ grid as an example again. Here is a modified version of our skew-symmetric matrix, with the signs of $a_{25}$ and $a_{52}$ changed:

$$K = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

This matrix gives us the correct answer! $\text{Pf}(K) = 3$. Let us try another example with the $2 \times 4$ grid. The skew-symmetric adjacency matrix of this grid is as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

$\text{Pf}(A) = 1$, while there are supposed to be 5 domino tilings. Here is a modified version of the skew-symmetric matrix:

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

$\text{Pf}(K) = 5$, which is correct. These types of matrices, where evaluating the Pfaffian doesn't lead to any sign issues, are called Kasteleyn matrices. Now that we have seen examples of Kasteleyn matrices, how do we go about finding them?

## 8 Finding Kasteleyn matrices

**Definition 8.1.** The *standard configuration* $C_0$ for an $m \times n$ grid, with $n$ even, is the domino tiling

$$\{\{p_1, p_2\}, \{p_3, p_4\}, \ldots \{p_{mn-1}, p_{mn}\}\}.$$

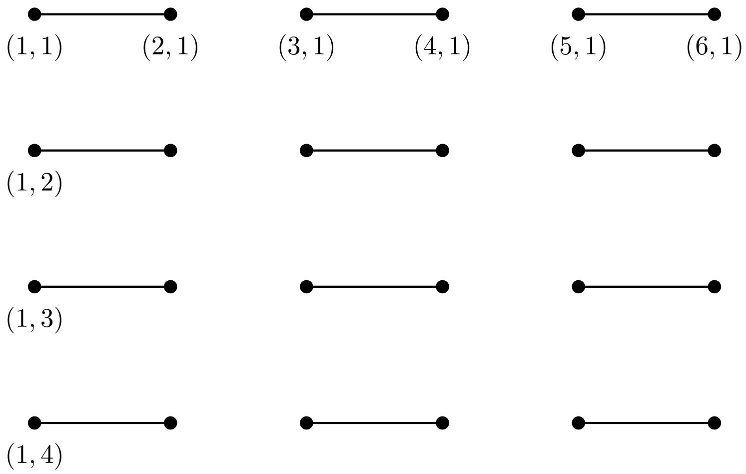


Figure 11: Graph of $C_0$ for a $4 \times 6$ grid

**Definition 8.2.** The *spanning tree* $T$ of a graph $G = (V, E)$ is a subgraph that is a tree which includes all the vertices of $G$.

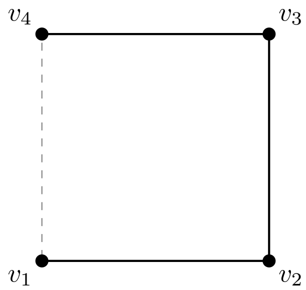


Figure 12: A spanning tree

**Definition 8.3.** A *planar graph* is a graph that can be drawn on a plane such that its edges only intersect at the graph's vertices.


(a) A planar graph

(b) A graph that isn't planar

We will show a method to construct a Kasteleyn matrix for any $m \times n$ grid, with $n$ even. From $C_0$, we can obtain any arbitrary configuration $C$ by doing the following. We can draw a diagram of the graph with every edge corresponding to a domino of $C_0$ drawn with a dotted line and every edge corresponding to a domino of $C$ drawn with a full line. Every vertex is at the endpoint of one line of each type, so the figure consists of pairs of vertices connected by both dotted and full lines and polygons consisting of vertices connected by alternating dotted and full lines. An example of a polygon is the square $WXYZ$ in Figure 14. If we then shift each of the polygons in $C_0$ clockwise or counterclockwise by one, $C_0$ becomes $C$. For example, the $C_0$ edges of square $WXYZ$ are $WX$ and $YZ$. When we shift the square clockwise by one, $WX$ goes to $ZW$ and $YZ$ goes to $XY$. The $C$ edges of the square are precisely $ZW$ and $XY$. We define $F$ as the number of internal square faces inside a polygon, $V_{\text{int}}$ as the number of vertices strictly inside a polygon, $E_{\text{int}}$ as the number of internal edges of a polygon, $E_{\text{boundary}}$ and $V_{\text{boundary}}$ as the number of edges and vertices on the boundary of a polygon, respectively, and $\omega(P)$ as the number of clockwise-oriented edges of polygon $P$.

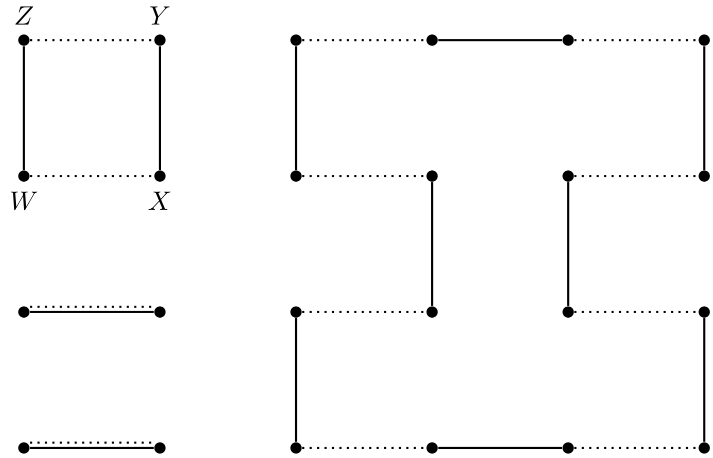


Figure 14: Construction of new configuration $C$ from $C_0$

**Theorem 8.1.** *(Kasteleyn, 1961) Every polygon contributes a factor of $-1$ to the permutation sign.*

*Proof.* $C \cup C_0$ forms a collection of disjoint polygons $\{P_1, P_2, \ldots, P_k\}$. We can evaluate the permutation that transforms $C_0$ to $C$ for a single isolated polygon $P$ of length $4r$. We will now calculate the contribution of each polygon to the permutation sign.

Within $C_0$, there are $2r$ edges belonging to $P$. When tracing $P$ in its cyclic orientation, some of these edges will be traced in their canonical orientation while others will be traced backward. Consider any column of edges, we will trace the edges on this column forward and backward the same number of times. Since this is true for any column, the polygon will contain $r$ forward steps and $r$ backward steps along $C_0$. This will be true for $C$ as well. To align these with the cyclic orientation, we must reverse the $r$ backward edges, which contributes a factor of $-1$ per reversed edge. In total, this contributes a factor of $(-1)^r$ to the sign of the permutation.

We can now rearrange the $2r$ edges of $C_0$ so that their endpoints match the sequence $(p_1, p_2, \ldots, p_{4r})$. Because this is a structural sorting of independent pairs into a continuous chain without changing internal edge orientations, it constitutes an even permutation of pairs, contributing a factor of 1 to the permutation sign.

We now need to shift each edge forward by 1. In configuration $C$, the endpoints match $(p_{4r}, p_1, p_2, \ldots, p_{4r-1})$. We can accomplish this by swapping the vertex $p_{4r}$ at the very end with its neighbors until it reaches the first vertex position. This requires a total of $4r - 1$ individual swaps. Each swap changes the permutation sign by an odd factor, contributing a factor of $(-1)^{4r-1}$ to the permutation sign.

We can now rearrange these $2r$ edges so that they are sorted into their original canonical matrix positions, which, as before, constitutes an even permutation and contributes a factor of 1. Finally, we reverse the $r$ uncanonical edges of $C$ to restore their original orientations, contributing a factor of $(-1)^r$ to the permutation sign.

In total, the permutation sign is determined by the product of these factors:

$$(-1)^r(1)(-1)^{4r-1}(1)(-1)^r = (-1)^{6r-1} = -1.$$

The resulting sign of the permutation is $-1$ for every polygon. □

Since different configurations have different numbers of these polygons, they have different signs, messing up the final sum. There is a way to change the signs of the graph so that each polygon contributes a factor of 1 to the parity instead of $-1$. Such a signing is called a Kasteleyn signing.

**Lemma 8.1.** *(Kasteleyn, 1961) For any planar graph $G$, there exists a valid edge orientation such that the parity of the number of clockwise-oriented edges of each internal face can be arbitrarily assigned.*

*Proof.* Let $T$ be a spanning tree of $G$ with its edges arbitrarily oriented. Then, add back the edges of $G$ to $T$ one at a time, enclosing exactly one face at a time. The orientation of that edge can then be assigned to satisfy the desired parity of clockwise-oriented edges for that face. By induction, all $F$ of the square faces can be signed to satisfy the requirement. □

**Lemma 8.2.** *(Kasteleyn, 1961) Let $P$ be a polygon on a planar grid graph $G$. If $\omega(f)$ denotes the number of clockwise-oriented edges of a face $f$, then:*

$$\omega(P) \equiv \left( \sum_{i=1}^{F} (\omega(f_i) + 1) \right) + V_{int} + 1 \ \textit{(mod 2)}.$$

*Proof.* We evaluate the total number of clockwise-oriented edges across all internal faces by summing $\omega(f_i)$ over all $F$ faces. When we sum these face by face, the internal edges are shared between two adjacent faces and are traversed in the clockwise direction exactly once. Therefore, internal edges will be counted once in the entire sum regardless of their orientation. Walking along the boundary edges is equivalent to walking along the polygon. Thus, these edges contribute $\omega(P)$ to the entire sum. Two equal quantities are by definition congruent. As a result,

$$\sum_{i=1}^{F} \omega(f_i) \equiv \omega(P) + E_{\text{int}} \pmod{2}.$$

Euler's formula tells us that $V - E + F = 2$, where $V, E, F$ are the number of vertices, edges, and faces of a planar graph respectively. Since we don't count the unbounded face outside of the grid as part of our polygon, this becomes $V - E + F = 1$. We can then substitute $V$ and $E$ with their internal and boundary components: $(V_{\text{int}} + V_{\text{boundary}}) - (E_{\text{int}} + E_{\text{boundary}}) + F = 1$. Since $P$ is a closed cycle, $V_{\text{boundary}} = E_{\text{boundary}}$. Thus,

$$V_{\text{int}} - E_{\text{int}} + F = 1 \implies E_{\text{int}} = V_{\text{int}} + F - 1.$$

We can substitute this into our previous sum:

$$\sum_{i=1}^{F} \omega(f_i) \equiv \omega(P) + V_{\text{int}} + F - 1 \pmod{2}.$$

We can then isolate $\omega(P)$:

$$\omega(P) \equiv \left(\sum_{i=1}^{F} \omega(f_i)\right) - V_{\text{int}} - F + 1 \pmod{2}.$$

Since addition is equivalent to subtraction modulo 2, we can group the face count $F$ into the summation and swap the sign of $V_{\text{int}}$:

$$\omega(P) \equiv \left(\sum_{i=1}^{F} (\omega(f_i) + 1)\right) + V_{\text{int}} + 1 \pmod{2}.$$

□

In a domino tiling, the total number of vertices $V$ and the number of boundary vertices $V_{\text{boundary}}$ are always even, and thus $V_{\text{int}}$ is also always even. Thus, we can have every single polygon have an odd number of clockwise edges by allowing each unit square have an odd number of clockwise edges by Lemma 8.2. But how does each polygon having an odd number of clockwise edges help us find a Kasteleyn matrix?

**Theorem 8.2.** *(Kasteleyn, 1961) If a polygon has an odd number of clockwise edges, then it will contribute a factor of 1 to the Pfaffian expansion of the configuration.*

*Proof.* Let $P$ be a polygon of $C$, where $M_e$ represents the matrix entries along the edges of the polygon. The total sign contribution of this polygon to the Pfaffian expansion is as follows:

$$\operatorname{sgn}(P) \prod_{e \in P} M_e.$$

By Theorem 8.1, $\operatorname{sgn}(P) = -1$. If we trace $P$ clockwise, the product of its edge weights will be

$$(1)^{\omega(P)}(-1)^{E_{\text{boundary}} - \omega(P)} = (-1)^{E_{\text{boundary}} - \omega(P)}.$$

Since $P$ is an alternating cycle of edges, $E_{\text{boundary}}$ is always even. If the polygon has an even number of clockwise edges, the product will reduce to 1 and the entire polygon will contribute $-1 \cdot 1 = -1$. However, if we have a polygon with an odd number of clockwise edges, then $(-1)^{E_{\text{boundary}} - \omega(P)} = -1$, and the entire polygon will contribute $-1 \cdot -1 = 1$ to the Pfaffian expansion. □

When we constructed our original skew-symmetric adjacency matrix, we used the rule $a_{ij} = 1 = -a_{ji}$ for $i < j$ if $p_i$ and $p_j$ were adjacent in our undirected graph. We can similarly construct a directed graph with the same edges as our undirected graph, but with the edges being directed canonically. Then, $a_{ij}$ would be equal to 1 if there was a directed edge from $p_i$ to $p_j$ and $-1$ if there was a directed edge from $j$ to $i$. Here is an example for our $2 \times 3$ grid:

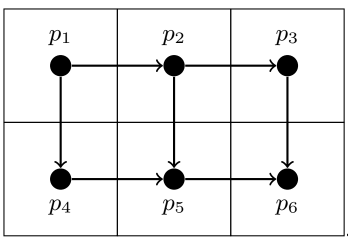


Figure 15: Example with a $2 \times 3$ grid

Each face has 2 clockwise edges. If we made it so every face had an odd number of clockwise edges, by Lemma 8.2 every polygon would have an odd number of clockwise edges, and then by Theorem 8.2 each polygon would contribute a factor of 1 to the Pfaffian expansion, which would get rid of our sign issues. We can accomplish this by turning every other vertical column of edges in the graph to face upwards instead of downwards, like so:

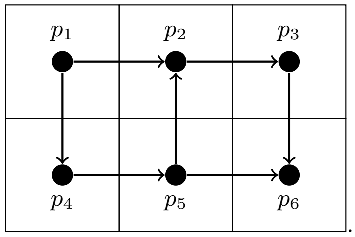


Figure 16: $2 \times 3$ grid with every other vertical column of edges reversed

Since all of the horizontal lines are oriented to the right, they contribute a total of 1 clockwise edge, and the alternating vertical lines will contribute 0 or 2 clockwise edges. Thus, each face will have an odd number of clockwise edges, and we have a Kasteleyn signing. A Kasteleyn matrix can then be found by taking the skew-symmetric adjacency matrix of this new graph. For example, the Kasteleyn matrix derived from the graph in Figure 16 is the same one seen in Section 7. We can generalize Kasteleyn matrices for all $m \times n$ grids.

**Theorem 8.3.** *(Kasteleyn, 1961) A Kasteleyn matrix $K$ of an $m \times n$ grid can be created from doing the following:*

$$k_{(i,j)(i+1,j)} = 1 \text{ for } 1 \leq i \leq m-1, 1 \leq j \leq n$$

$$k_{(i,j)(i,j+1)} = (-1)^{(i+1)} \text{ for } 1 \leq i \leq m, 1 \leq j \leq n-1$$

$$k_{ij} = -k_{ji} \text{ for } 1 \leq i \leq mn, 1 \leq j \leq mn$$

$$k_{(i,j)(i',j')} = 0 \text{ otherwise.}$$

*Proof.* Let an arbitrary square face $f$ be defined by its four vertices $(i,j), (i,j+1), (i+1,j), (i+1,j+1)$, as seen in Figure 17. We now evaluate the orientation of the four edges bounding $f$. $(i,j)$ to $(i+1,j)$ is rightwards since $K_{(i,j)(i+1,j)} = 1$, and since it is the top horizontal edge it is clockwise. $(i,j+1)$ to $(i+1,j+1)$ is similarly rightwards but since it is the bottom horizontal edge it is counterclockwise. The orientation of $(i,j)$ to $(i,j+1)$ is determined by $K_{(i,j)(i,j+1)} = (-1)^{i+1}$. If $i$ is odd, then this evaluates to 1, leading to a downwards edge. Since this is the left edge of the face, this would be counterclockwise. Similarly, if $i$ is even, this would be a clockwise edge. Similarly for $(i+1,j)$ to $(i+1,j+1)$, this would be a clockwise edge if $i$ is even and a counterclockwise edge if $i$ is odd. We can now find the total number of clockwise edges of $f$ by looking at the two cases for the parity of $i$. If $i$ is odd, the top edge is the only clockwise edge and $f$ would have 1 clockwise edge. If $i$ is even, the top, right, and left edges will all be clockwise, and $f$ would have 3 clockwise edges. In both cases, every square face $f$ would have an odd number of clockwise edges. By Lemma 8.2, every polygon would have an odd number of clockwise edges, and then by Theorem 8.2 each polygon will contribute a factor of 1 to the Pfaffian expansion. Thus, $K$ is a valid Kasteleyn matrix. $\square$

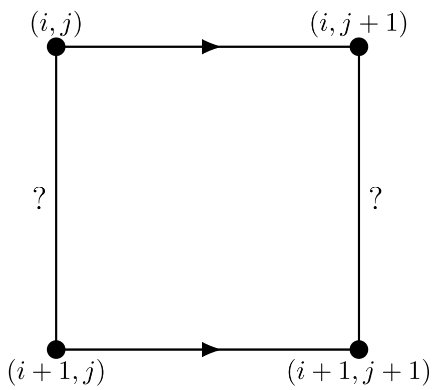


Figure 17: An arbitrary face $f$

Notice that the Kasteleyn matrices generated by Theorem 8.3 are not the only valid Kasteleyn matrices for any grid. We can just as easily create new Kasteleyn matrices by reversing vertical columns of edges starting from the first column instead of the second column or by reversing horizontal rows of edges instead of vertical columns. Also note that we can create Kasteleyn matrices using Theorem 8.3 for $m \times n$ grids where $mn$ is odd. The Pfaffian of these matrices reduce to 0, which is consistent with Theorem 3.1.

# 9 Evaluation of the Pfaffian

**Definition 9.1.** If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is a $p \times q$ matrix, then the *Kronecker product* $\mathbf{A} \otimes \mathbf{B}$ is the following $pm \times qn$ matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}.$$

If $m = n$ and $p = q$ and the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are $\lambda_i$ and $\mu_j$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., p$ respectively, then the eigenvalues of $\mathbf{A} \otimes \mathbf{B}$ are $\lambda_i \mu_j$. Another property is that for matrices $W, X, Y,$ and $Z$, $(W \otimes X)(Y \otimes Z) = (WY \otimes XZ)$.

**Definition 9.2.** The *Cartesian product* of two graphs $G$ and $H$ is an operation that combines them into a higher dimension graph $G \square H$. The vertex set of this new graph is the Cartesian product of the vertex sets of the two component graphs, and two vertices $(u, v)$ and $(u', v')$ are adjacent in the new graph if and only if either $u = u'$ and $v$ is adjacent to $v'$ or $v = v'$ and $u$ is adjacent to $u'$.

The adjacency matrix of $G \square H$, $A$, where $A_1$ and $A_2$ are the adjacency matrices of $G$ and $H$, where $I_k$ is an identity matrix of dimensions $k \times k$, and where $G$ and $H$ have $m$ and $n$ vertices respectively, can be found as so:

$$A = (A_1 \otimes I_n) + (I_m \otimes A_2).$$

If the eigenvalues of $A_1$ and $A_2$ are $\lambda_i$ and $\mu_j$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ respectively, then the eigenvalues of $A$ are $\lambda_i + \mu_j$.

Now that we have Kasteleyn matrices for our grids, we can find the number of domino tilings by calculating the Pfaffian of the matrix. Recalling Definition 6.3, the Pfaffian can be easily calculated by finding the absolute value of the determinant and taking the square root of it. However, finding the determinant of such large matrices is not straightforward. The determinant of a matrix can be calculated by finding the product of all of its eigenvalues. If $K$ were a periodic matrix, we could easily find the eigenvalues through a straightforward diagonalization. However, $K$ is not periodic due to the fact that the edges of the graph don't wrap around like a torus. Fortunately, it is still possible to find the eigenvalues in a relatively easy way.

**Definition 9.3.** A *path graph* $P_k$ is a graph such that all of its $k$ vertices and $k-1$ edges lie on a straight line, forming a single chain with no branches or cycles.

We can notice that the graph of a normal $m \times n$ grid without our changed signs can be written as a Cartesian product of two directed path graphs:

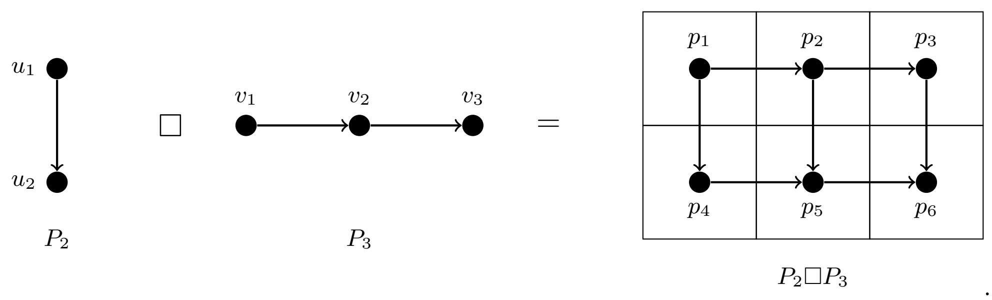


Figure 18: The Cartesian product $P_2 \square P_3$ generating the $2 \times 3$ grid graph.

The $k \times k$ skew-symmetric adjacency matrix $Q_k$ of a path graph with $k$ vertices with all the edges pointing the same direction looks like this:

$$
Q_k = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{bmatrix}.
$$

Such matrices with nonzero elements present only on the main diagonal, superdiagonal, and subdiagonal are also called tridiagonal matrices. Then, the adjacency matrix $A$ of our grid can be written as $(Q_n \otimes I_m) + (I_n \otimes Q_m)$. We cannot directly do this for our Kasteleyn matrices though, since every other row of vertical edges is reversed. Instead, we can write our Kasteleyn matrix as $K = (Q_n \otimes I_m) + (F_n \otimes Q_m)$, where $F_n$ is an $n \times n$ identity matrix with every other term along the diagonal negated:

$$
F_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (-1)^{n-1} \end{bmatrix}.
$$

The matrix $I_m$ is already a diagonal matrix, so all of its eigenvalues are equal to 1. Thus, the eigenvalues of $(Q_n \otimes I_m)$ will just be the eigenvalues of $Q_n$. The matrix $F_n$ is also a diagonal matrix, but with eigenvalues 1 and $-1$. The eigenvalues of $(F_n \otimes Q_m)$ will then be the eigenvalues of $Q_m$ and their negations. Now we will find the eigenvalues of $Q_m$ and $Q_n$.

**Theorem 9.1.** *(Rutherford, 1951) The eigenvalues of $Q_k$ are given by*

$$
\lambda_j = 2i \cos\left(\frac{\pi j}{k+1}\right) \text{ for } j = 1, 2, 3, \dots, k
$$

*where $i = \sqrt{-1}$.*

*Proof.* Let $\lambda$ be an eigenvalue of $Q_k$ and let $v$ be its corresponding eigenvector. The eigenvalue equation $Q_k v = \lambda v$ expands row by row to yield the following systems of equations:

$$
v_2 = \lambda v_1
$$

$$
-v_{j-1} + v_{j+1} = \lambda v_j \text{ for } 1 < j < k
$$

$$
-v_{k-1} = \lambda v_k.
$$

By defining auxilliary boundary conditions $v_0 = 0$ and $v_{k+1} = 0$, we get the following recurrence relation:

$$
v_{j+1} - \lambda v_j - v_{j-1} = 0 \text{ for } j = 1, 2, \dots, k.
$$

To solve this recurrence relation, we can substitute $v_j = r^j$ for some complex $r$ to simplify the recurrence into a quadratic. This yields the following:

$$
r^2 - \lambda r - 1 = 0.
$$

Let $r_1$ and $r_2$ be the roots of this equation. By Vieta's formulas, we have $r_1 + r_2 = \lambda$ and $r_1 r_2 = -1 \Rightarrow r_2 = -1/r_1$. The general solution for the components of the eigenvector for distinct complex constants $c_1, c_2$ is $v_j = c_1 r_1^j + c_2 r_2^j$. Since $v_0 = 0$, $c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$. Without loss of generality, we set $c_1 = 1$. Then, $v_j = r_1^j - r_2^j = r_1^j - (-1)^j r_1^{-j}$. Since $v_{k+1} = 0$, we have

$$
v_{k+1} = r_1^{k+1} - (-1)^{k+1} r_1^{-(k+1)} = 0 \implies r_1^{2k+2} = (-1)^{k+1}.
$$

We can express the right side using Euler's identity ($e^{i\pi} = -1$): $r_1^{2k+2} = e^{i\pi(k+1)+i2\pi j} = e^{i\pi(k+1+2j)}$ for $j \in \mathbb{Z}$. Then, we take the $2k+2$-th root: $r_1 = e^{\frac{i\pi(k+1+2j)}{2k+2}} = e^{i(\frac{\pi}{2}+\frac{\pi j}{k+1})} = ie^{\frac{i\pi j}{k+1}}$. Since $r_2 = -1/r_1$, we have $r_2 = ie^{-\frac{i\pi j}{k+1}}$. We can find the eigenvalue $\lambda$ by summing $r_1$ and $r_2$ and applying the identity $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$: $\lambda_j = 2i\cos(\frac{\pi j}{k+1})$. Choosing $j = 1, 2, ..., k$ will give us the $k$ distinct eigenvalues of $Q_k$. Thus, the eigenvalues of $Q_k$ are

$$\lambda_j = 2i\cos\left(\frac{\pi j}{k+1}\right) \text{ for } j = 1, 2, 3, \ldots, k.$$

□

With this result, the eigenvalues of $(Q_n \otimes I_m)$ can be easily found as $2i\cos\left(\frac{\pi j}{n+1}\right)$. Since the eigenvalues of $F_n$ are $\pm 1$, the eigenvalues of $F_n \otimes Q_m$ are $\pm 2i\cos\left(\frac{\pi k}{m+1}\right)$. Unfortunately, we cannot just add the eigenvalues of $(Q_n \otimes I_m)$ and $(F_n \otimes Q_m)$ together, as they do not share an eigenbasis due to the alternating signs of $F_n$. Instead, we must utilize another method.

**Theorem 9.2.** *The eigenvalues of $K$ are*

$$\pm 2i\sqrt{\cos^2\left(\frac{\pi j}{m+1}\right) + \cos^2\left(\frac{\pi k}{n+1}\right)} \text{ for } j = 1, 2, ..., m, k = 1, 2, ..., n.$$

*Proof.* If we square $K$, we get $K^2 = (Q_n^2 \otimes I_m) + (I_n \otimes Q_m^2)$. The alternating terms in $F_n$ go away because $(-1)^2 = 1$. The cross terms vanish since $Q_m$ and $F_m$ anticommute ($Q_m F_m = -F_m Q_m$). Let $v_k$ be an eigenvector of $Q_n$ with eigenvalue $\lambda_k$, and $w_j$ be an eigenvector of $Q_m$ with eigenvalue $\mu_j$. Thus, $Q_n^2 v_k = \lambda_k^2 v_k$ and $Q_m^2 w_j = \mu_j^2 w_j$. Then, we can create combined eigenvectors for the entire matrix $K^2$ using the Kronecker product $u_{kj} = v_k \otimes w_j$. We can then multiply $K^2$ by these new eigenvectors:

$$K^2(v_k \otimes w_j) = (Q_n^2 \otimes I_m)(v_k \otimes w_j) + (I_n \otimes Q_m^2)(v_k \otimes w_j)$$

$$K^2(v_k \otimes w_j) = (Q_n^2 v_k \otimes I_m w_j) + (I_n v_k \otimes Q_m^2 w_j).$$

We can substitute our eigenvalue equations from earlier and utilize the fact that $I_n$ and $I_m$ are identity matrices:

$$K^2(v_k \otimes w_j) = (\lambda_k^2 v_k \otimes w_j) + (v_k \otimes \mu_j^2 w_j) = \lambda_k^2(v_k \otimes w_j) + \mu_j^2(v_k \otimes w_j)$$

$$K^2(v_k \otimes w_j) = (\lambda_k^2 + \mu_j^2)(v_k \otimes w_j).$$

Thus, the corresponding eigenvalue for $K^2$ is $\lambda_k^2 + \mu_j^2$. Then, the corresponding eigenvalue for $K$ is $\pm\sqrt{\lambda_k^2 + \mu_j^2}$. We know by Theorem 9.1 that the eigenvalues of $Q_n$ are $\lambda_k = 2i\cos\left(\frac{\pi k}{n+1}\right)$ for $k = 1, 2, ..., n$ and the eigenvalues of $Q_m$ are $\mu_j = 2i\cos\left(\frac{\pi j}{m+1}\right)$ for $j = 1, 2, ..., m$. The eigenvalues of $K$ can now be calculated:

$$\pm 2i\sqrt{\cos^2\left(\frac{\pi j}{m+1}\right) + \cos^2\left(\frac{\pi k}{n+1}\right)} \text{ for } j = 1, 2, ..., m, k = 1, 2, ..., n.$$

□

Then, we may think that the determinant of $K$ can be found using the eigenvalues from Theorem 9.2:

$$\det(K) = \prod_{j=1}^{m}\prod_{k=1}^{n}\left(2i\sqrt{\cos^2\left(\frac{\pi j}{m+1}\right) + \cos^2\left(\frac{\pi k}{n+1}\right)}\right) \cdot \left(-2i\sqrt{\cos^2\left(\frac{\pi j}{m+1}\right) + \cos^2\left(\frac{\pi k}{n+1}\right)}\right).$$

However, the $\pm$ term causes each eigenvalue to appear twice. We really only need one of the terms:

$$\det(K) = \left|\prod_{j=1}^{m}\prod_{k=1}^{n} 2i\sqrt{\cos^2\left(\frac{\pi j}{m+1}\right) + \cos^2\left(\frac{\pi k}{n+1}\right)}\right|.$$

Since $\cos(\theta) = -\cos(\pi - \theta)$, the contributing term for $(j, k), (m+1-j, k), (j, n+1-k),$ and $(m+1-j, n+1-k)$ will be the about the same, with half of them containing an extra factor of $-1$. Note that this isn't true if $j$ or $k$ is odd, since then points would overlap, but we don't care about that since then the cosines would become 0 and wouldn't affect our calculation. We also know that the Pfaffian is the square root of the absolute value of the determinant. Thus, we can group them up and take the root so we don't need to painfully expand a fourth power:

$$\text{Pf}(K) = \sqrt{|\det(K)|} = \sqrt{\prod_{j=1}^{\lceil \frac{m}{2} \rceil}\prod_{k=1}^{\lceil \frac{n}{2} \rceil}\left(2i\sqrt{\cos^2\left(\frac{\pi j}{m+1}\right) + \cos^2\left(\frac{\pi k}{n+1}\right)}\right)^4}$$

$$\mathrm{Pf}(K) = \sqrt{\prod_{j=1}^{\lceil \frac{m}{2} \rceil} \prod_{k=1}^{\lceil \frac{n}{2} \rceil} 16 \left( \sqrt{\cos^2\left(\frac{\pi j}{m+1}\right) + \cos^2\left(\frac{\pi k}{n+1}\right)} \right)^4}$$

$$\mathrm{Pf}(K) = \prod_{j=1}^{\lceil \frac{m}{2} \rceil} \prod_{k=1}^{\lceil \frac{n}{2} \rceil} 4 \left( \sqrt{\cos^2\left(\frac{\pi j}{m+1}\right) + \cos^2\left(\frac{\pi k}{n+1}\right)} \right)^2$$

$$\mathrm{Pf}(K) = \prod_{j=1}^{\lceil \frac{m}{2} \rceil} \prod_{k=1}^{\lceil \frac{n}{2} \rceil} \left( 4\cos^2\left(\frac{\pi j}{m+1}\right) + 4\cos^2\left(\frac{\pi k}{n+1}\right) \right).$$

We finally have a formula for the Pfaffian of a Kasteleyn matrix. Since the Pfaffian is equal to the domino tilings, we have

$$\boxed{T(m,n) = \prod_{j=1}^{\lceil \frac{m}{2} \rceil} \prod_{k=1}^{\lceil \frac{n}{2} \rceil} \left( 4\cos^2\left(\frac{\pi j}{m+1}\right) + 4\cos^2\left(\frac{\pi k}{n+1}\right) \right)}.$$

## 10 Conclusion and Acknowledgements

Our formula has some interesting properties. It correctly reduces to 0 domino tilings when $mn$ is odd, which is in accordance with Theorem 3.1. Another property is that $2^k | T(2k, 2k)$. For some more properties, check out sequence A004003 on OEIS.

There are also other ways to count the number of domino tilings of $m \times n$ grids. Instead of changing the signs of the skew-symmetric matrix to calculate the Pfaffian using the determinant, one can instead change the signs of the biadjacency matrix to calculate the permanent using the determinant. There are also many alternative methods to find the determinant of the Kasteleyn matrix, such as changing the sign of every other column of $Q_k$ instead of $I_k$, which allows for the two terms of $K$ to share an eigenbasis. Counting the number of domino tilings of grids with small $m$ is also a common problem in computer science and is a good example of dynamic programming. Relating to this, grids with small $m$ may also be solved with recursion, beyond just $m = 2$ as we discussed in Theorem 4.2. For example, $T(3,0) = 1, T(3,2) = 3,$ and $T(3,n) = 4T(3,n-2) - T(3,n-4)$. However, it is nowhere as simple to find as it was with $m = 1, 2$.

There are also many variations on the tiling of a rectangular grid. For example, a *tatami* is a domino that can only be placed in a grid such that exactly 3 tatami meet at any corner. Some grids require use of $1 \times 1$ tiles in order to be tiled with tatami. There are also studies of the macroscopic behavior of the tilings of infinitely large rectangular grids, which also has applications in physics. One may also count the domino tilings of tori and other spaces.

There are also many other interesting problems related to domino tiling. One particularly interesting problem is tiling the Aztec diamond:

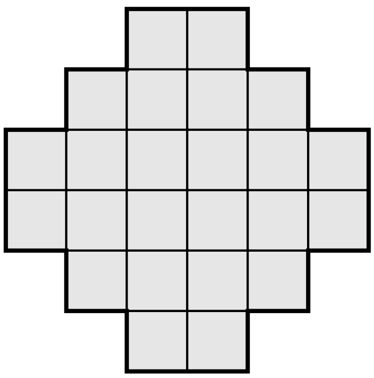

Figure 19: An Aztec diamond of order 3

The number of ways to tile an Aztec diamond of order $n$ is $2^{\frac{n(n+1)}{2}}$. There are around 12 proofs of this now, none of which are as simple as the result may suggest. [1]

An interesting theorem relating to the Aztec diamond is the Arctic Circle Theorem, which states that a random tiling of an Aztec diamond tends to be “frozen” outside of a certain circle. Here is an example:

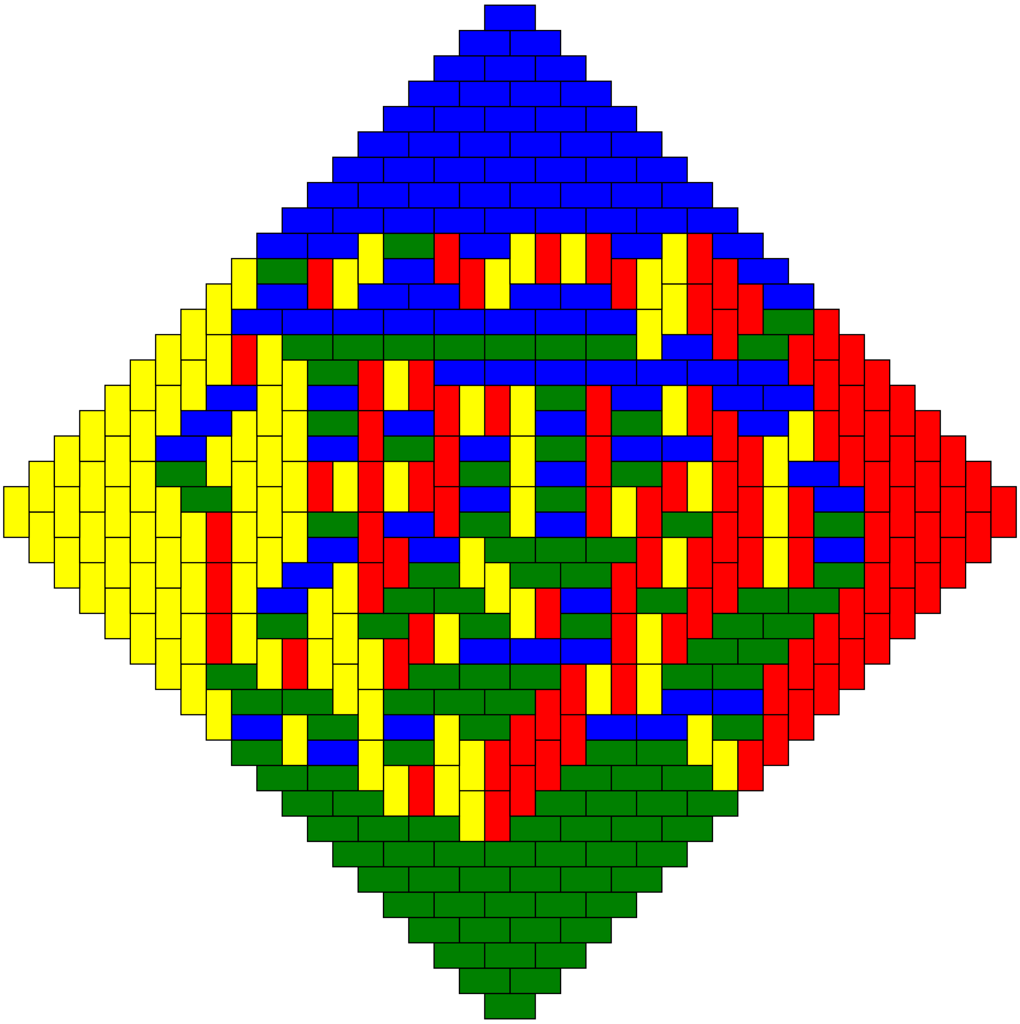

Figure 20: A random tiling of an Aztec diamond of order 20
[2]

There are plenty of variations of Aztec diamond to study, such as those with 1 or 3 center rows.

Besides dominoes, there are many other geometrical shapes that can be used in tilings. One may try using rectangles or other polygons to try to tile a grid. A popular form of this is to use polyominoes.

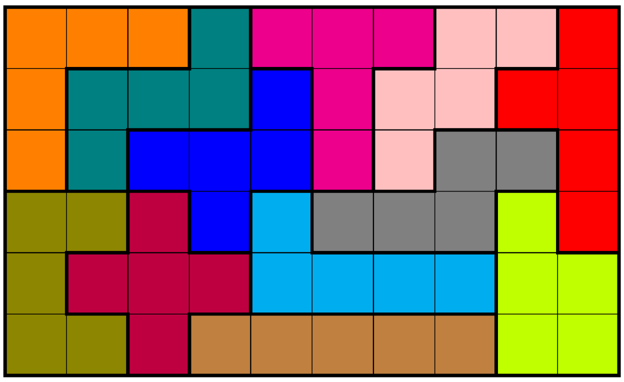

Figure 21: A $6 \times 10$ rectangle tiled with unique pentominoes

One can also consider tilings on non-square grids, such as say equilateral triangles or hexagons. One could even try tiling in higher dimensions, say with $1 \times 1 \times 2$ dominoes. The realm of tiling has innumerable possibilities.

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[1] Federico Ardila and Richard P Stanley. Tilings. *The Mathematical Intelligencer*, 32(4):32–43, 2010.

[2] Leonid Petrov. Domino tilings of the Aztec diamond, 2025.