Stokes' Theorem & de Rham Cohomology

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July 2026

A Familiar Pattern

Throughout calculus we encounter theorems of the form:

$$\int_{\text{region}} (\text{some function}) = \int_{\text{boundary of region}} (\text{another function}).$$

Much like the Fundamental Theorem of Calculus in $\mathbb{R}$:

Green's Theorem: $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{\partial D} P\, dx + Q\, dy,$$

Classical Stokes' Theorem: $$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r},$$

Divergence Theorem: $$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n}\, dS.$$

Question:

$\boxed{\text{Can all of these theorems be unified?}}$

The Goal

To answer this question we need:

1. Spaces on which calculus can be performed
2. Objects that can be integrated
3. A generalized notion of differentiation

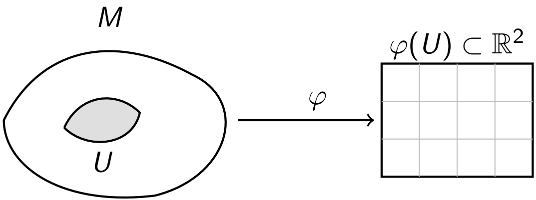
We will cover:

- Manifolds
- Tangent Vectors and Covectors
- Differential Forms
- Closed and Exact Forms

All allowing us to reach de Rham Cohomology and Stokes' Theorem.

Manifolds

A manifold is a space that locally looks like Euclidean space.



Examples:

- Curves
- Surfaces
- Spheres
- Torus

Although globally curved, manifolds are locally flat enough for calculus.

Tangent Vectors and Covectors

At each point $p \in M$, the tangent space of manifold M at p is denoted

$$T_pM.$$

Elements of T_pM are tangent vectors, which represent directions through the point p.

The *dual space* of T_pM is

$$T_p^*M = \{\omega : T_pM \to \mathbb{R} \mid \omega \text{ is linear}\}.$$

Elements of T_p^*M are called *covectors* and take a vector as input, outputting a real number:

$$\omega : T_pM \to \mathbb{R}.$$

If $\omega = \begin{bmatrix} 3 & 4 \end{bmatrix}$ and $v = \begin{bmatrix} x \\ y \end{bmatrix}$, then:

$$\omega(v) = 3x + 4y.$$

Coordinate Covectors

Under the definition A tangent vector $v = \langle P, Q \rangle$ can be written as

$$v = P\,\frac{\partial}{\partial x} + Q\,\frac{\partial}{\partial y},$$

whereas a covector $\omega = \begin{bmatrix} M & N \end{bmatrix}$ can be written as

$$\omega = M\,dx + N\,dy.$$

Coordinate covectors satisfy

$$dx\left(\frac{\partial}{\partial x}\right) = 1, \qquad dx\left(\frac{\partial}{\partial y}\right) = 0,$$

$$dy\left(\frac{\partial}{\partial x}\right) = 0, \qquad dy\left(\frac{\partial}{\partial y}\right) = 1,$$

So once again,

$$\omega(v) = [M, N]\begin{bmatrix} P \\ Q \end{bmatrix} = MP + NQ.$$

The Wedge Product

To build higher-dimensional objects from covectors, we introduce the wedge product:

$$\alpha \wedge \beta.$$

Combining two covectors produces a 2-form.

$$dx \wedge dy$$

represents an oriented area element, while

$$dx \wedge dy \wedge dz$$

represents an oriented volume element.

The wedge product is antisymmetric:

$$dx \wedge dy = -dy \wedge dx.$$

Therefore orientation matters:

$$\alpha \wedge \alpha = 0.$$

Differential Forms

Differential forms generalize covectors.

A k-form is an object that can be integrated over a k-dimensional domain. Examples:

- 0-form: $x^2 \sin(yz)$
- 1-form: $2\,dx + x\,dy + |z|\,dz$
- 2-form: $\sin(z)\,dx \wedge dy$
- 3-form: $xyz\,dx \wedge dy \wedge dz$

We denote the collection of all smooth differential k-forms on M by

$$\Omega^k(M).$$

The Exterior Derivative

The exterior derivative is a map

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

that generalizes differentiation to differential forms.

Examples:

- If $f = x^2 y$, then
$$df = 2xy\,dx + x^2\,dy.$$
- If
$$\omega = P\,dx + Q\,dy,$$
then
$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

Applying the exterior derivative *twice* always gives zero:

$$d(d\omega) = 0.$$

Integrating Differential Forms

A differential k-form can be integrated over a k-dimensional oriented manifold.

$$\int_C \omega$$

for a 1-form along a curve.

$$\int_S \eta$$

for a 2-form over a surface.

$$\int_V \mu$$

for a 3-form over a volume.
Forms generalize the objects integrated in calculus.

Closed and Exact Forms

Given some k-form ω, we call it closed if $d\omega = 0$.

And exact if for some $(k-1)$-form η, we have $\omega = d\eta$.

Since $d^2 = 0$, every exact form is closed.

Question:

$\boxed{\text{When is every closed form exact?}}$

de Rham Cohomology

Closed forms satisfy $d\omega = 0$, and exact forms satisfy $\omega = d\eta$. Since every exact form is closed, we form the quotient

$$H^k_{dR}(M) = \frac{\ker d}{\operatorname{im} d}$$

where the left side represents the k-th de Rham cohomology group of the manifold M.

de Rham cohomology measures closed forms that are not exact.

Why Cohomology Matters

Consider the punctured plane

$$\mathbb{R}^2 \setminus \{(0,0)\}.$$

A hole exists at the origin.

The 1-form

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

satisfies $d\omega = 0$, so ω is closed.

However, ω is not exact, i.e., there exists no smooth function f such that $df = \omega$.

The obstruction is precisely the hole at the origin.

$$\boxed{\text{Topology creates closed forms that are not exact.}}$$

Generalized Stokes Theorem

For a compact oriented manifold M,

$$\boxed{\int_M d\omega = \int_{\partial M} \omega.}$$

This single theorem contains:

- Fundamental Theorem of Calculus
- Green's Theorem
- Classical Stokes' Theorem
- Divergence Theorem

Stokes' Proof: Pt. 1 – Local

Let $\Omega = [a, b] \times [c, d]$ and $\omega = f(x, y)\, dx$. Then

$$d\omega = -\frac{\partial f}{\partial y}\, dx\, dy.$$

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Applying the Fundamental Theorem of Calculus,

$$\int_\Omega d\omega = \int_a^b f(x, c)\,dx - \int_a^b f(x, d)\,dx.$$

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Applying the Fundamental Theorem of Calculus,

$$\int_\Omega d\omega = \int_a^b f(x, c)\, dx - \int_a^b f(x, d)\, dx.$$

The first term is the integral along the bottom edge, and the second is the integral along the top edge (with boundary orientation), so

$$\int_\Omega d\omega = \int_{\text{bottom}} \omega + \int_{\text{top}} \omega = \int_{\partial\Omega} \omega.$$

The vertical edges contribute 0 since $dx = 0$ there.

Stokes' Proof: Pt. 2 – Global

We scale up from local boxes to a global (curved) manifold Ω using a **Partition of Unity** ($\sum \rho_i = 1$):

$$\omega = \left(\sum_i \rho_i\right)\omega = \sum_i (\rho_i \omega)$$

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Each isolated piece ($\rho_i \omega$) is neatly trapped inside a flat local coordinate chart. By linearity of derivatives and integrals:

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Apply the local box proof in pt. 1 to every individual chart piece:

$$\sum_i \int_\Omega d(\rho_i \omega) = \sum_i \int_{\partial\Omega} \rho_i \omega = \int_{\partial\Omega} \left(\sum_i \rho_i \omega\right) = \int_{\partial\Omega} \omega \quad \blacksquare$$

Conclusion

1. Differential forms generalize integration.
2. The exterior derivative generalizes differentiation.
3. Closed and exact forms lead to cohomology.
4. Cohomology detects topology.
5. Stokes unifies classical calculus.

Thank you.