

STOKES' THEOREM AND DE RHAM COHOMOLOGY THROUGH THE LANGUAGE OF FORMS

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ABSTRACT. The Fundamental Theorem of Calculus, Green's theorem, the classical Stokes theorem, and the divergence theorem each relate differentiation on a region to integration on its boundary. This paper develops the language of smooth manifolds and differential forms in which these results become instances of a single statement. After introducing tangent and cotangent spaces, alternating forms, the wedge product, and the exterior derivative, we define integration of top-degree forms on oriented manifolds. We then state the generalized Stokes theorem and give a proof sketch assembled globally by a partition of unity. Finally, we introduce closed and exact forms and de Rham cohomology. The standard angular form on the punctured plane gives a concrete example of a closed form that is not exact, showing how cohomology detects a topological hole.

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1. INTRODUCTION

Calculus contains a recurring pattern:

$$\int_{\text{region}} (\text{derivative}) = \int_{\text{boundary of the region}} (\text{original object}).$$

The Fundamental Theorem of Calculus says

$$\int_a^b f'(x) dx = f(b) - f(a).$$

The right-hand side can be regarded as an integral over the oriented boundary of the interval: the endpoint b has sign $+1$ and the endpoint a has sign -1 . In higher-dimensional calculus the same pattern appears in the forms

$$\begin{aligned} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \oint_{\partial D} P dx + Q dy && \text{(Green's theorem),} \\ \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} && \text{(classical Stokes),} \\ \iiint_V (\nabla \cdot \mathbf{F}) dV &= \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS && \text{(divergence theorem).} \end{aligned}$$

Each theorem differentiates something on a region and integrates the result; the original object is then integrated on the boundary. The purpose of this paper is to explain the common structure behind these theorems. The main result is the following. The discussion is aimed at readers who know multivariable calculus and are meeting manifolds and differential forms for the first time; the emphasis is therefore on geometric meaning and the calculations that recover the familiar theorems.

Theorem 1.1 (Generalized Stokes theorem). *Let M be a compact oriented smooth n -manifold with boundary, and let $\omega \in \Omega^{n-1}(M)$. Then*

$$\int_M d\omega = \int_{\partial M} \omega,$$

where ∂M has the boundary orientation.

The notation in [Theorem 1.1](#) will be developed in the next sections. The route mirrors the familiar calculus story: manifolds supply domains for calculus, differential forms supply integrands, and the exterior derivative supplies a coordinate-independent derivative. In particular, the formal objects needed to unify the four displayed theorems are: spaces on which calculus can be performed, objects that can be integrated over those spaces, and a generalized differentiation operation.

2. SMOOTH MANIFOLDS

2.1. Topological background. Before introducing manifolds, we recall just the topological language needed for their definition. It makes precise what it means for a space to be locally Euclidean.

Definition 2.1. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X , called *open sets*, such that:

- (i) $\emptyset$ and X belong to $\mathcal{T}$;
- (ii) the union of any collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$;
- (iii) the intersection of finitely many sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is a *topological space*. An open set U containing p is called an *open neighborhood* of p .

In $\mathbb{R}^n$, the usual topology is generated by open balls. A manifold uses the same notion of local neighborhood, but it need not be a global subset of a single $\mathbb{R}^n$.

Definition 2.2. A topological space X is *Hausdorff* if, for every two distinct points $p, q \in X$, there are disjoint open sets U and V with $p \in U$ and $q \in V$.

The Hausdorff condition ensures that distinct points can be separated by neighborhoods, as they can in Euclidean space.

Definition 2.3. A collection $\mathcal{B}$ of open subsets of X is a *basis* for the topology if every open set is a union of members of $\mathcal{B}$. The space is *second countable* if it has a countable basis.

For example, open balls with rational centers and radii form a countable basis of $\mathbb{R}^n$. This mild condition holds for the usual geometric examples and supports the global constructions used later, such as partitions of unity.

Definition 2.4. A map $f : X \rightarrow Y$ between topological spaces is *continuous* if $f^{-1}(U)$ is open in X whenever U is open in Y . A bijection $f : X \rightarrow Y$ is a *homeomorphism* if both f and f^{-1} are continuous.

A homeomorphism may bend or stretch a space, but cannot tear it apart or glue points together. Thus local homeomorphism to $\mathbb{R}^n$ is the rigorous meaning of “locally Euclidean.”

2.2. Charts, atlases, and smoothness.

Definition 2.5. An *n-dimensional topological manifold* is a Hausdorff, second-countable topological space M such that every $p \in M$ has an open neighborhood U and a homeomorphism $\varphi : U \rightarrow V$, where $V \subseteq \mathbb{R}^n$ is open. The pair (U, φ) is a *chart*.

The set $U \subset M$ is called a *coordinate patch*; its image $V = \varphi(U) \subseteq \mathbb{R}^n$ is the *coordinate domain*. A chart is therefore a local translation between the manifold and a familiar Euclidean region. The coordinates $x^1, \dots, x^n$ on V can be used to do calculus on U . One chart usually cannot describe the entire manifold: for example, the sphere has no single chart covering all of S^2 , but every point of the sphere has a neighborhood that can be flattened into a planar coordinate domain.

Definition 2.6. An *atlas* for a manifold M is a collection of charts $\{(U_i, \varphi_i)\}_{i \in I}$ whose patches cover M . If two patches overlap, the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

is the *transition map* from the i th coordinates to the j th coordinates.

A topological manifold is *smooth* when it has an atlas whose transition maps are C^∞ . Smooth transition maps ensure that a derivative computed in one chart agrees with the derivative computed in another. Thus a manifold may be curved globally while looking like Euclidean space in every sufficiently small chart, with all local calculations fitting together consistently.

The familiar examples illustrate the local-versus-global distinction. A smooth curve is a one-dimensional manifold, and a smooth surface is a two-dimensional manifold. The sphere S^2 and a torus are both locally modelled on open subsets of $\mathbb{R}^2$, even though neither is globally a single copy of the plane. The charts and their transition maps provide the local coordinates needed to perform calculus without pretending that the entire space is flat.

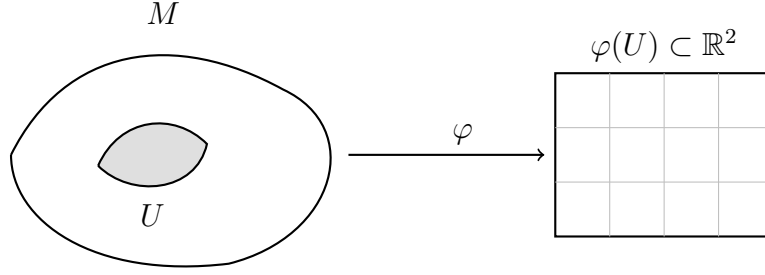


FIGURE 1. A chart identifies a local patch of a manifold with an open Euclidean set.

Manifolds with boundary are defined similarly, except that local charts are allowed to map into the half-space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$. Points mapped to $x_n = 0$ form ∂M . This is the setting required by Stokes' theorem.

3. TANGENT VECTORS AND COVECTORS

Let $p \in M$. A tangent vector at p can be represented by the velocity $\gamma'(0)$ of a smooth curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ with $\gamma(0) = p$, where two curves represent the same vector if their coordinate velocities agree in one, hence every, chart. The resulting vector space is the *tangent space* $T_p M$. In coordinates $(x^1, \dots, x^n)$, it has basis

$$\left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p.$$

Thus, in a two-dimensional coordinate patch, a tangent vector has the form

$$v = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}.$$

Definition 3.1. The *cotangent space* at p is the dual vector space

$$T_p^* M = \{\omega : T_p M \rightarrow \mathbb{R} : \omega \text{ is linear}\}.$$

Its elements are called *covectors*.

The dual basis is written $dx^1, \dots, dx^n$ and satisfies

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta_j^i.$$

For instance, in $\mathbb{R}^2$, if $v = P \partial_x + Q \partial_y$ and $\omega = M dx + N dy$, then

$$\omega(v) = MP + NQ.$$

This distinction matters: vectors give directions, while covectors measure vectors and return numbers. In particular,

$$dx(\partial_x) = 1, \quad dx(\partial_y) = 0, \quad dy(\partial_x) = 0, \quad dy(\partial_y) = 1.$$

These identities show that dx and dy extract the x - and y -components of a tangent vector. Consequently the coordinate expression $M dx + N dy$ is not merely suggestive notation: it is exactly the covector whose value on $P \partial_x + Q \partial_y$ is $MP + NQ$.

4. DIFFERENTIAL FORMS AND THE EXTERIOR DERIVATIVE

Definition 4.1. A k -form at p is an alternating k -linear map $(T_p M)^k \rightarrow \mathbb{R}$. A smooth assignment of a k -form to each $p \in M$ is a *differential k -form*; the vector space of such forms is denoted $\Omega^k(M)$.

The wedge product combines an r -form and an s -form into an $(r + s)$ -form. It is bilinear and graded-commutative:

$$\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha.$$

In particular, $dx \wedge dy = -dy \wedge dx$ and $\alpha \wedge \alpha = 0$ for a one-form α . The form $dx \wedge dy$ represents an oriented area element; $dx \wedge dy \wedge dz$ represents an oriented volume element.

Example 4.2 (Forms of different degrees). On a coordinate region in $\mathbb{R}^3$, the following are respectively a 0-, 1-, 2-, and 3-form:

$$x^2 \sin(yz), \quad 2dx + xdy + z^2 dz, \quad \sin(z) dx \wedge dy, \quad xyz dx \wedge dy \wedge dz.$$

A k -form is the type of object that can be integrated over an oriented k -dimensional domain. This degree matching is the differential-form version of choosing dx , dA , or dV in elementary calculus.

Every k -form on a coordinate neighborhood has the form

$$\omega = \sum_{i_1 < \dots < i_k} f_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

The *exterior derivative* is the linear map

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

determined in coordinates by

$$d\omega = \sum_{i_1 < \dots < i_k} df_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

It obeys the graded product rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta \quad \text{and} \quad d^2 = 0.$$

For $f = x^2y$, $df = 2xy dx + x^2 dy$; and if $\omega = P dx + Q dy$, then

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

The second formula is the two-dimensional curl expression occurring in Green's theorem. The identity $d^2 = 0$ will also be the reason that every exact form is closed in [Section 8](#).

5. INTEGRATION AND ORIENTATION

An orientation selects which coordinate bases are positive. On an oriented n -manifold, an n -form can be integrated by transporting it to positively oriented coordinate charts, integrating there in the usual multivariable sense, and combining the pieces with a partition of unity. This construction is independent of the chosen oriented atlas. More generally, a k -form is integrated over an oriented k -dimensional manifold or submanifold.

For example, if C is an oriented curve and ω is a 1-form, then $\int_C \omega$ is a line integral. If S is an oriented surface and η is a 2-form, then $\int_S \eta$ is a surface integral. On an oriented

three-dimensional region V , a 3-form μ can be integrated as $\int_V \mu$. These examples explain why the degree of a form must agree with the dimension of its domain of integration.

If M has boundary, its orientation is defined by the “outward normal first” rule: $(v_1, \dots, v_{n-1})$ is positive in $T_p \partial M$ when $(\nu, v_1, \dots, v_{n-1})$ is positive in $T_p M$, where ν points outward. This convention supplies the signs in Stokes’ theorem.

6. THE CLASSICAL THEOREMS IN THE LANGUAGE OF FORMS

The power of differential forms is not that they replace the familiar vector calculus theorems with unrelated notation. Rather, they put the quantities already used in multivariable calculus into a single system. The following three calculations make that dictionary explicit.

6.1. Green’s theorem. Let $D \subset \mathbb{R}^2$ be an oriented region with positively oriented boundary, and let $\mathbf{F} = (P, Q)$ be a smooth planar vector field. Associate to $\mathbf{F}$ the 1-form

$$\alpha = P dx + Q dy.$$

The exterior derivative is

$$\begin{aligned} d\alpha &= dP \wedge dx + dQ \wedge dy \\ &= (P_x dx + P_y dy) \wedge dx + (Q_x dx + Q_y dy) \wedge dy \\ &= (Q_x - P_y) dx \wedge dy. \end{aligned}$$

Here $dx \wedge dx = dy \wedge dy = 0$ and $dy \wedge dx = -dx \wedge dy$ explain the minus sign. With the standard orientation of the plane, $dx \wedge dy$ is the usual area element dA . In addition, integrating α around a parametrized boundary curve gives

$$\int_{\partial D} \alpha = \oint_{\partial D} P dx + Q dy.$$

Therefore generalized Stokes’ theorem for the pair (D, α) is exactly

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{\partial D} P dx + Q dy,$$

which is Green’s theorem. The usual counterclockwise direction is precisely the boundary orientation induced by the standard orientation of $\mathbb{R}^2$.

6.2. Classical Stokes’ theorem. Let $S \subset \mathbb{R}^3$ be an oriented surface and write $\mathbf{F} = (P, Q, R)$. This time associate to $\mathbf{F}$ the 1-form

$$\alpha = P dx + Q dy + R dz.$$

The orientation of ∂S is the one for which an outward-pointing vector in the surface, followed by a positively directed tangent vector to ∂S , gives the chosen orientation of S . This is the surface version of the boundary-orientation convention used in generalized Stokes. Computing as above gives

$$\begin{aligned} d\alpha &= (R_y - Q_z) dy \wedge dz + (P_z - R_x) dz \wedge dx \\ &\quad + (Q_x - P_y) dx \wedge dy. \end{aligned}$$

The three coefficients are the components of $\nabla \times \mathbf{F}$. To see how the form records the usual surface integral, let $\mathbf{r}(u, v)$ be an orientation-preserving parametrization of S . A direct substitution shows that

$$\mathbf{r}^*(d\alpha) = ((\nabla \times \mathbf{F})(\mathbf{r}(u, v))) \cdot (\mathbf{r}_u \times \mathbf{r}_v) du \wedge dv.$$

Thus $\int_S d\alpha$ is exactly the flux integral of the curl. On a boundary curve $\mathbf{r}(t)$, the pullback of α is

$$\mathbf{r}^*\alpha = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt,$$

so $\int_{\partial S} \alpha$ is the circulation integral. Generalized Stokes' theorem consequently becomes

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

6.3. The divergence theorem. For the same vector field $\mathbf{F} = (P, Q, R)$, the appropriate object is now the 2-form

$$\beta = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy.$$

This is the *flux form* of $\mathbf{F}$. Each wedge term measures the component of $\mathbf{F}$ normal to the complementary coordinate plane. For example, $P dy \wedge dz$ measures flux through a surface whose normal points in the x -direction. Taking the exterior derivative yields

$$\begin{aligned} d\beta &= P_x dx \wedge dy \wedge dz + Q_y dy \wedge dz \wedge dx + R_z dz \wedge dx \wedge dy \\ &= (P_x + Q_y + R_z) dx \wedge dy \wedge dz \\ &= (\nabla \cdot \mathbf{F}) dV. \end{aligned}$$

On an oriented boundary surface, $\int_{\partial V} \beta$ is the usual outward flux integral. Applying generalized Stokes to (V, β) gives

$$\iiint_V (\nabla \cdot \mathbf{F}) dV = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS,$$

the divergence theorem. The change from a 1-form in classical Stokes to a 2-form here reflects the different quantity being integrated over the boundary: circulation along a curve versus flux across a surface.

7. GENERALIZED STOKES THEOREM

We now justify [Theorem 1.1](#). The underlying local computation is the Fundamental Theorem of Calculus. For example, on a rectangle $R = [a, b] \times [c, d]$, take $\omega = f(x, y) dx$. Then

$$d\omega = -\frac{\partial f}{\partial y} dx \wedge dy,$$

so Fubini's theorem and the Fundamental Theorem give

$$\int_R d\omega = \int_a^b (f(x, c) - f(x, d)) dx = \int_{\partial R} \omega.$$

The vertical sides contribute zero because dx vanishes on their tangent vectors; the signs on the horizontal sides are exactly the boundary orientation. This elementary calculation is already a complete instance of generalized Stokes: the derivative has increased the degree from a 1-form to a 2-form, and the 2-dimensional region has a 1-dimensional boundary.

Proof sketch of Theorem 1.1. Choose finitely many oriented boundary charts and interior charts covering the compact manifold M , together with a smooth partition of unity $\{\rho_i\}$ subordinate to that cover. Since $\sum_i \rho_i = 1$,

$$\int_M d\omega = \sum_i \int_M d(\rho_i \omega).$$

In each chart, the preceding Euclidean Stokes computation (in its n -dimensional form) applies to the compactly supported form $\rho_i \omega$. The interior-chart boundary terms vanish, and the boundary-chart terms carry the induced boundary orientation. Hence

$$\sum_i \int_M d(\rho_i \omega) = \sum_i \int_{\partial M} \rho_i \omega = \int_{\partial M} \left(\sum_i \rho_i \right) \omega = \int_{\partial M} \omega.$$

□

This proof reduces the global statement to the Euclidean Stokes theorem in each coordinate chart. The existence of the partition of unity and the Euclidean theorem are standard results; the rectangle computation above shows the local mechanism behind them.

The theorem recovers familiar theorems by choosing n and ω appropriately: $n = 1$ gives the Fundamental Theorem of Calculus; a 1-form on a planar region gives Green's theorem; and suitable 1- and 2-forms in $\mathbb{R}^3$ give the classical Stokes and divergence theorems.

8. DE RHAM COHOMOLOGY

Definition 8.1. A form $\omega \in \Omega^k(M)$ is *closed* if $d\omega = 0$ and *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$. The k th de Rham cohomology group is

$$H_{\text{dR}}^k(M) = \frac{\ker(d : \Omega^k(M) \rightarrow \Omega^{k+1}(M))}{\text{im}(d : \Omega^{k-1}(M) \rightarrow \Omega^k(M))}.$$

Because $d^2 = 0$, every exact form is closed, so the quotient is well defined. It records precisely the obstruction to a closed form being a derivative. In other words, the numerator contains all forms with no further exterior derivative, while the denominator identifies those which already arose as an exterior derivative of a lower-degree form.

Example 8.2 (The punctured plane). On $M = \mathbb{R}^2 \setminus \{(0, 0)\}$ let

$$\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

At a point $(x, y) \neq (0, 0)$, this form measures infinitesimal change in the polar angle. It is tempting to call it $d\theta$; locally that is correct, but no single smooth angle function θ is defined on the entire punctured plane. The following computation makes the obstruction precise.

Write $\omega = P dx + Q dy$, where

$$P = \frac{-y}{x^2 + y^2} \quad \text{and} \quad Q = \frac{x}{x^2 + y^2}.$$

Using the coordinate formula for the exterior derivative gives

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

The two derivatives are

$$\frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad \text{and} \quad \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

They agree, so $d\omega = 0$. Thus ω is closed everywhere on M .

To test whether it is exact, integrate around the unit circle

$$\gamma : [0, 2\pi] \longrightarrow M, \quad \gamma(t) = (\cos t, \sin t).$$

Along this curve, $dx = -\sin t \, dt$, $dy = \cos t \, dt$, and $x^2 + y^2 = 1$. Therefore

$$\begin{aligned}\gamma^*\omega &= (-\sin t)(-\sin t \, dt) + (\cos t)(\cos t \, dt) \\ &= (\sin^2 t + \cos^2 t)dt = dt.\end{aligned}$$

Consequently,

$$\int_{\gamma} \omega = \int_0^{2\pi} dt = 2\pi.$$

If $\omega = df$ for a globally defined smooth function f on M , then the Fundamental Theorem of Calculus along the closed curve would give

$$\int_{\gamma} \omega = \int_{\gamma} df = f(\gamma(2\pi)) - f(\gamma(0)) = 0,$$

because $\gamma(0) = \gamma(2\pi)$. This contradicts the value 2π above, so ω cannot be exact. Therefore $[\omega] \neq 0$ in $H_{\text{dR}}^1(M)$. The nonzero integral measures one trip around the missing origin: the hole creates a closed form that is not the derivative of any globally defined function.

9. CONCLUSION

Differential forms make both differentiation and integration intrinsic to a manifold. The exterior derivative satisfies $d^2 = 0$, which produces de Rham cohomology and lets topology appear in calculus. Generalized Stokes' theorem then unifies the major boundary-integral theorems of elementary and multivariable calculus. A natural next step is de Rham's theorem, which identifies de Rham cohomology with singular cohomology with real coefficients.

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