Geometric Inequalities

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Overview

- AM-GM Inequality
- Isoperimetric Inequality For Triangles
- Definition of Parametric Curves and Simple Closed Curves
- Arc length and Area of Closed Curves
- Isoperimetric Inequality Proof

AM-GM Inequality

- AM-GM inequality states that $\frac{\sum_{i=1}^{n} a_i}{n} \geq \sqrt[n]{\prod_{i=1}^{n} a_i}$

Introduction

- The isoperimetric inequality states, for the length L of a closed plane curve and the area A of the region that it encloses, that

$$4\pi A \leq L^2 \tag{1}$$

and that equality holds if the closed curve is circle.

Isoperimetric Inequality For Triangles

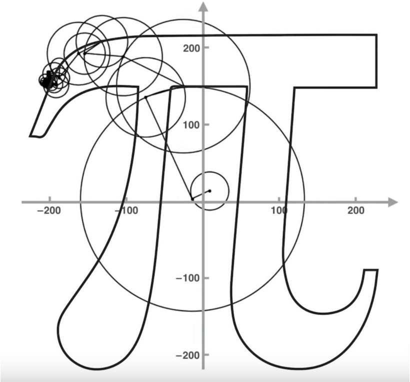
- We will use Heron Formula and AM-GM inequality
- In the end, we will get $L^2 \geq 12\sqrt{3} \cdot A$

Simple Closed Curve

- A simple closed curve is a continuous, loop-like path in a plane that begins and ends at the exact same point but does not cross or intersect itself anywhere along the way.

Parametric Curves

- Parametric Curves are curves with parameter.
- We will use this to prove isoperimetric inequality



Arc Length and The Area of the Closed Curve

- We will use this equation to find the perimeter of the closed curve

$$L = \int_0^L \sqrt{x'(s)^2 + y'(s)^2}ds$$

- We will use this equation to find the area of the closed curve

$$A = \frac{1}{2}\int_0^L (x(s)y'(s) - y(s)x'(s))ds$$

- We will use these equations to prove isoperimetric inequality

Conclusion

- After some calculations, we will get the inequality $L^2 \geq 4\pi A$.
- We will also get circle has the greatest area.

Applications

- Cauchy–Schwarz Inequality
- Dido's Problem
- Curves With Constant Width

Cauchy–Schwarz Inequality

- Cauchy-Schwarz inequality states that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

- We can prove this by simply using AM-GM inequality.

Dido's Problem

- We are trying to maximize the area bounded by a line (the coastline) and a curve of fixed length
- We can view the coastline as the reflection symmetry and reflect over it and we will get a circle. Given circle has the greatest area with the fixed perimeter so semi-circle has the greatest area.

Curves with constant width

- Let a convex plane curve have constant width w. We will get the exact perimeter $L = \pi w$
- This tells how far any constant width shape is from the optimal area for its fixed width.

Thank You!