Combinatorial Methods for Determinant Evaluation

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Historical Context

The ideas in this talk were developed during different periods of mathematics.

- **1683–1812:** Determinants developed from methods for solving systems of equations, through the work of Seki, Leibniz, Vandermonde, and Cauchy.
- **1896–1916:** Percy MacMahon developed the theory of plane partitions and discovered remarkable product formulas for counting them.
- **1973:** Bernt Lindström interpreted certain determinants using families of disjoint paths.
- **1985:** Ira Gessel and Gérard Viennot developed this approach into a major tool of enumerative combinatorics.

Historical significance

The LGV approach gives a modern combinatorial explanation for formulas that were discovered long before the path method existed.

The Guiding Question

Question

Why do some determinants with complicated entries simplify into clean product formulas or small counting formulas?

The answer is often hidden cancellation.

- Algebraically, a determinant is a signed sum over permutations.
- Combinatorially, those signed terms can represent objects.
- A sign-reversing involution pairs off the "bad" objects.
- The determinant then counts the surviving "good" objects.

determinant $\longrightarrow$ **signed objects** $\longrightarrow$ **cancellation** $\longrightarrow$ **positive count**

Why Determinants Need Structure

For an $n \times n$ matrix $A = (a_{ij})$, the determinant is

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

- There are $n!$ terms, so direct expansion becomes useless very quickly.
- Each term chooses one entry from every row and every column.
- The signs are not random: they encode the parity of the permutation.

Motivation for special determinants

Instead of expanding everything, we search for matrix families whose entries already count something structured. Then the determinant can become a signed count of those structures.

The Cancellation Pattern

Algebraically

$$\sum_{x \in \mathcal{S}} \mathrm{sign}(x)\ \mathrm{wt}(x)$$

Positive and negative terms appear together.

Combinatorially: Find a map $\phi : \mathcal{S} \to \mathcal{S}$ such that

$$\phi(\phi(x)) = x, \qquad \mathrm{sign}(\phi(x)) = -\,\mathrm{sign}(x).$$

Then paired objects cancel.
The fixed points are the objects that cannot be canceled. These become the final count.

The LGV Lemma: Determinants Become Paths

Let G be a directed acyclic graph with origins $o_1, \ldots, o_n$ and destinations $d_1, \ldots, d_n$. Let

$$m_{ij} = \#\{\text{directed paths from } o_i \text{ to } d_j\}.$$

Then $M = (m_{ij})$ is the path-counting matrix.

Lindström–Gessel–Viennot lemma

$$\det(M) = \text{signed count of path families.}$$

In planar nonpermutable cases, this becomes

$$\det(M) = \#\{\text{nonintersecting path families from } o_i \text{ to } d_i\}.$$

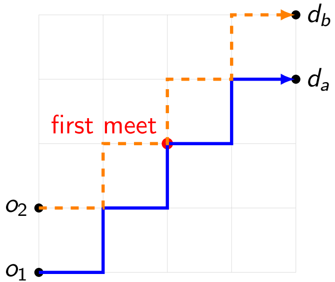
A determinant can enforce nonintersection automatically.

LGV switching tails

Expanding the determinant gives

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) m_{1,\sigma(1)} \cdots m_{n,\sigma(n)}.$$

- The product counts path families with endpoint pattern σ.
- If two paths intersect, switch their tails after the first intersection.
- This changes the permutation by a transposition, so the sign flips.
- Intersecting families cancel; nonintersecting families remain.



Motivation

This is the combinatorial version of the negative signs in the determinant.

Example 1: Vandermonde Measures Separation

A Vandermonde matrix has rows built from powers:

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$

$$\det(V) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Motivation

If $x_i = x_j$, then rows i and j are identical, so the determinant must vanish. Therefore every pairwise difference $x_j - x_i$ appears as a factor.

Vandermonde detects collision, and grows with separation.

Example 2: Cauchy Balances Growth and Shrinkage

The Cauchy matrix is

$$C = \left(\frac{1}{x_i + y_j} \right)_{1 \le i,j \le n}.$$

Its determinant is

$$\det(C) = \frac{\displaystyle\prod_{1 \le i < j \le n} (x_j - x_i)(y_j - y_i)}{\displaystyle\prod_{1 \le i,j \le n} (x_i + y_j)}.$$

- The numerator contains two Vandermonde-type products.
- The determinant still vanishes when two x's or two y's collide.
- The denominator reflects the reciprocal structure of the entries.

Motivation

Cauchy shows the same separation principle as Vandermonde, but it is balanced by a global denominator that makes the determinant shrink when the pairwise sums grow.

Factorization to Counting

Vandermonde and Cauchy determinants: The determinant simplifies because row and column structure forces certain factors.
LGV and Hankel determinants: The determinant simplifies because its entries count paths, and cancellation removes intersecting families.

The next example makes the counting interpretation explicit.

Hankel Determinants and Catalan Numbers

A Hankel matrix has entries depending only on $i + j$:

$$H_n = (a_{i+j})_{0 \le i,j \le n-1} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-2} \end{pmatrix}.$$

Why Catalan numbers are natural:
The Catalan number

$$C_m = \frac{1}{m+1}\binom{2m}{m}$$

counts lattice paths from $(0, 0)$ to (m, m) using right and up steps that stay on or below $y = x$.

Hankel structure: one sequence $\to$ many path counts arranged in a matrix.

Building the Catalan Path Matrix

For $t \geq 0$, define origins and destinations by

$$o_i = (-i, -i), \qquad d_j = (j + t, j + t), \qquad 0 \leq i, j \leq n - 1.$$

A path from o_i to d_j takes $i + j + t$ right steps and $i + j + t$ up steps. If it stays below $y = x$, the number of such paths is

$$C_{i+j+t}.$$

Thus the path matrix is the Catalan Hankel matrix

$$M_n^t = (C_{i+j+t})_{0 \leq i,j \leq n-1}.$$

Motivation

The entry depends only on $i + j$ because moving the origin back and the destination forward both add the same number of right and up steps.

Applying LGV to Catalan Hankel Determinants

By LGV,

$$\det(M_n^t) = \text{signed count of nonintersecting Catalan path systems.}$$

- The origins and destinations lie in a fixed planar order.
- A nonintersecting family cannot switch destinations.
- Therefore every surviving family corresponds to the identity permutation.
- So the signed count becomes an ordinary positive count.

Key point

The determinant is no longer just simplifying the matrix. It is counting the possible nonintersecting route systems.

Algebraic determinant = geometric nonintersection rule.

Special Cases: Why the Answers Are Small

For

$$M_n^t = (C_{i+j+t})_{0\le i,j\le n-1},$$

we get the following clean formulas:

Shift t	Determinant	Why it happens
0	$\det(M_n^0) = 1$	one forced nested system
1	$\det(M_n^1) = 1$	still forced by nonintersection
2	$\det(M_n^2) = n+1$	choose one avoided point
3	$\det(M_n^3) = \sum_{k=1}^{n+1} k^2$	choose avoided points on two diagonals

Motivation

The small closed forms are not coincidences. Nonintersection makes the route systems so rigid that they are determined by a tiny amount of data.

A Larger Application for Macmahon's Theorem

MacMahon's theorem counts plane partitions in an $a \times b \times c$ box:

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

- A plane partition can be drawn as a stack of cubes.
- The same object can be viewed as a lozenge tiling of a hexagon.
- A lozenge tiling can be translated into nonintersecting lattice paths.

How it fits the paper

MacMahon's theorem is a large-scale version of the same strategy:

$$\begin{aligned}&\text{tilings} \to \text{paths}\\ &\qquad \to \text{determinants} \to \text{product.}\end{aligned}$$

This shows why LGV is not just a trick, but a general method.

Conclusion

- The Leibniz formula makes determinants into signed sums over permutations.
- Sign-reversing involutions explain how bad objects cancel.
- The LGV Lemma turns path-counting matrices into counts of nonintersecting path families.
- Vandermonde and Cauchy determinants show how structure creates clean factorization.
- Catalan Hankel determinants show the full combinatorial story: a determinant can count rigid path systems.

Combinatorial determinant evaluation turns algebraic cancellation into visible counting.