COMBINATORIAL METHODS FOR DETERMINANT EVALUATION

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1. Introduction

This paper focuses on studying three cases of combinatorics of determinants, discussing the non-intersecting lattice paths theorem; MacMahon's theorem and Hexagons with Holes; Hankel determinants and orthogonal polynomials. Combinatorics provides clean, visual proofs for determinants. The main motivation for this paper is to evaluate the determinants of specific matrices or families of matrices using combinatorial structures, where we primarily employ the Lindström–Gessel–Viennot (LGV) Lemma. The central idea is that many determinant identities look mysterious when viewed only algebraically, but become more natural when the terms of the determinant are interpreted as counting objects. In particular, the positive and negative signs in the Leibniz formula can often be understood as a cancellation process. After this cancellation, the remaining objects frequently have a simple combinatorial description. This is the main reason why combinatorial methods are useful for determinant evaluation: they turn complicated algebraic expressions into visual counting problems.

Determinants were originally developed as algebraic tools of linear equations. Over time, mathematicians began studying special structured determinants, such as Cauchy determinants and vandermonde determinants because the Leibniz formula expansions often provided elegant formulas. This created a new study where researchers tried to find closed forms for determinants of certain matrix families.

A major combinatorial interpretation of certain determinants as non intersecting paths was appeared in Lindström's work in 1973. The idea was later developed by Gessel and Viennot, shaping what is not called the Lindström-Gessel-Viennot lemma. The lemma connects the cancellation in determinants with the combinatorial cancellation among intersecting path families.

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2. Determinants

2.1. A Multilinear Definition of the Determinant. The determinant is often introduced through the permutation formula, but that formula can make the determinant look like an artificial expression. A more conceptual definition comes from multilinear algebra. In this viewpoint, the determinant is the scalar by which a linear map scales a top-degree alternating multilinear form. [6]

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let

$$n = \dim V.$$

An m-linear form on V is a function

$$\alpha : V^m \to \mathbb{F}$$

that is linear in each input separately. Thus, if all inputs except one are fixed, then α behaves like a linear function of the remaining input.

An m-linear form α is called *alternating* if

$$\alpha(v_1, \ldots, v_m) = 0$$

whenever two of the vectors $v_1, \ldots, v_m$ are equal. We denote the vector space of alternating m-linear forms on V by

$$\mathrm{Alt}^m(V).$$

The key fact is that when $m = n = \dim V$, the space $\mathrm{Alt}^n(V)$ is one-dimensional. In other words, up to multiplying by a scalar, there is only one alternating n-linear form on V. This is why the determinant can be defined without first choosing coordinates.

Now let $T \in \mathrm{Lin}(V)$ be a linear operator. If $\alpha \in \mathrm{Alt}^m(V)$, define a new function α_T by applying T to each input before evaluating α:

$$\alpha_T(v_1, \ldots, v_m) = \alpha(Tv_1, \ldots, Tv_m).$$

This new function is again an alternating m-linear form. Indeed, if $v_j = v_k$ for some $j \neq k$, then $Tv_j = Tv_k$, so the alternating property of α gives

$$\alpha_T(v_1, \ldots, v_m) = 0.$$

Therefore $\alpha_T \in \mathrm{Alt}^m(V)$.

The important case is $m = n$. Since $\mathrm{Alt}^n(V)$ is one-dimensional, the map

$$\alpha \mapsto \alpha_T$$

is a linear map from a one-dimensional vector space to itself. Every such map is multiplication by a unique scalar. That scalar is the determinant of T.

Definition 2.1. Let V be an n-dimensional vector space over $\mathbb{F}$, and let $T \in \operatorname{Lin}(V)$. The determinant of T, denoted $\det T$, is the unique scalar in $\mathbb{F}$ such that

$$\alpha_T = (\det T)\alpha$$

for every $\alpha \in \operatorname{Alt}^n(V)$.

This definition says that the determinant measures how T acts on oriented n-dimensional volume. An alternating n-linear form may be thought of as a way of measuring oriented volume. Applying T to each input changes that volume, and the determinant is exactly the factor by which it changes.

2.2. **Basic Consequences.** This definition immediately gives the usual simple examples. If I is the identity operator on V, then

$$\alpha_I(v_1, \dots, v_n) = \alpha(Iv_1, \dots, Iv_n) = \alpha(v_1, \dots, v_n).$$

Thus $\alpha_I = \alpha$, so

$$\det I = 1.$$

If $\lambda \in \mathbb{F}$, then

$$\alpha_{\lambda I}(v_1, \dots, v_n) = \alpha(\lambda v_1, \dots, \lambda v_n).$$

By multilinearity, one factor of λ comes out of each of the n inputs. Hence

$$\alpha_{\lambda I}(v_1, \dots, v_n) = \lambda^n \alpha(v_1, \dots, v_n),$$

and therefore

$$\det(\lambda I) = \lambda^n.$$

More generally, for any $T \in \operatorname{Lin}(V)$,

$$\det(\lambda T) = \lambda^n \det T.$$

The definition also explains the eigenvalue formula. Suppose $e_1, \dots, e_n$ is a basis of V consisting of eigenvectors of T, with

$$Te_i = \lambda_i e_i$$

for $i = 1, \dots, n$. Let $\alpha \in \operatorname{Alt}^n(V)$ be nonzero. Then

$$\begin{aligned} \alpha_T(e_1, \dots, e_n) &= \alpha(Te_1, \dots, Te_n) \\ &= \alpha(\lambda_1 e_1, \dots, \lambda_n e_n) \\ &= (\lambda_1 \cdots \lambda_n)\alpha(e_1, \dots, e_n). \end{aligned}$$

Since $\alpha_T = (\det T)\alpha$, it follows that

$$\det T = \lambda_1 \cdots \lambda_n.$$

Thus, when a linear operator has a basis of eigenvectors, its determinant is the product of its eigenvalues.

2.3. **The Determinant of a Matrix.** We now pass from operators to matrices. Let A be an $n \times n$ matrix with entries in $\mathbb{F}$. Let $T_A \in \mathrm{Lin}(\mathbb{F}^n)$ be the linear operator defined by

$$T_A x = Ax.$$

The determinant of the matrix A is defined to be the determinant of this associated operator.

Definition 2.2. If $A \in M_n(\mathbb{F})$, then

$$\det A = \det T_A,$$

where $T_A : \mathbb{F}^n \to \mathbb{F}^n$ is the linear map $T_A x = Ax$.

This agrees with the expected formulas for simple matrices. If $A = I_n$, then T_A is the identity map, so

$$\det I_n = 1.$$

If A is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_n$, then the standard basis vectors are eigenvectors of T_A. Hence

$$\det A = \lambda_1 \cdots \lambda_n.$$

The same definition also explains why the determinant is alternating and multilinear in the columns of a matrix. Let $v_1, \ldots, v_n \in \mathbb{F}^n$, and let

$$(v_1 \;\; \cdots \;\; v_n)$$

denote the matrix whose kth column is v_k. Choose an alternating n-linear form $\omega \in \mathrm{Alt}^n(\mathbb{F}^n)$ such that

$$\omega(e_1, \ldots, e_n) = 1,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{F}^n$.

Let T be the linear map satisfying

$$Te_k = v_k$$

for each $k = 1, \ldots, n$. The matrix of T in the standard basis is exactly $(v_1 \;\; \cdots \;\; v_n)$. By the definition of determinant,

$$\omega_T = (\det T)\omega.$$

Evaluating both sides at $(e_1, \ldots, e_n)$ gives

$$\begin{aligned} \omega(v_1, \ldots, v_n) &= \omega(Te_1, \ldots, Te_n) \\ &= (\det T)\omega(e_1, \ldots, e_n) \\ &= \det T. \end{aligned}$$

Thus

$$\det(v_1 \;\; \cdots \;\; v_n) = \omega(v_1, \ldots, v_n).$$

Since ω is alternating and multilinear, the determinant is also alternating and multilinear in the columns. This gives a conceptual explanation for familiar determinant rules. If two columns are equal, then the determinant is zero. If one column is multiplied by a scalar, then the determinant is multiplied by that scalar. If one column is replaced by a sum, then the determinant splits into the corresponding sum of determinants.

2.4. **Recovering the Permutation Formula.** The usual Leibniz formula follows from the alternating multilinear property. Let $A = (A_{j,k})$ be an $n \times n$ matrix, and let v_k be its kth column. Then

$$v_k = \sum_{j=1}^{n} A_{j,k} e_j.$$

Using multilinearity in the columns,

$$\det A = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} A_{j_1,1} \cdots A_{j_n,n} \det(e_{j_1} \ \cdots \ e_{j_n}).$$

If any two indices among $j_1, \ldots, j_n$ are equal, then two columns are equal, so the corresponding determinant is zero. Therefore the only surviving terms are those in which

$$(j_1, \ldots, j_n)$$

is a permutation of $(1, \ldots, n)$. Hence

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} A_{\sigma(k),k}.$$

This is the Leibniz formula written in column form. Equivalently, one may write

$$\det A = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)},$$

which is the more common row version.

As an example, suppose A is upper triangular with diagonal entries $\lambda_1, \ldots, \lambda_n$. In the column version of the formula, a term can be nonzero only if

$$\sigma(k) \leq k$$

for every k. The only permutation satisfying all these inequalities is the identity permutation. Therefore the only surviving term is the product of the diagonal entries:

$$\det A = \lambda_1 \cdots \lambda_n.$$

Thus the multilinear definition and the permutation formula describe the same object. The multilinear definition explains what the determinant means: it is the scalar by which a linear map acts on top-degree alternating multilinear forms. The permutation formula is then a concrete coordinate expression for that same scalar. [2]

3. Vandermonde determinant

A vandermonde determinant is a specific way of computing determinants on a vandermonde matrix. The Vandermonde determinant is one of the simplest examples where a structured matrix produces a clean product formula. Its importance comes from the fact that the determinant vanishes whenever two variables are equal, since two rows of the matrix become identical. This forces the determinant to contain factors of the form $x_j - x_i$. Thus, the Vandermonde determinant shows how symmetry and cancellation in a determinant can produce a compact factorized expression.

Definition 3.1. A Vandermonde matrix is a matrix where there is a geometric progression. The Vandermonde determinant is important combinatorially because its Leibniz expansion is an alternating sum over permutations. This alternating structure causes cancellation and produces the compact product formula. In a $n \times n$matrix,

$$V = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

The Vandermonde determinant equation:

$$\det(X_i^{j-1})_{1\leq i,j\leq n} = \prod_{1\leq i<j\leq n} (X_j - X_i).$$

Proof by induction: first start by the case of $n = 2$

$$A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$$

then, the determinant of A would be

$$\det A = x_2 - x_1$$

now suppose the claim holds for $n-1$. Let A be a Vandermonde matrix of order n. Then

$$\det A = \det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 & \cdots & x_2^{n-2} - x_1^{n-2} & x_2^{n-1} - x_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & x_n - x_1 & x_n^2 - x_1^2 & \cdots & x_n^{n-2} - x_1^{n-2} & x_n^{n-1} - x_1^{n-1} \end{pmatrix}_{n\times n}$$

$$= \det \begin{pmatrix} x_2 - x_1 & x_2^2 - x_1^2 & \cdots & x_2^{n-2} - x_1^{n-2} & x_2^{n-1} - x_1^{n-1} \\ x_3 - x_1 & x_3^2 - x_1^2 & \cdots & x_3^{n-2} - x_1^{n-2} & x_3^{n-1} - x_1^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_n - x_1 & x_n^2 - x_1^2 & \cdots & x_n^{n-2} - x_1^{n-2} & x_n^{n-1} - x_1^{n-1} \end{pmatrix}_{(n-1)\times(n-1)}$$

$$= \det\left(\begin{pmatrix} x_2 - x_1 & 0 & \cdots & 0 \\ 0 & x_3 - x_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_n - x_1 \end{pmatrix} \begin{pmatrix} 1 & x_2 + x_1 & \cdots & \sum_{i=0}^{n-2} x_2^{n-2-i} x_1^i \\ 1 & x_3 + x_1 & \cdots & \sum_{i=0}^{n-2} x_3^{n-2-i} x_1^i \\ \vdots & \vdots & & \vdots \\ 1 & x_n + x_1 & \cdots & \sum_{i=0}^{n-2} x_n^{n-2-i} x_1^i \end{pmatrix} \right)$$

by the determinant product rule, we get that

$$= \prod_{j=2}^{n} (x_j - x_1) \det \begin{pmatrix} 1 & x_2 + x_1 & \cdots & \sum_{i=0}^{n-2} x_2^{n-2-i} x_1^i \\ 1 & x_3 + x_1 & \cdots & \sum_{i=0}^{n-2} x_3^{n-2-i} x_1^i \\ \vdots & \vdots & & \vdots \\ 1 & x_n + x_1 & \cdots & \sum_{i=0}^{n-2} x_n^{n-2-i} x_1^i \end{pmatrix}$$

$$= \prod_{j=2}^{n} (x_j - x_1) \det \left(\begin{pmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} \\ 0 & 1 & x_1 & \cdots & x_1^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \right)$$

again, by product determinant rule, we get that

$$= \prod_{j=2}^{n} (x_j - x_1) \det \begin{pmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} \end{pmatrix}$$

and by the inductive hypothesis, we get

$$= \prod_{j=2}^{n} (x_j - x_1) \prod_{2\le i<j\le n} (x_j - x_i) = \prod_{1\le i<j\le n} (x_j - x_i)$$

The equation:

$$\prod_{1\le i<j\le n} (x_j - x_i)$$

vanishes whenever two variables (x_i) and (x_j) are equal. This reflects the fact that two rows of the Vandermonde matrix become identical. Thus, the Vandermonde determinant grows larger when the variables become more spread apart, since the product of their pairwise differences becomes larger. In this sense, the Vandermonde determinant directly measures separation between variables. [3]

4. Cauchy determinant

The Cauchy determinant is another important example of a structured determinant with a surprisingly simple closed form. Unlike the Vandermonde matrix, whose entries are powers of variables, the Cauchy matrix has entries of the form $\frac{1}{x_i+y_j}$. At first, this matrix looks harder to evaluate because of the fractions. However, the determinant still factors into simple products involving differences between the x_i's and differences between the y_j's. This makes the Cauchy determinant a useful example of how special algebraic structure can lead to strong cancellation.

$$M = M(a,b) = (\frac{1}{a_i + b_j})_{1\leq i,j\leq n}$$

where $a, b \in \mathbb{C}^n$, and $a_i \neq b_j$ for all i, j. This will look like,

$$\begin{pmatrix} \dfrac{1}{x_1+y_1} & \dfrac{1}{x_1+y_2} & \cdots & \dfrac{1}{x_1+y_n} \\ \dfrac{1}{x_2+y_1} & \dfrac{1}{x_2+y_2} & \cdots & \dfrac{1}{x_2+y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{x_n+y_1} & \dfrac{1}{x_n+y_2} & \cdots & \dfrac{1}{x_n+y_n} \end{pmatrix}$$

The determinant of it is defined by,

$$D_n = \frac{\prod\limits_{1\leq i<j\leq n} (x_j - x_i)(y_i - y_j)}{\prod\limits_{1\leq i,j\leq n} (x_i - y_j)}.$$

Proof:
subtract column 1 from each of columns 2 to n. Thus:

$$a_{ij} \leftarrow \frac{1}{x_i + y_j} - \frac{1}{x_i + y_1}$$

$$= \frac{(x_i + y_1) - (x_i + y_j)}{(x_i + y_j)(x_i + y_1)}$$

$$= \frac{(x_i + y_1) - (x_i + y_j)}{(x_i + y_j)(x_i + y_1)}$$

From Multiple of Row Added to Row of Determinant this will have no effect on the value of the determinant. Now, we will extract the factor $\frac{1}{x_i+y_1}$ from each row $1 \leq i \leq n$, and extract the factor $y_1 - y_j$ from each column $2 \leq j \leq n$.Thus from Determinant with Row Multiplied by Constant we have:

$$D_n = \left(\prod_{i=1}^{n} \frac{1}{x_i + y_1} \right) \left(\prod_{j=2}^{n} (y_1 - y_j) \right) \det \begin{pmatrix} 1 & \frac{1}{x_1 + y_2} & \frac{1}{x_1 + y_3} & \cdots & \frac{1}{x_1 + y_n} \\ 1 & \frac{1}{x_2 + y_2} & \frac{1}{x_2 + y_3} & \cdots & \frac{1}{x_2 + y_n} \\ 1 & \frac{1}{x_3 + y_2} & \frac{1}{x_3 + y_3} & \cdots & \frac{1}{x_3 + y_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{x_n + y_2} & \frac{1}{x_n + y_3} & \cdots & \frac{1}{x_n + y_n} \end{pmatrix}.$$

Now, subtract row 1 from each of rows 2 to n. Column 1 will go to 0 for all but the first row, columns 2 to n will become:

$$a_{ij} \leftarrow \frac{1}{x_i + y_j} - \frac{1}{x_1 + y_j}$$

$$= \frac{(x_1 + y_j) - (x_i + y_j)}{(x_i + y_j)(x_1 + y_j)}$$

$$= \left(\frac{x_1 - x_i}{x_1 + y_j} \right) \left(\frac{1}{x_i + y_j} \right)$$

From Multiple of Row added to Row of Determinant, this will have no effect on the value of the determinant. We will then extract the factor $x_1 - x_i$ from each row $2 \leq i \leq n$ and extract the factor $\frac{1}{x_1+y_j}$ from each column $2 \leq j \leq n$. From the determinant with row multiplied by constant, we have:

$$D_n = \left(\prod_{i=1}^{n} \frac{1}{x_i + y_1} \right) \left(\prod_{j=1}^{n} \frac{1}{x_1 + y_j} \right) \left(\prod_{i=2}^{n} (x_1 - x_i) \right) \left(\prod_{j=2}^{n} (y_1 - y_j) \right) \begin{vmatrix} 1 & 1 & 1 & \cdots \\ 0 & \frac{1}{x_2 + y_2} & \frac{1}{x_2 + y_3} & \cdots \\ 0 & \frac{1}{x_3 + y_2} & \frac{1}{x_3 + y_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & \frac{1}{x_n + y_2} & \frac{1}{x_n + y_3} & \cdots \end{vmatrix}$$

From Determinant with Unit element in otherwise zero row and fixing up the products, we get:

$$D_n = \frac{\prod_{i=2}^{n}(x_i - x_1)(y_i - y_1)}{\left(\prod_{i=1}^{n}(x_i + y_1)\right)\left(\prod_{j=1}^{n}(x_1 + y_j)\right)} \begin{vmatrix} \frac{1}{x_2+y_2} & \frac{1}{x_2+y_3} & \cdots & \frac{1}{x_2+y_n} \\ \frac{1}{x_3+y_2} & \frac{1}{x_3+y_3} & \cdots & \frac{1}{x_3+y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n+y_2} & \frac{1}{x_n+y_3} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix}.$$

We then repeat the process for the remaining rows and columns 2 to n and we get:

$$D_n = \det\left(\frac{1}{x_i + y_j}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{1\le i,j\le n}(x_i + y_j)}.$$

In the equation:

$$\frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{i=1}^{n}\prod_{j=1}^{n}(x_i + y_j)}.$$

The numerator contains two Vandermonde-type products, one in the (x)-variables and one in the (y)-variables, so the determinant still detects when variables collide. However, the denominator contains all pairwise sums $(x_i + y_j)$, which makes the determinant shrink when these sums become large. Therefore, the Vandermonde determinant mainly grows with separation, while the Cauchy determinant balances this same separation effect against a reciprocal shrinking effect. [4]

5. The Lindström–Gessel–Viennot Lemma

The Lindström–Gessel–Viennot Lemma is one of the main tools used to connect determinants with combinatorics. The motivation behind the lemma is that the determinant is a signed sum over permutations, while a family of paths from several origins to several destinations can also be organized by permutations. Each permutation describes which origin is connected to which destination. Because of this, the determinant of a path-counting matrix naturally becomes a signed count of path families.

The important idea is that most intersecting path families cancel out. This cancellation comes from a sign-reversing involution, where an intersecting family of paths is paired with another intersecting family of opposite sign. After this cancellation, only the nonintersecting path families remain. Therefore, the LGV Lemma gives a powerful way to evaluate determinants by turning them into counting problems involving nonintersecting paths.

This lemma is useful because nonintersecting paths are often much easier to visualize than the full algebraic expansion of a determinant. Instead of directly expanding a determinant with many terms, we can construct a directed acyclic graph, count paths between chosen vertices, and then interpret the determinant as the number of nonintersecting path systems. [1]

6. Hankel Determinants

Hankel determinants are especially natural from a combinatorial point of view because their entries come from a single sequence. Instead of depending separately on the row and column indices, the (i, j)-entry depends only on the sum $i + j$. When the sequence is a counting sequence, such as the Catalan numbers, the entries of the matrix count paths of different lengths. Therefore, the determinant of a Hankel matrix can sometimes be interpreted as counting families of nonintersecting paths.

Definition 6.1. We define a lattice path to be an ordered sequence of vertices on the integer points in $\mathbb{Z}^2$ in the cartesian graph that follows:

- We let $L(m, n)$ be the number of these lattice paths. Initial conditions will be $L(m, 0) = L(0, n) = 1$
- We assign the symbol East to a $(1, 0)$-steps and North to $(0, 1)$-step, where they are restricted to these transformations.

Definition 6.2. A Hankel matrix is an $n \times n$ matrix whose entries are constant along each ascending skew-diagonal. Equivalently, a matrix $A = (a_{i,j})$ is Hankel if each entry depends only on the sum $i + j$. A general Hankel matrix has the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & a_4 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-2} \end{pmatrix}.$$

Hankel determinants appear naturally when the entries of a matrix are given by a sequence. In this section, we focus on Hankel matrices whose entries are Catalan numbers. Recall that the mth Catalan number is

$$C_m = \frac{1}{m+1}\binom{2m}{m}.$$

One important interpretation of C_m is that it counts the number of lattice paths from $(0,0)$ to (m,m) using only right steps and up steps, while never going above the diagonal line $y=x$.

For $n \geq 1$ and $t \geq 0$, define the Catalan digraph D_n^t as follows. The vertices of D_n^t are the integer lattice points on or below the line $y=x$. Each edge is directed either one unit to the right or one unit upward. We define the origins and destinations by

$$o_i = (-i,-i), \qquad d_j = (j+t, j+t),$$

where $0 \leq i, j \leq n-1$.

A directed path from o_i to d_j must take

$$i+j+t$$

steps to the right and

$$i+j+t$$

steps upward. Since the path must stay on or below the line $y=x$, the number of such paths is the Catalan number

$$C_{i+j+t}.$$

Therefore, the path matrix associated with D_n^t is

$$M_n^t = \left(C_{i+j+t}\right)_{0\leq i,j\leq n-1}.$$

Written explicitly,

$$M_n^t = \begin{pmatrix} C_t & C_{t+1} & C_{t+2} & \cdots & C_{t+n-1} \\ C_{t+1} & C_{t+2} & C_{t+3} & \cdots & C_{t+n} \\ C_{t+2} & C_{t+3} & C_{t+4} & \cdots & C_{t+n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{t+n-1} & C_{t+n} & C_{t+n+1} & \cdots & C_{t+2n-2} \end{pmatrix}.$$

This is a Hankel matrix because each entry depends only on the sum $i+j$. This is the point where the Hankel structure becomes useful. The entry C_{i+j+t} counts paths from the origin o_i to the destination d_j, so the entire Hankel matrix can be viewed as a path-counting matrix. Therefore, taking the determinant of this matrix is the same as applying the Lindström-Gessel-Viennot Lemma to these paths. The determinant is no longer just an algebraic expression; it becomes a way to count nonintersecting Catalan path families. By the Lindström-Gessel-Viennot Lemma, the determinant

$$\det(M_n^t)$$

has a combinatorial interpretation as a signed count of families of nonintersecting paths. More specifically, the determinant counts the number of even nonintersecting n-routes minus the number of odd nonintersecting n-routes.

In the Catalan digraph D_n^t, the origins and destinations are ordered along the same diagonal. Because of this planar ordering, any nonintersecting n-route must send each origin o_i to its corresponding destination d_i. If two paths attempted to switch destinations, then the paths would be forced to intersect. Thus every nonintersecting n-route corresponds to the identity permutation, which is even. Therefore,

$$\det(M_n^t)$$

counts the total number of nonintersecting n-routes in D_n^t.

We now consider several special cases.

First, when $t = 0$, we have

$$M_n^0 = (C_{i+j})_{0 \le i,j \le n-1}.$$

In this case,

$$\det(M_n^0) = 1.$$

Combinatorially, this occurs because there is only one possible nonintersecting route system. The innermost path is forced, and each path outside it is forced to wrap around the previous one without intersecting it. Hence there is exactly one nonintersecting n-route.

Similarly, when $t = 1$, we have

$$M_n^1 = (C_{i+j+1})_{0 \le i,j \le n-1}.$$

Again,

$$\det(M_n^1) = 1.$$

Although the destinations are shifted one step farther along the diagonal, the nonintersection condition still forces a unique nested family of paths.

The first more interesting case occurs when $t = 2$. In this case,

$$M_n^2 = (C_{i+j+2})_{0 \le i,j \le n-1}.$$

The determinant is

$$\det(M_n^2) = n + 1.$$

To see this combinatorially, consider the diagonal line

$$y = 2 - x.$$

Every nonintersecting n-route in D_n^2 must pass through exactly n of the $n + 1$ possible lattice points on this line. Therefore, one of these $n+1$ lattice points must be avoided. Once the avoided point is chosen, the entire nonintersecting route system is determined. Since there are $n+1$ choices for the avoided point, we obtain

$$\det(M_n^2) = n + 1.$$

For $t = 3$, the same method gives a larger count. In this case,

$$M_n^3 = (C_{i+j+3})_{0 \le i,j \le n-1},$$

and

$$\det(M_n^3) = \sum_{k=1}^{n+1} k^2.$$

Using the formula for the sum of squares, this becomes

$$\det(M_n^3) = \frac{(n+1)(n+2)(2n+3)}{6}.$$

Combinatorially, the square appears because the nonintersecting route system is determined by choosing avoided points on two related diagonal lines. For a fixed value of k, there are k choices on one diagonal and k choices on the other, giving k^2 possible route systems. Summing over all possible values of k gives

$$\det(M_n^3) = 1^2 + 2^2 + \cdots + (n+1)^2.$$

These examples show why Hankel determinants with Catalan number entries have natural combinatorial explanations. The Hankel structure appears because the number of paths from o_i to d_j depends only on $i+j$. The determinant then counts nonintersecting families of Catalan paths. Thus determinant identities such as

$$\det(M_n^0) = 1, \qquad \det(M_n^1) = 1, \qquad \det(M_n^2) = n + 1$$

can be understood visually through the structure of nonintersecting lattice paths. The main lesson from these examples is that the determinant is doing more than simplifying a matrix. It is enforcing a nonintersection condition. Algebraically, this condition appears through cancellation between positive and negative permutation terms. Combinatorially, the same cancellation removes intersecting path families and leaves only the nonintersecting ones. This explains why Catalan Hankel determinants often have simple closed forms: the determinant is secretly counting highly constrained path systems. [5]

7. MacMahon's Theorem and Plane Partitions

MacMahon's theorem is a classical result in enumerative combinatorics that counts plane partitions inside a box. One reason this theorem is important is that it has several different interpretations. A plane partition can be viewed algebraically as an array of integers, geometrically as a stack of cubes, or combinatorially as a lozenge tiling of a hexagon. This makes MacMahon's theorem a strong example of how one counting problem can appear in many different forms. The connection to determinants comes from the lozenge tiling interpretation. Certain lozenge tilings can be represented using families of nonintersecting lattice paths. Once the tilings are translated into path systems, the Lindström–Gessel–Viennot Lemma can be applied. This turns the problem of counting tilings into the problem of evaluating a determinant.

Thus, MacMahon's theorem fits naturally into the theme of this paper. The theorem shows how a difficult enumeration problem can be simplified by translating it into nonintersecting paths and then using determinant methods. In this way, MacMahon's theorem provides a major example of the power of combinatorial determinant evaluation.

8. Conclusion

The examples in this paper demonstrate that determinants can be understood as more than formulas for performing algebraic calculations. From the multilinear viewpoint, the determinant measures the action of a linear transformation on oriented volume, while the Leibniz formula expresses it as a signed sum over permutations. This alternating structure explains why determinants naturally produce cancellation and why matrices with special patterns can have unexpectedly simple evaluations.

The Vandermonde and Cauchy determinants illustrate how algebraic structure forces clean factorizations. The Lindström–Gessel–Viennot Lemma gives this cancellation a direct combinatorial meaning: intersecting path families are paired with families of the opposite sign, leaving only nonintersecting families after cancellation. For the Catalan Hankel matrices, this interpretation explains formulas such as

$$\det(M_n^0) = 1, \qquad \det(M_n^1) = 1, \qquad \det(M_n^2) = n + 1.$$

These values are small not because of accidental algebraic simplification, but because the nonintersection condition makes the surviving path systems highly constrained. MacMahon's theorem shows that the same approach extends to more complicated objects. Plane partitions can be represented as lozenge tilings, which can then be translated into nonintersecting lattice paths and studied using determinants. Therefore, the LGV Lemma is not merely a technique for one special family of matrices, but part of a broader method in enumerative combinatorics. Overall, combinatorial determinant evaluation replaces difficult algebraic expansion with a clearer sequence of interpretations.

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