The Number $\frac{32}{27}$: A Mathematical Mystery

Chromatic Polynomials and Their Roots

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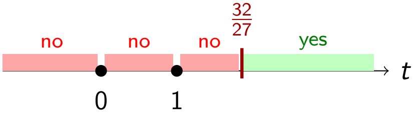
Where We Are Going



Red zones: something can **never** happen here. The sharp boundary between red and green is $\frac{32}{27} \approx 1.185$.

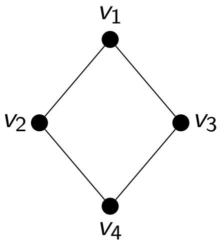
By the end of this talk you will know exactly what this means and why it is true.

Overview

What is a Graph?

Definition

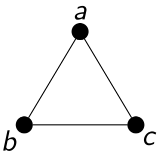
A **graph** $G = (V, E)$ has vertices V and edges E, where each edge is an unordered pair of distinct vertices.



- $n = |V(G)|$ = number of vertices
- **Adjacent** = joined by an edge
- **Leaf** = vertex with exactly one edge
- **Bridge** = edge whose removal disconnects G

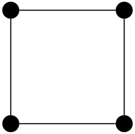
Three Key Graphs

Triangle K_3



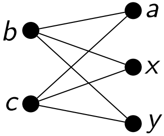
Every pair adjacent

Square C_4



Ring of 4

$K_{2,3}$

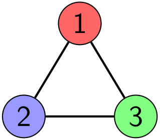


Two groups

Proper Colourings

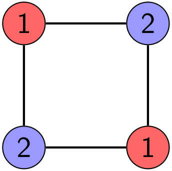
Definition

A **proper** t**-colouring** assigns one of t colours to every vertex so adjacent vertices never share a colour. The **chromatic number** $\chi(G)$ is the smallest such t.



$\chi(K_3) = 3$

2 colours fail: c touches both



$\chi(C_4) = 2$

Opposite vertices not adjacent

Counting Colourings: $P(G, t)$

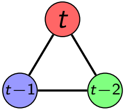
Definition

$P(G, t)$ = number of distinct proper t-colourings of G.

Compute $P(K_3, t)$**:**

- Colour a: t choices
- Colour b: $t - 1$ choices
- Colour c: $t - 2$ choices

$P(K_3, t) = t(t-1)(t-2)$



t	P
1	0
2	0
3	6

Key shift

Polynomial in t: plug in any real t.

It Is Always a Polynomial

Theorem

For every simple graph G on n vertices, $P(G, t)$ is a polynomial of degree n with leading coefficient 1.

Key idea. Same-colour vertices form an **independent set** (no edges between them). Let e_k = ways to split $V(G)$ into k nonempty independent sets. Then:

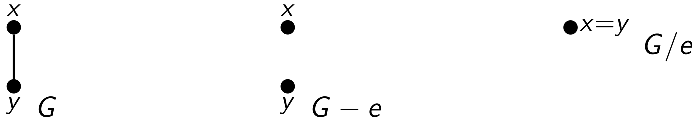
$$P(G, t) = \sum_{k=1}^{n} e_k \cdot t(t-1)\cdots(t-k+1)$$

Each term is a polynomial in t. A finite sum of polynomials is a polynomial. Since $e_n = 1$, the leading term is t^n. $\square$

Deletion-Contraction

Two operations on edge $e = xy$

$G - e$: delete the edge G/e: merge x and y



Theorem (Deletion-Contraction)

$P(G, t) = P(G - e,\ t) - P(G/e,\ t)$

Proof. Split colourings of $G - e$:

- x, y **differ**: valid for G. Count $= P(G, t)$.
- x, y **match**: equivalent to G/e. Count $= P(G/e, t)$.

So $P(G - e, t) = P(G, t) + P(G/e, t)$. Rearrange. □

Chromatic Roots

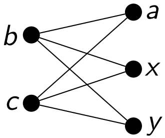
Definitions

A **chromatic root** is a value of t where $P(G, t) = 0$.
$t = 0$ and $t = 1$ are **trivial** roots.
All others are **nontrivial**.

A nontrivial root exists:
$P(K_{2,3}, t) = t(t-1)(t^3 - 5t^2 + 10t - 7)$ The cubic has a real root at $t \approx 1.431$.

Central question

Can nontrivial roots get close to 1?

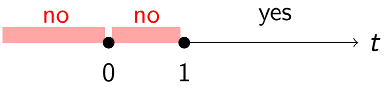


$K_{2,3}$: root at ≈ 1.431

Tutte's Theorem

Theorem (Tutte, 1970s)

No graph has a chromatic root in $(-\infty, 0)$ *or in* $(0, 1)$.



Proof idea. Track the sign of $(-1)^{n+c}P(G,t)$ through deletion-contraction induction. This sign is permanently positive in both forbidden regions, so $P(G,t)$ can never be zero there.

Consequence: every nontrivial root satisfies $t \geq 1$.

Is There Another Gap Above 1?

Tutte: every nontrivial root satisfies $t \geq 1$.
$K_{2,3}$ has a root at ≈ 1.431, so roots DO exist above 1.

Can roots get arbitrarily close to 1 from above?

Tutte raised this in 1974 but left it open.

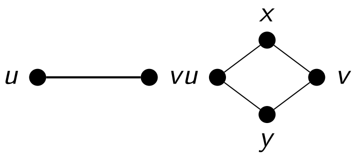
Jackson's answer (1993)

No. There is another forbidden zone above 1, with sharp boundary $\frac{32}{27} \approx 1.185$.

Double Subdivision and Generalized Triangles

Double Subdivision of uv

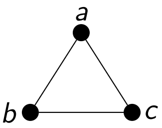
Delete uv. Add vertices x, y. Add edges ux, xv, uy, yv.

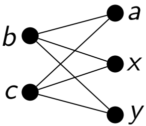


Generalized Triangle

K_3, or any graph obtained from K_3 by repeated double subdivisions.

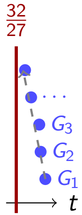
One subdivision of K_3 gives $K_{2,3}$:





Roots Converging to $\frac{32}{27}$

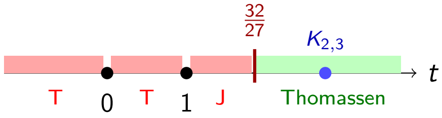
Graph	$\lvert V \rvert$	Smallest root
K_3	3	none in $(1, 2)$
$G_1 = K_{2,3}$	5	≈ 1.4302
G_2	7	≈ 1.3611
G_3	9	≈ 1.3351
$\vdots$		$\downarrow$
		$\to \frac{32}{27}$



Theorem (Jackson, 1993)

No graph has a chromatic root in $(1, \frac{32}{27})$. *This bound is sharp.*

The Complete Picture



- $(-\infty, 0)$ and $(0, 1)$: no roots **[T-Tutte]**
- $(1, \frac{32}{27})$: no roots **[J-Jackson]**
- $(\frac{32}{27}, \infty)$: roots dense, no further gaps **[Thomassen 1997]**

Restricted Families and Open Questions

Dong-Koh (2010):

Forbidden minor	Threshold
K_3, $K_4 - e$, $K_{2,3}$	2
K_4	$\frac{32}{27}$
$K_{2,4}$	≈ 1.431

Thomassen (2000):
Graphs with a Hamiltonian path have no root below $t_0 \approx 1.296$.

Open problems:

1. Perrett's Conjecture: threshold $> \frac{32}{27}$ iff family omits some generalized triangle.
2. Does $\frac{5}{4}$ threshold extend to all 3-connected graphs?
3. Complex roots connect to the Potts model in statistical physics.
4. Analogous threshold for flow polynomials?

References

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[P] T. Perrett, *Chromatic roots and minor-closed families*, arXiv:1603.08255 (2018).

Scan for the animation

"Mathematics is the art of giving the same name to different things." – **Henri Poincaré**