The Euler-Maclaurin Summation Formula

Foundations, Mechanics, and Applications

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The Objective of Euler-Maclaurin

Understanding the Gap

Mathematics often distinguishes between two types of operations:

- **Discrete Sums ($\sum$):** Step-by-step additions of individual, isolated points.
- **Continuous Integrals ($\int$):** Smooth, fluid measurements of the area under a curve.

The **Euler-Maclaurin formula** acts as the explicit bridge between these operations. It calculates the exact discrepancy between a discrete sum and its continuous integral counterpart.

Foundations: Bernoulli Polynomials

To translate bars into smooth curves, the formula uses a custom-engineered family of polynomials, $B_n(x)$, defined by strict calculus properties:

1. The Derivative Property

Each polynomial maps directly to the power below it when differentiated:

$$B_n'(x) = nB_{n-1}(x)$$

This ensures predictable behavior during calculus operations.

2. The Zero Net-Area Property

Across a single standard 1-unit interval, the average height is zero:

$$\int_0^1 B_n(x)\,dx = 0 \quad (\text{for } n \geq 1)$$

This prevents the formula from adding extra numbers.

Foundations: Mathematical Induction

- The Euler-Maclaurin formula uses a repeating, infinite chain of higher-order derivatives $(f'(x), f''(x), f'''(x), \dots)$.
- To validate this infinite expansion without manually computing lines of text into infinity, we rely on **mathematical induction**.

The Two Step Process

1. **Base Case:** Prove the relationship manually for the very first derivative interval.
2. **Inductive Step:** Show that if the structural relationship holds for the k-th derivative, running Integration by Parts automatically locks the $(k + 1)$-th derivative into place.

Defining the Domain: Complex Numbers

To prepare for advanced applications, the input variables must step off the standard real number line and expand into the **complex plane**.

Complex Coordinate Definition

A complex number z expands our field of view from 1D lines to a 2D coordinate space:

$$z = a + bi$$

- **Real Part (a):** Moves horizontally along the traditional standard axis.
- **Imaginary Part (b):** Moves vertically along the imaginary axis ($i = \sqrt{-1}$).

In this domain, exponents no longer just scale values up or down; they introduce geometric rotations and fluid wave motions.

Proof: Starting with Base Case

The formal proof begins on a single, narrow interval from $x = 0$ to $x = 1$. We integrate the product of the function's slope and the first Bernoulli polynomial:

$$\int_0^1 f'(x) B_1(x)\, dx$$

Applying the standard formula for Integration by Parts ($\int u\, dv = uv - \int v\, du$) yields:

$$\int_0^1 f'(x) B_1(x)\, dx = [f(x) B_1(x)]_0^1 - \int_0^1 f(x) B_1'(x)\, dx$$

Substituting our explicit definitions ($B_1(0) = -\frac{1}{2}$, $B_1(1) = \frac{1}{2}$, and $B_1'(x) = 1$) isolates our target terms:

$$\int_0^1 f'(x) B_1(x)\, dx = \left(\frac{f(0) + f(1)}{2}\right) - \int_0^1 f(x)\, dx$$

Proof: Telescoping Sums

To expand across a macro interval from $x = a$ to $x = b$, we line up individual 1-unit blocks back-to-back and add them together. This triggers a cascading cancellation known as a **telescoping sum**:

- **Integrals Combine Seamlessly:**

$$\int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^4 f(x)dx = \int_1^4 f(x)dx$$

- **Interior Derivatives Collapse to Zero:**

$$\cdots + \left(\cancel{f'(2)} - f'(1)\right) + \left(\cancel{f'(3)} - \cancel{f'(2)}\right) + \left(f'(4) - \cancel{f'(3)}\right) + \dots$$

Every internal measurements cancels out completely. The only values required to compute the entire system are the data points left at the absolute **boundaries** (a and b).

The Complete Euler-Maclaurin Formula

Bringing all structural components together yields the full equation:

$$\sum_{i=a}^{b} f(i) = \int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2} + \sum_{k=2}^{m} \frac{B_k}{k!}\left(f^{(k-1)}(b) - f^{(k-1)}(a)\right) + R_m$$

Component	Notation	Functional Role
Discrete Sum	$\sum f(i)$	Target operation to be evaluated
Macro Integral	$\int_a^b f(x)dx$	Smooth continuous baseline approximation
Trapezoidal Shift	$\frac{f(a)+f(b)}{2}$	Linear correction factor for boundary edges
Bernoulli Strings	$\frac{B_k}{k!}\Delta f^{(k-1)}$	High-precision slope refinements via boundaries

Application: The Riemann Zeta Function

The Riemann Zeta Function is defined as an infinite sum of complex variables:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

Divergence

If the real part is $s \leq 1$, the terms do not shrink, causing the sum to explode to infinity. For example, plugging in $s = -1$ yields the divergent string:

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = \infty$$

To evaluate this, we deploy Euler-Maclaurin on $f(x) = x^{-s}$.

Application: Bypassing Infinity via Limits

Applying the formula allows us to split the sum into distinct visual segments:

$$\sum_{n=1}^{N} \frac{1}{n^s} = \left(\frac{1}{s-1} - \frac{N^{1-s}}{s-1} \right) + \frac{1 + N^{-s}}{2} + \left(\frac{s}{12} - \frac{sN^{-s-1}}{12} \right) + \dots$$

When we push the upper tracking boundary to infinity ($N \to \infty$):

- Every single variable pinned to N cleanly collapses to zero ($N^{1-s} \to 0, N^{-s} \to 0$).
- The runaway infinity of the integral is systematically isolated and stripped away.

This yields a stable, alternative analytically continued equation:

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \dots$$

The Evaluation

Using our stable analytic continuation formula, we can now safely compute $s = -1$:

$$\zeta(-1) = \frac{1}{(-1) - 1} + \frac{1}{2} + \frac{(-1)}{12} - 0$$

Because the function simplifies down to a first-degree line ($f(x) = x^1$), all higher derivative terms beyond $\frac{s}{12}$ become exactly zero, cleanly truncating the equation:

$$\begin{aligned} \zeta(-1) &= -\frac{1}{2} + \frac{1}{2} - \frac{1}{12} \\ &= 0 - \frac{1}{12} \\ &= -\frac{1}{12} \end{aligned}$$

The Euler-Maclaurin formula acts as an analytical filter, getting rid of the infinity and giving us a nice tangible value.

Conclusion

- **Unified Framework:** The Euler-Maclaurin formula establishes a structural link between discrete arithmetic sums ($\sum$) and continuous integrals ($\int$).
- **Boundary Efficiency:** By coordinating Bernoulli polynomials with mathematical induction, it reduces complex macro-interval calculations down to data localized exclusively at the boundaries.
- **Analytical Filter:** In advanced applications, it provides the mathematical foundation for analytic continuation, allowing us to stabilize divergent functions on the complex plane.

Thank you! Questions?