

# Analysis of the Euler-Maclaurin Summation Formula

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## Abstract

This paper explores the mechanics and applications of the Euler-Maclaurin summation formula, focusing on its ability to connect discrete arithmetic sums ( $\sum$ ) with continuous calculus integrals ( $\int$ ). We examine the foundational concepts required to understand this relationship, including mathematical induction, the algebraic properties of Bernoulli polynomials, and the geometric structure of the complex plane. Through a step-by-step derivation of the telescoping sum over a finite interval, we demonstrate how the formula uses boundary data to bridge the gap between discrete sequences and continuous curves. Finally, we analyze how this framework serves as a practical tool for the analytic continuation of the Riemann Zeta function. By breaking the function into a continuous integral and localized boundary corrections, the Euler-Maclaurin formula provides a reliable method to evaluate the series beyond its standard domain of convergence, offering a clear mathematical explanation for the evaluation of  $\zeta(-1) = -\frac{1}{12}$ .

# Contents

|          |                                                                       |           |
|----------|-----------------------------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction and Historical Context</b>                            | <b>4</b>  |
| 1.1      | Bernoulli and the Higher Power Problem . . . . .                      | 4         |
| 1.2      | Euler's Extension . . . . .                                           | 4         |
| 1.3      | Maclaurin's Independent Convergence . . . . .                         | 4         |
| 1.4      | Geometric Intuition . . . . .                                         | 5         |
| <b>2</b> | <b>Mathematical Foundations</b>                                       | <b>5</b>  |
| 2.1      | Bernoulli Numbers . . . . .                                           | 5         |
| 2.2      | Bernoulli Polynomials . . . . .                                       | 5         |
| 2.2.1    | Property 1: The Base Case . . . . .                                   | 5         |
| 2.2.2    | Property 2: The Power Rule . . . . .                                  | 6         |
| 2.2.3    | Property 3: Perfect Balance . . . . .                                 | 6         |
| 2.2.4    | Proof: Building the Polynomials from the Properties . . . . .         | 6         |
| 2.3      | Mathematical Induction . . . . .                                      | 8         |
| 2.3.1    | The Domino Analogy . . . . .                                          | 8         |
| 2.3.2    | The Two Steps of an Inductive Proof . . . . .                         | 8         |
| 2.3.3    | Why Induction is Essential for the Euler-Maclaurin Proof . . . . .    | 8         |
| 2.4      | Complex Numbers? . . . . .                                            | 9         |
| 2.4.1    | Negative Square Root . . . . .                                        | 9         |
| 2.4.2    | The Imaginary Unit . . . . .                                          | 9         |
| 2.4.3    | The Structure and Algebra of Complex Numbers . . . . .                | 10        |
| 2.4.4    | The Complex Plane and Geometric Rotations . . . . .                   | 10        |
| 2.4.5    | Complex Functions and Imaginary Exponents . . . . .                   | 11        |
| 2.5      | Analytic Continuation . . . . .                                       | 11        |
| 2.5.1    | Example: The Infinite Geometric Series . . . . .                      | 12        |
| 2.5.2    | Finding A Formula . . . . .                                           | 12        |
| 2.5.3    | Getting a Real Number . . . . .                                       | 12        |
| 2.5.4    | The Identity Theorem . . . . .                                        | 13        |
| <b>3</b> | <b>The Euler-Maclaurin Summation Formula</b>                          | <b>13</b> |
| <b>4</b> | <b>Proof via Inductive Integration by Parts</b>                       | <b>13</b> |
| 4.1      | Step 1: The Base Case on a Single Interval . . . . .                  | 13        |
| 4.2      | Step 2: Higher Derivatives . . . . .                                  | 14        |
| 4.3      | Step 3: Telescoping . . . . .                                         | 14        |
| 4.3.1    | Example: $x = 1$ to $x = 4$ . . . . .                                 | 14        |
| 4.3.2    | Component 1: The Integrals Combine . . . . .                          | 15        |
| 4.3.3    | Component 2: The Derivatives Telescope . . . . .                      | 15        |
| 4.3.4    | Component 3: The Discrete Sum Accumulates . . . . .                   | 15        |
| 4.3.5    | The Final Assembly . . . . .                                          | 16        |
| <b>5</b> | <b>Applications in Modern Mathematics: The Riemann Zeta Function</b>  | <b>16</b> |
| 5.1      | The Zeta Function . . . . .                                           | 16        |
| 5.2      | Using the Euler-Maclaurin Formula . . . . .                           | 17        |
| 5.2.1    | Step 1: Evaluating the Continuous Integral . . . . .                  | 17        |
| 5.2.2    | Step 2: Evaluating the First Derivative Boundary Correction . . . . . | 17        |
| 5.2.3    | Step 3: Putting it Back Together . . . . .                            | 18        |
| 5.3      | Achieving Analytic Continuation . . . . .                             | 18        |
| 5.3.1    | Resolving the Paradox . . . . .                                       | 18        |



# 1 Introduction and Historical Context

The boundary between discrete mathematics and continuous calculus was one of the most fiercely debated frontiers of 18th-century mathematics. Although the newly invented tools of integration allowed mathematicians to find smooth areas under curves with immense speed, evaluating discrete sums—adding up individual values one by one—remained a tedious brute-force endeavor.

The Euler-Maclaurin summation formula emerged not as an isolated theory, but as the answer to a deeply frustrating practical question: *How can we use the continuous architecture of calculus to shortcut the grueling arithmetic of discrete progressions?*

## 1.1 Bernoulli and the Higher Power Problem

The historical lineage of the formula begins with Jakob Bernoulli in the late 1700s. Mathematicians had long sought an efficient way to calculate the sum of the first  $n$  integers raised to a specific power  $k$ :

$$S_k(n) = 1^k + 2^k + 3^k + \cdots + n^k = \sum_{i=1}^n i^k \quad (1)$$

For small powers like  $k = 1$ ,  $k = 2$ , and  $k = 3$ , formulas were well-known. However, finding the sum of tenth powers ( $k = 10$ ) up to  $n = 1,000$  required hours of manual calculation. In his 1713 masterpiece *Ars Conjectandi*, Bernoulli discovered a master pattern within the coefficients of these formulas. He demonstrated that the sums of any higher power could be expressed through a single polynomial framework, powered by a sequence of unique constant weights—now known as the *Bernoulli numbers* ( $B_n$ ). Bernoulli famously boasted that he could use these numbers to compute the sum of the tenth powers of the first 1,000 integers “in less than half a quarter of an hour.”

## 1.2 Euler’s Extension

While Bernoulli successfully mapped the higher-power sums of integers, he left a massive question unanswered: Could this pattern be generalized for functions other than perfect polynomials?

In the 1730s, Leonhard Euler took up this challenge. Euler was deeply obsessed with approximating infinite series that converged agonizingly slowly. He was determined to find the exact value of the harmonic series expansion and the infamous Basel Problem:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad (2)$$

To compute these values to dozens of decimal places by hand was functionally impossible using standard addition. Euler realized that if he treated the discrete terms of a series as snapshots along a continuous curve  $f(x)$ , he could use an integral ( $\int f(x)dx$ ) as a baseline approximation.

Euler’s core motivating question became: *What is the exact, universal mathematical “correction tape” needed to bridge the geometric gap between a discrete sum and a continuous integral?* By applying integration by parts inductively, Euler discovered that the error leftover between a sum and an integral was dictated entirely by the function’s derivatives at its boundaries, scaled precisely by Bernoulli’s mysterious numbers.

## 1.3 Maclaurin’s Independent Convergence

Unbeknownst to Euler due to the slow speed of academic publishing and communication across 18th-century Europe, the Scottish mathematician Colin Maclaurin was tackling the

exact same problem. Working in Edinburgh, Maclaurin approached the dilemma from a purely geometric standpoint. He was seeking a rigorous way to defend Isaac Newton’s calculus against contemporary critics who viewed it as a collection of loose approximations.

In his 1742 textbook *Treatise of Fluxions*, Maclaurin published the exact same summation formula. When the mathematical community realized that both giants had converged on the same mathematical truth from two entirely different directions, the formula was permanently co-authored as the *Euler-Maclaurin summation formula*.

## 1.4 Geometric Intuition

Geometrically, a standard discrete summation represents the combined area of flat-topped rectangular blocks of width 1. Superimposing these blocks over a smooth, bending curve highlights the fundamental friction between the discrete and continuous worlds: visible, jagged triangular gaps.

The Euler-Maclaurin formula elegantly proves that these gaps are not chaotic noise. Because the blocks are glued together back-to-back, the complex curvature behavior in the middle of the graph undergoes a massive cascading cancellation. Ultimately, the total area of thousands of microscopic middle slivers is governed entirely by how the curve tilts (first derivative) and bends (higher-order derivatives) at the absolute starting point  $a$  and ending point  $b$ .

# 2 Mathematical Foundations

## 2.1 Bernoulli Numbers

Discovered by Jakob Bernoulli while hunting for a shortcut to calculate sums of powers, the Bernoulli numbers  $B_n$  are a sequence of rational numbers. Mathematically, they are defined as the coefficients of the Taylor series expansion of the following function:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (3)$$

Expanding the first few terms yields the values outlined in Table 1.

| Table 1: The First Few Bernoulli Numbers |                |               |       |                 |       |                |       |                 |
|------------------------------------------|----------------|---------------|-------|-----------------|-------|----------------|-------|-----------------|
| $B_0$                                    | $B_1$          | $B_2$         | $B_3$ | $B_4$           | $B_5$ | $B_6$          | $B_7$ | $B_8$           |
| 1                                        | $-\frac{1}{2}$ | $\frac{1}{6}$ | 0     | $-\frac{1}{30}$ | 0     | $\frac{1}{42}$ | 0     | $-\frac{1}{30}$ |

An essential fact to note for our upcoming proof is that every odd Bernoulli number after  $B_1$  is exactly equal to zero ( $B_{2k+1} = 0$  for  $k \geq 1$ ).

## 2.2 Bernoulli Polynomials

### 2.2.1 Property 1: The Base Case

**The Rule:**

$$B_0(x) = 1 \quad (4)$$

Every mathematical family tree needs a common ancestor. When creating a sequence of changing curves, we begin with the simplest geometric shape possible: a perfectly flat, constant horizontal line at  $y = 1$ . This serves as the starting foundation from which all higher-order polynomials ( $B_1(x), B_2(x), B_3(x), \dots$ ) are systematically generated.

### 2.2.2 Property 2: The Power Rule

**The Rule:** For any polynomial index  $n \geq 1$ , the derivative obeys the relation:

$$B'_n(x) = nB_{n-1}(x) \quad (5)$$

In introductory calculus, the standard shortcut for finding a derivative is the Power Rule, which states that the exponent drops down to become a multiplier, and the power decreases by one (e.g., the derivative of  $x^3$  is  $3x^2$ ). Bernoulli polynomials are explicitly constructed to mirror this exact behavior. For example, if we differentiate the third polynomial, it collapses into exactly three times the second polynomial:  $B'_3(x) = 3B_2(x)$ .

Because the Euler-Maclaurin proof relies on repeating Integration by Parts over and over again, we need a set of functions that degrade in a highly predictable pattern when differentiated. This property guarantees that every derivative step moves smoothly down the chain without creating chaotic fractions.

### 2.2.3 Property 3: Perfect Balance

For any polynomial index  $n \geq 1$ , the definite integral over a single unit interval is always zero:

$$\int_0^1 B_n(x) dx = 0 \quad (6)$$

This property means that between  $x = 0$  and  $x = 1$ , the curve spends an exactly equal amount of time above the  $x$ -axis as it does below it. The positive area created above the axis and the negative area created beneath the axis balance each other out perfectly, canceling the net area to zero.

This balancing act is what forces the Euler-Maclaurin formula to only care about the absolute endpoints ( $a$  and  $b$ ). Because the areas under these curves always collapse to zero in the middle of any integer interval, they naturally “get out of the way” during integration, leaving behind only the clean boundary data.

### 2.2.4 Proof: Building the Polynomials from the Properties

To prove how these three properties dictate the exact algebraic structure of the Bernoulli polynomials, we can use them like a puzzle to build the first two polynomials step-by-step.

**Deriving  $B_1(x)$ :** We begin with our anchor,  $B_0(x) = 1$ . According to Property 2, the derivative of our first polynomial must be:

$$B'_1(x) = 1 \cdot B_0(x) = 1 \quad (7)$$

To undo this derivative and find  $B_1(x)$ , we take the indefinite integral of both sides, which introduces an unknown constant of integration,  $C$ :

$$B_1(x) = \int 1 dx = x + C \quad (8)$$

To solve for the exact value of  $C$ , we apply Property 3, which states that the integral of  $B_1(x)$  from 0 to 1 must equal 0:

$$\int_0^1 (x + C) dx = 0 \quad (9)$$

Using standard calculus integration rules, we find the antiderivative:

$$\left[ \frac{1}{2}x^2 + Cx \right]_0^1 = 0 \quad (10)$$

Now, we explicitly evaluate this by plugging in the upper bound ( $x = 1$ ) and subtracting the lower bound ( $x = 0$ ):

$$\begin{aligned}\left(\frac{1}{2}(1)^2 + C(1)\right) - \left(\frac{1}{2}(0)^2 + C(0)\right) &= 0 \\ \frac{1}{2} + C &= 0 \\ C &= -\frac{1}{2}\end{aligned}$$

Substituting  $C = -\frac{1}{2}$  back into our equation proves the exact structure of the first Bernoulli polynomial:

$$B_1(x) = x - \frac{1}{2} \quad (11)$$

**Deriving  $B_2(x)$ :** We can repeat this exact logical loop to build the next layer. According to Property 2, the derivative of the second polynomial is tied to the first polynomial:

$$B_2'(x) = 2 \cdot B_1(x) = 2 \left(x - \frac{1}{2}\right) = 2x - 1 \quad (12)$$

We integrate both sides to undo the derivative and find  $B_2(x)$ , adding a new integration constant,  $C$ :

$$B_2(x) = \int (2x - 1) dx = x^2 - x + C \quad (13)$$

Once again, we invoke Property 3 to force the net area from 0 to 1 to equal zero, allowing us to isolate  $C$ :

$$\int_0^1 (x^2 - x + C) dx = 0 \quad (14)$$

Finding the continuous anti-derivative gives us:

$$\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + Cx\right]_0^1 = 0 \quad (15)$$

We evaluate the bounds explicitly by plugging in 1 and subtracting 0:

$$\begin{aligned}\left(\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 + C(1)\right) - (0) &= 0 \\ \frac{1}{3} - \frac{1}{2} + C &= 0 \\ -\frac{1}{6} + C &= 0 \\ C &= \frac{1}{6}\end{aligned}$$

Plugging  $C = \frac{1}{6}$  back into our equation constructs the second Bernoulli polynomial:

$$B_2(x) = x^2 - x + \frac{1}{6} \quad (16)$$

This construction serves as a definitive proof of the properties. The constants generated at the very end of this integration process ( $-\frac{1}{2}$  and  $\frac{1}{6}$ ) are the exact values of the first two Bernoulli numbers ( $B_1$  and  $B_2$ ) used in the final summation formula.

## 2.3 Mathematical Induction

Throughout this paper, terms like “inductive proof” or “repeating this process inductively” are used to describe how the Euler-Maclaurin formula is built. In everyday English, to be “inductive” means to look at a few specific examples and make a reasonable guess about a general rule. In mathematics, however, a guess is not enough.

Mathematical induction is a technique used to prove that a specific statement or formula is absolutely true for an infinite sequence of numbers ( $n = 1, 2, 3, 4, \dots$ ). Rather than checking every single number individually into infinity—which is physically impossible—induction allows us to prove an entire infinite sequence all at once using a mathematical chain reaction.

### 2.3.1 The Domino Analogy

The easiest way to visualize mathematical induction is to imagine a line of millions of dominoes standing upright, stretching far out of sight. If you want to prove that every single domino in the infinite line will eventually fall, you do not need to walk down the line and push each one over individually. You only need to establish two conditions:

1. **The First Push:** You must prove that you can knock down the very first domino (called Domino 1).
2. **The Structural Link:** You must prove that the dominoes are spaced closely enough together so that *if* any arbitrary domino (let us call it Domino  $k$ ) falls, it will structurally guarantee the collapse of the next domino in line (Domino  $k + 1$ ).

If both conditions are met, a flawless chain reaction is locked into place. Knocking over the first domino automatically knocks over the second, which knocks over the third, cascading seamlessly into infinity.

### 2.3.2 The Two Steps of an Inductive Proof

Translating this geometric analogy into formal mathematics requires two distinct algebraic steps:

- **Step 1: The Base Case (The First Push):** You manually plug the very first number (usually  $n = 1$  or  $n = 0$ ) into your formula and show that the left side of the equation equals the right side perfectly.
- **Step 2: The Inductive Step (The Structural Link):** This is where the magic happens.
  1. First, you make a safe assumption called the *inductive hypothesis*: you assume the formula works perfectly for an unknown integer  $k$ .
  2. Second, you use that assumption to mathematically prove that the formula must also work for the next number,  $k + 1$ .

Algebraically, you show that the structure of the equation for step  $k$  naturally opens up an entry point to calculate step  $k + 1$ :

$$\text{Statement}(k) \implies \text{Statement}(k + 1) \tag{17}$$

### 2.3.3 Why Induction is Essential for the Euler-Maclaurin Proof

When we prove the Euler-Maclaurin formula in Section 4, we face a massive structural hurdle: the formula utilizes an infinite string of higher-order derivatives ( $f'(x), f''(x), f'''(x), \dots$ ). We cannot manually calculate Integration by Parts an infinite number of times on paper.

Instead, we use induction. We establish our Base Case by running Integration by Parts once to isolate the first derivative. Then, for the Inductive Step, we demonstrate that running

Integration by Parts on the  $k$ -th derivative cleanly generates the  $(k+1)$ -th derivative. By proving this underlying structural link, the domino effect takes over, validating the infinite expansion of the Euler-Maclaurin formula in one definitive, elegant sweep.

## 2.4 Complex Numbers?

To appreciate how advanced mathematical formulas navigate across equations, one must understand the environment in which they operate. For centuries, mathematics was confined to the “real” number line—the familiar continuum of positive numbers, negative numbers, fractions, and decimals. However, this one-dimensional worldview possessed a structural limitation: it could not handle the square root of a negative number.

### 2.4.1 Negative Square Root

In algebra, the square root of a number asks: “What number, multiplied by itself, gives this result?” For instance,  $\sqrt{4} = 2$  because  $2 \times 2 = 4$ . Crucially, if we attempt to find  $\sqrt{-4}$  on the traditional number line, we hit an immediate dead end:

- $2 \times 2 = 4$  (not  $-4$ )
- $-2 \times -2 = 4$  (not  $-4$ , because a negative times a negative yields a positive)

Because no real number multiplied by itself can ever produce a negative value, early mathematicians declared equations requiring negative square roots to be entirely “impossible” or “absurd.”

### 2.4.2 The Imaginary Unit

To break through this barrier, mathematicians in the 16th and 17th centuries took a leap of faith. They invented a brand-new mathematical building block called the **imaginary unit**, denoted by the symbol  $i$ . It is defined by a single, revolutionary rule:

$$i = \sqrt{-1} \implies i^2 = -1 \tag{18}$$

The term “imaginary” was originally coined by René Descartes as a historical insult, intended to mock the idea of numbers that did not exist in the physical world. However,  $i$  is no more imaginary than a negative number or a fraction; it is simply a tool that expands the mathematical universe.

One of the most elegant features of  $i$  is its perfectly cyclical nature when raised to higher powers. If we continuously multiply by  $i$ , the answers lock into a repeating pattern of four distinct outcomes:

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= i^2 \times i = (-1) \times i = -i \\ i^4 &= i^3 \times i = (-i) \times i = -i^2 = -(-1) = 1 \\ i^5 &= i^4 \times i = 1 \times i = i \end{aligned}$$

This pattern  $(i, -1, -i, 1)$  repeats endlessly into infinity, acting like a mathematical clock face.

### 2.4.3 The Structure and Algebra of Complex Numbers

A complex number is born when we combine a traditional real number with an imaginary number. It is written in the standard form:

$$z = a + bi \quad (19)$$

where  $a$  is the real number part and  $bi$  is the imaginary number part (with  $b$  representing a real coefficient). For example, in the complex number  $z = 3 + 4i$ , the real part is 3 and the imaginary part is  $4i$ .

While working with these hybrid numbers might seem intimidating, they obey the standard rules of high school algebra, provided we remember to replace  $i^2$  with  $-1$  whenever it appears.

**Addition Step-by-Step:** To add two complex numbers, we simply group the real components together and the imaginary components together, just like combining like terms in algebra:

$$\begin{aligned}(3 + 4i) + (2 + 5i) &= (3 + 2) + (4i + 5i) \\ &= 5 + 9i\end{aligned}$$

**Multiplication Step-by-Step:** To multiply two complex numbers, we expand the brackets using the standard FOIL method (First, Outer, Inner, Last):

$$\begin{aligned}(3 + 2i)(1 + 4i) &= (3 \times 1) + (3 \times 4i) + (2i \times 1) + (2i \times 4i) \\ &= 3 + 12i + 2i + 8i^2\end{aligned}$$

Now, we combine the middle imaginary terms ( $12i + 2i = 14i$ ) and apply our foundational rule ( $i^2 = -1$ ), turning the  $+8i^2$  into a real  $-8$ :

$$\begin{aligned}&= 3 + 14i + 8(-1) \\ &= 3 + 14i - 8 \\ &= -5 + 14i\end{aligned}$$

### 2.4.4 The Complex Plane and Geometric Rotations

The ultimate conceptual breakthrough occurs when we view complex numbers geometrically. Because a complex number  $z = a + bi$  consists of two separate pieces of data ( $a$  and  $b$ ), it cannot fit on a one-dimensional, left-to-right number line. Instead, it requires a two-dimensional grid called the complex plane (or Argand diagram).

In this system, the horizontal axis represents the Real Axis, and the vertical axis represents the Imaginary Axis. The complex number  $z = 3 + 4i$  is mapped exactly like a coordinate on a map: 3 steps along the real horizontal line, and 4 steps up along the imaginary vertical line.

This spatial transformation unlocks a beautiful geometric meaning behind multiplication. On a standard real number line, multiplying a number by  $-1$  represents a 180-degree reflection across zero (e.g.,  $+5$  flips over to the opposite side to become  $-5$ ).

In the complex plane, multiplying a number by  $i$  represents a perfect 90-degree counterclockwise rotation. Consider the following progression:

1. Start at the real number 1 on the horizontal axis.
2. Multiply by  $i$ :  $1 \times i = i$ . Geometrically, this rotates the point 90 degrees upward onto the vertical imaginary axis.
3. Multiply by  $i$  again:  $i \times i = i^2 = -1$ . This rotates the point another 90 degrees, landing it perfectly on  $-1$  on the horizontal real axis.

This explains the old mathematical rule of why  $i^2 = -1$ . Two consecutive 90-degree spatial rotations naturally combine to create a 180-degree turn, reversing your direction on the grid. By transforming static numbers into moving spatial coordinates, the complex plane allows advanced formulas to smoothly curve, pivot, and navigate around numerical dead-ends that would otherwise break standard arithmetic.

### 2.4.5 Complex Functions and Imaginary Exponents

A standard mathematical function, such as  $f(x) = x^2$ , can be viewed as an input-output machine: you drop in a real number like 3, and it spits out 9. A complex function, denoted as  $f(z)$ , operates on the exact same principle, but it accepts a two-dimensional complex coordinate as an input, processes it through an algebraic rule, and maps it to a brand-new coordinate on the complex plane.

This becomes vital when analyzing functions defined by exponents, such as the individual terms of the Riemann Zeta function, which take the general form  $n^{-s}$ , where the input exponent is a complex number  $s = a + bi$ . Using standard algebraic laws of exponents, we can split this expression into two distinct parts:

$$n^{-s} = n^{-(a+bi)} = n^{-a} \cdot n^{-bi} = \frac{1}{n^a} \cdot n^{-bi} \quad (20)$$

By separating the real and imaginary components, we can decode exactly what this complex operation does to a number step-by-step:

1. **The Real Number Component ( $\frac{1}{n^a}$ ):** This behaves according to traditional arithmetic. It scales the magnitude (the raw size) of the number. For example, if  $a = 2$ , then  $\frac{1}{2^2} = \frac{1}{4}$ . It dictates how close or far our coordinate sits relative to the origin  $(0, 0)$ .
2. **The Imaginary Number Component ( $n^{-bi}$ ):** This introduces a purely geometric behavior. Via Euler's Formula ( $e^{ix} = \cos x + i \sin x$ ), raising a base to a purely imaginary power does not alter the size of the number. Instead, it acts as a spatial steering wheel, spinning the coordinate in a perfect circle around the origin of the complex plane at a specific angle determined by the value of  $b$ .

Consequently, when a complex number  $s$  is passed into a sequence, it transforms every static term of the sum into a moving vector possessing both a specific physical length and a spatial rotation. This dynamic framework is what allows the Euler-Maclaurin formula to translate rigid, discrete blocks into fluidly rotating geometric pathways across the complex plane.

## 2.5 Analytic Continuation

Imagine you are an explorer mapping a newly discovered island. Stand on the shore, and you can see the coastline perfectly. However, a thick, heavy fog rolls in, completely obscuring the interior of the island. While you cannot see inland directly, you can look closely at the trajectories, shapes, and speeds of the rivers where they breach the coastline. By studying the mathematical patterns visible at the boundary, you can deduce exactly where those hidden rivers must flow deeper inland. You are using a visible territory to map an invisible one.

In advanced mathematics, analytic continuation is this exact process. Many of the most powerful formulas in calculus are highly restrictive; they only function properly within a small, safe boundary of allowed numbers (known as the function's *domain*). If you attempt to plug a number from outside this safe zone into the equation, the math completely breaks down, spitting out an impossible answer or cascading into infinity.

Analytic continuation is the rigorous technique of finding a *new, alternative equation* that gives the exact same answers as the old equation inside the safe zone, but remains stable, functional, and predictable when you start using numbers outside this safe zone.

### 2.5.1 Example: The Infinite Geometric Series

To see exactly how this works, let us look at a classic infinite sum. Consider the following progression, which we will call  $S(x)$ :

$$S(x) = 1 + x + x^2 + x^3 + x^4 + \dots \quad (21)$$

Let us test this equation by plugging in a safe number, like  $x = \frac{1}{2}$ :

$$\begin{aligned} S\left(\frac{1}{2}\right) &= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots \\ &= 1 + 0.5 + 0.25 + 0.125 + \dots \\ &= 2 \end{aligned}$$

The math behaves perfectly. Because each term gets progressively smaller, the infinite sum converges cleanly on the number 2.

Now, let us watch the equation completely collapse. What happens if we try to plug in a number outside the safe zone, such as  $x = 2$ ?

$$S(2) = 1 + 2 + 4 + 8 + 16 + \dots = \infty \quad (22)$$

The sum explodes to infinity. The equation cannot give us a meaningful number for  $x = 2$ . The safe zone for this specific infinite sum is strictly trapped between  $x = -1$  and  $x = 1$ .

### 2.5.2 Finding A Formula

To perform analytic continuation, we must find a different algebraic representation of this function that bypasses the infinite addition. Let us use a standard algebraic trick on our original sum:

$$S(x) = 1 + x + x^2 + x^3 + \dots$$

$$\text{Multiply both sides by } x : \quad x \cdot S(x) = x + x^2 + x^3 + x^4 + \dots$$

Now, let us subtract the second equation from the first equation:

$$S(x) - x \cdot S(x) = (1 + x + x^2 + x^3 + \dots) - (x + x^2 + x^3 + \dots)$$

Notice the massive, cascading cancellation on the right side. Every single higher-power term disappears, leaving behind only the number 1:

$$S(x)(1 - x) = 1 \quad (23)$$

Isolating  $S(x)$  gives us our alternative, closed-form equation:

$$S(x) = \frac{1}{1 - x} \quad (24)$$

### 2.5.3 Getting a Real Number

Equation 24 is a great example of analytic continuation. Let us test it inside our original safe zone at  $x = \frac{1}{2}$ :

$$S\left(\frac{1}{2}\right) = \frac{1}{1 - 1/2} = \frac{1}{1/2} = 2 \quad (25)$$

It yields 2, matching our infinite sum perfectly.

But look at what happens when we plug in our forbidden number,  $x = 2$ , which completely broke the original infinite sum:

$$S(2) = \frac{1}{1-2} = \frac{1}{-1} = -1 \quad (26)$$

Instead of exploding to infinity, our new equation smoothly processes the input and spits out a definitive, real answer:  $-1$ .

Mathematicians say that  $\frac{1}{1-x}$  is the *analytic continuation* of the infinite sum  $1 + x + x^2 + \dots$ . It acts as a bridge that allows us to assign meaningful, structured values to regions of space where the original math completely fails.

#### 2.5.4 The Identity Theorem

A common point of confusion is the question of whether this is just a trick. Can you invent any random equation to escape a boundary?

The answer is a strict no, governed by a foundational law in complex calculus called the Identity Theorem. The Identity Theorem states that if two complex functions give the exact same output values over any tiny, overlapping region of space, then they must be identical across the entire universe of numbers.

There is only *one unique way* to extend a function smoothly without creating jagged, unnatural breaks. It is this rigid uniqueness that allows the Euler-Maclaurin formula to take a discrete sum, transform it into a continuous calculus landscape, and extend it into the complex plane to redefine functions like the Riemann Zeta function.

### 3 The Euler-Maclaurin Summation Formula

Let  $f(x)$  be a sufficiently differentiable function on an interval of integers  $[a, b]$ . The formula states that the exact discrete sum of  $f(i)$  is given by:

**Theorem 3.1** (Euler-Maclaurin Summation).

$$\sum_{i=a}^b f(i) = \int_a^b f(x) dx + \frac{f(a) + f(b)}{2} + \sum_{k=1}^m \frac{B_{2k}}{(2k)!} \left( f^{(2k-1)}(b) - f^{(2k-1)}(a) \right) + R_m \quad (27)$$

where  $B_{2k}$  represents the even Bernoulli numbers,  $f^{(2k-1)}$  indicates the  $(2k-1)$ -th derivative of the function, and  $R_m$  represents the remainder error term.

### 4 Proof via Inductive Integration by Parts

The proof relies heavily on the concept of Integration by Parts (IBP),  $\int u dv = uv - \int v du$ , applied over unit intervals.

#### 4.1 Step 1: The Base Case on a Single Interval

Consider the continuous integral of a differentiable function  $f(x)$  over the single integer interval  $[0, 1]$ . We cleverly rewrite the integrand by introducing the first Bernoulli polynomial,  $B_1(x) = x - \frac{1}{2}$ , noting that  $B_1'(x) = 1$ .

$$\int_0^1 f(x) dx = \int_0^1 f(x) B_1'(x) dx \quad (28)$$

Applying IBP by setting  $u = f(x)$  and  $dv = B_1'(x)dx$ :

$$\int_0^1 f(x) dx = \left[ f(x)B_1(x) \right]_0^1 - \int_0^1 f'(x)B_1(x) dx \quad (29)$$

Evaluating the boundary term using  $B_1(1) = \frac{1}{2}$  and  $B_1(0) = -\frac{1}{2}$ :

$$\left[ f(x)B_1(x) \right]_0^1 = f(1) \left( \frac{1}{2} \right) - f(0) \left( -\frac{1}{2} \right) = \frac{f(0) + f(1)}{2} \quad (30)$$

Substituting this back and isolating the Trapezoid rule average yields:

$$\frac{f(0) + f(1)}{2} = \int_0^1 f(x) dx + \int_0^1 f'(x) \left( x - \frac{1}{2} \right) dx \quad (31)$$

Equation 31 demonstrates that the trapezoidal block equals the true integral plus a leftover curved sliver error.

## 4.2 Step 2: Higher Derivatives

To calculate the value of the leftover error integral, we run IBP on it a second time. We integrate  $(x - \frac{1}{2})$  to generate a quadratic helper polynomial  $g(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$ , where  $g(1) = g(0) = \frac{1}{12} = \frac{B_2}{2!}$ .

Running IBP passes the derivative onto  $f'(x) \rightarrow f''(x)$  and isolates our first derivative boundary correction:

$$\frac{f(0) + f(1)}{2} = \int_0^1 f(x) dx + \frac{1}{12} (f'(1) - f'(0)) - \int_0^1 f''(x)g(x) dx \quad (32)$$

Repeating this process inductively generates the sequence of higher-order boundary derivatives.

## 4.3 Step 3: Telescoping

In Steps 1 and 2, we successfully proved the Euler-Maclaurin relationship for a single, narrow block of width 1 (from  $x = 0$  to  $x = 1$ ). However, real-world problems require calculating massive graphs across a large interval, spanning from a starting integer  $a$  to an ending integer  $b$ .

To scale our formula up, we use a technique called stacking. Imagine lining up a series of narrow, 1-unit-wide blocks back-to-back along the  $x$ -axis, like building blocks. We write out the equation for each individual block and add them all together. When we do this, a telescoping sum occurs, causing almost every middle term to cleanly cancel out.

### 4.3.1 Example: $x = 1$ to $x = 4$

To see exactly how this works, let us look at a concrete example. Suppose we want to apply our formula across a curve from a starting point of  $a = 1$  to an ending point of  $b = 4$ . This space is made up of exactly three individual, 1-unit-wide intervals:

1. **Interval 1:** From  $x = 1$  to  $x = 2$
2. **Interval 2:** From  $x = 2$  to  $x = 3$
3. **Interval 3:** From  $x = 3$  to  $x = 4$

Let us write out the exact equation we derived in Step 2 for each of these three individual blocks, placing them one after the other:

**Block 1 (Interval  $[1, 2]$ ):**

$$\frac{f(1) + f(2)}{2} = \int_1^2 f(x) dx + \frac{1}{12}(f'(2) - f'(1)) - \text{Error}_1 \quad (33)$$

**Block 2 (Interval  $[2, 3]$ ):**

$$\frac{f(2) + f(3)}{2} = \int_2^3 f(x) dx + \frac{1}{12}(f'(3) - f'(2)) - \text{Error}_2 \quad (34)$$

**Block 3 (Interval  $[3, 4]$ ):**

$$\frac{f(3) + f(4)}{2} = \int_3^4 f(x) dx + \frac{1}{12}(f'(4) - f'(3)) - \text{Error}_3 \quad (35)$$

Now, let us add all three equations together. We will group the results into three separate math categories to watch how they behave: the continuous integrals, the boundary derivatives, and the discrete sum.

#### 4.3.2 Component 1: The Integrals Combine

When we add the continuous areas under the curve for each individual block, they fit together seamlessly like pieces of a puzzle:

$$\int_1^2 f(x) dx + \int_2^3 f(x) dx + \int_3^4 f(x) dx = \int_1^4 f(x) dx \quad (36)$$

Because the end of the first block connects perfectly to the start of the next, the three small integrals merge into one single, clean macro-integral covering the entire distance from 1 to 4.

#### 4.3.3 Component 2: The Derivatives Telescope

Next, let us look at what happens when we add the first-derivative correction terms together. Factoring out the shared  $\frac{1}{12}$  fraction, the combined string looks like this:

$$\frac{1}{12} \left[ (f'(2) - f'(1)) + (f'(3) - f'(2)) + (f'(4) - f'(3)) \right] \quad (37)$$

Notice the immediate cascading cancellation. We have a positive  $f'(2)$  and a negative  $-f'(2)$ , which cancel out to zero. We also have a positive  $f'(3)$  and a negative  $-f'(3)$ , which likewise cancel out to zero:

$$\frac{1}{12} \left[ \cancel{f'(2)} - f'(1) + \cancel{f'(3)} - \cancel{f'(2)} + f'(4) - \cancel{f'(3)} \right] \quad (38)$$

Every single derivative measurement taken in the interior middle of the graph vanishes completely. The only two terms that survive this algebraic wipeout are the absolute boundaries of our map:

$$\frac{1}{12} (f'(4) - f'(1)) \quad (39)$$

#### 4.3.4 Component 3: The Discrete Sum Accumulates

Finally, let us add the fractions on the left-hand side of our equations. These represent the heights of our discrete blocks:

$$\frac{f(1) + f(2)}{2} + \frac{f(2) + f(3)}{2} + \frac{f(3) + f(4)}{2} \quad (40)$$

Let us split these fractions apart to see how many of each term we have:

$$\left(\frac{1}{2}f(1) + \frac{1}{2}f(2)\right) + \left(\frac{1}{2}f(2) + \frac{1}{2}f(3)\right) + \left(\frac{1}{2}f(3) + \frac{1}{2}f(4)\right) \quad (41)$$

Grouping like terms together yields:

$$\frac{1}{2}f(1) + \left(\frac{1}{2}f(2) + \frac{1}{2}f(2)\right) + \left(\frac{1}{2}f(3) + \frac{1}{2}f(3)\right) + \frac{1}{2}f(4) \quad (42)$$

Because  $\frac{1}{2} + \frac{1}{2} = 1$ , the interior terms combine into whole integers, while the outer boundaries remain halved:

$$\frac{1}{2}f(1) + f(2) + f(3) + \frac{1}{2}f(4) \quad (43)$$

To turn this into a clean, traditional mathematical summation ( $\sum$ ), we can add another half of  $f(1)$  and another half of  $f(4)$  to make them whole, and then subtract them right back out at the end so we don't change the true value:

$$\sum_{i=1}^4 f(i) - \frac{f(1) + f(4)}{2} \quad (44)$$

#### 4.3.5 The Final Assembly

Bringing all three components back together into a single master equation gives us the complete Euler-Maclaurin formula for our interval from 1 to 4:

$$\sum_{i=1}^4 f(i) = \int_1^4 f(x) dx + \frac{f(1) + f(4)}{2} + \frac{1}{12}(f'(4) - f'(1)) - \text{Total Error} \quad (45)$$

By replacing the numbers 1 and 4 with the general variables  $a$  and  $b$ , we arrive at the classic, universal formula used by mathematicians worldwide:

$$\sum_{i=a}^b f(i) = \int_a^b f(x) dx + \frac{f(a) + f(b)}{2} + \sum_{k=2}^m \frac{B_k}{k!} (f^{(k-1)}(b) - f^{(k-1)}(a)) + R_m \quad (46)$$

This step completes the proof. We have explicitly demonstrated how a massive string of discrete additions can be compressed entirely into a smooth continuous integral, a simple boundary correction, and a series of edge-only derivatives.  $\square$

## 5 Applications in Modern Mathematics: The Riemann Zeta Function

Now that we have mapped out the mechanics of the Euler-Maclaurin formula, we can explore an application: where the formula serves as the foundational bridge that unlocked the Riemann Zeta function and altered our understanding of number theory.

### 5.1 The Zeta Function

The Riemann Zeta function, denoted by the Greek letter  $\zeta$ , is an infinite series that takes a complex number input  $s$  and processes it across the following rule:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (47)$$

As established in our foundations section, when the input  $s$  is a complex number, it converts every single fraction in this sum into a tiny, spinning arrow (or vector) on the complex plane.

If we plug in a number where the real part of  $s$  is greater than 1, the arrows shrink rapidly in size as they spin, causing the entire infinite web to loop tightly around a single, clean coordinate point. For instance, if we plug in  $s = 2$ , the series converges perfectly:

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6} \approx 1.6449 \quad (48)$$

However, if we attempt to pass a number where the real part of  $s$  is less than or equal to 1, the function goes to infinity. Consider what happens if we plug in  $s = -1$ :

$$\zeta(-1) = \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \dots = 1 + 2 + 3 + 4 + \dots = \infty \quad (49)$$

Because the numbers keep getting bigger, the sum explodes to infinity. To study the deep properties hidden within this function, Bernhard Riemann needed a solution to bypass this infinite addition. He found that solution by using the Euler-Maclaurin formula.

## 5.2 Using the Euler-Maclaurin Formula

To see exactly how the formula acts as a stabilizer, let us choose our function to be  $f(x) = x^{-s}$ . Instead of trying to add up terms to infinity all at once, let us use the Euler-Maclaurin formula to compute a finite sum from a starting block of  $n = 1$  to an ending block of  $N$ :

$$\sum_{n=1}^N n^{-s} = \int_1^N x^{-s} dx + \frac{1^{-s} + N^{-s}}{2} + \frac{1}{12}(f'(N) - f'(1)) + \dots \quad (50)$$

Let us calculate each component of this formula explicitly step-by-step so we can see how it restructures our problem.

### 5.2.1 Step 1: Evaluating the Continuous Integral

The first term on the right side transitions our discrete steps into a fluid, continuous calculus integral. Using the standard power rule for integration ( $\int x^n dx = \frac{x^{n+1}}{n+1}$ ), we find:

$$\int_1^N x^{-s} dx = \left[ \frac{x^{1-s}}{1-s} \right]_1^N \quad (51)$$

Evaluating this at the upper bound  $N$  and subtracting the lower bound 1 gives:

$$\int_1^N x^{-s} dx = \frac{N^{1-s}}{1-s} - \frac{1^{1-s}}{1-s} = \frac{N^{1-s}}{1-s} - \frac{1}{1-s} \quad (52)$$

To make this visually cleaner, we can factor out a negative sign from the denominators, changing  $1-s$  into  $s-1$ :

$$\int_1^N x^{-s} dx = \frac{1}{s-1} - \frac{N^{1-s}}{s-1} \quad (53)$$

### 5.2.2 Step 2: Evaluating the First Derivative Boundary Correction

Next, let us look at the first-derivative correction term. To find  $f'(x)$  for our function  $f(x) = x^{-s}$ , we bring the exponent down to the front and decrease the power by one:

$$f'(x) = -s \cdot x^{-s-1} = -\frac{s}{x^{s+1}} \quad (54)$$

Now, we evaluate this derivative at our two absolute boundaries,  $N$  and 1:

$$\begin{aligned}\text{At upper bound } N : \quad f'(N) &= -s \cdot N^{-s-1} \\ \text{At lower bound } 1 : \quad f'(1) &= -s \cdot (1)^{-s-1} = -s\end{aligned}$$

Subtracting the lower bound from the upper bound and multiplying by our Bernoulli fraction ( $\frac{1}{12}$ ) yields:

$$\frac{1}{12}(f'(N) - f'(1)) = \frac{1}{12}(-sN^{-s-1} - (-s)) = \frac{s}{12} - \frac{sN^{-s-1}}{12} \quad (55)$$

### 5.2.3 Step 3: Putting it Back Together

Let us plug our freshly calculated integral and derivative components back into the whole Euler-Maclaurin equation:

$$\sum_{n=1}^N \frac{1}{n^s} = \left( \frac{1}{s-1} - \frac{N^{1-s}}{s-1} \right) + \frac{1+N^{-s}}{2} + \left( \frac{s}{12} - \frac{sN^{-s-1}}{12} \right) + \dots \quad (56)$$

On the left side, we have our rigid, step-by-step summation. On the right side, the math has been entirely split into two distinct visual categories: static stable fractions ( $\frac{1}{s-1}$ ,  $\frac{1}{2}$ ,  $\frac{s}{12}$ ) and terms containing the tracking variable  $N$  ( $N^{1-s}$ ,  $N^{-s}$ ,  $N^{-s-1}$ ).

## 5.3 Achieving Analytic Continuation

We are now ready to perform **analytic continuation**. Let us send our tracking variable to infinity ( $N \rightarrow \infty$ ) to map out the entire infinite Zeta function.

If we assume we are in the original safe zone ( $\text{Real}(s) > 1$ ), any variable raised to a negative power collapses to zero as it grows to infinity (e.g.,  $N^{-2} = \frac{1}{N^2} \rightarrow 0$ ). Therefore, look at what happens to all the  $N$  terms in Equation 56 as  $N \rightarrow \infty$ :

- $N^{1-s} \rightarrow 0$
- $N^{-s} \rightarrow 0$
- $N^{-s-1} \rightarrow 0$

Every single term pinned to the moving upper boundary vanishes into nothingness. We are left with an alternative, smooth equation for the Zeta function:

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \dots \quad (57)$$

According to the laws of the Identity Theorem we proved in the foundations section, because this new equation matches our original sum perfectly inside the safe zone, it must represent the true, un-fogged map of the function across the entire universe of numbers.

But notice a monumental change: Equation 57 has completely shed the restriction of the divergence wall! There are no longer any expanding fractions or infinite loops on the right side. We can now plug in numbers that were previously completely forbidden.

### 5.3.1 Resolving the Paradox

Let us test our new analytic continuation formula by plugging in our forbidden input,  $s = -1$ , which originally caused the series to explode into  $1 + 2 + 3 + 4 \dots = \infty$ :

$$\zeta(-1) = \frac{1}{(-1)-1} + \frac{1}{2} + \frac{(-1)}{12} - 0 \quad (58)$$

(Note: All higher Bernoulli correction terms after  $\frac{s}{12}$  contain higher derivatives of  $x^{-s}$ . For  $s = -1$ , our function is  $f(x) = x^1$ , meaning its second derivative and all derivatives beyond it are exactly zero, causing the rest of the infinite string to drop off perfectly.)

Let us run the basic arithmetic explicitly:

$$\begin{aligned}\zeta(-1) &= \frac{1}{-2} + \frac{1}{2} - \frac{1}{12} \\ &= -\frac{1}{2} + \frac{1}{2} - \frac{1}{12} \\ &= 0 - \frac{1}{12} \\ &= -\frac{1}{12}\end{aligned}$$

Through the architectural machinery of the Euler-Maclaurin summation formula, we have proven that the analytic continuation of the string of all positive integers yields exactly  $-\frac{1}{12}$ .

By using calculus to subtract away the infinity of the continuous integral ( $\frac{N^{1-s}}{1-s}$ ), the Euler-Maclaurin formula acts as a mathematical filter, getting rid of the infinity and giving us a nice tangible value. Without this specific formula acting as the initial bridge to smooth out discrete sequences, Bernhard Riemann could never have mapped the complex coordinates of this function, leaving the distribution of prime numbers forever obscured behind the divergence wall.

## 6 Conclusion

The transition from adding up individual numbers to working with continuous calculus curves shows how closely connected different areas of mathematics really are. The Euler-Maclaurin summation formula provides a direct link between discrete sums ( $\sum$ ) and continuous integrals ( $\int$ ), showing that we can use the tools of calculus to analyze and simplify standard arithmetic additions.

As demonstrated throughout this paper, the formula works because its core pieces are designed to coordinate with each other. By using Bernoulli polynomials, the formula ensures that the derivatives break down in a predictable pattern during integration by parts. Combined with mathematical induction, this structure allows us to evaluate a long or infinite series of numbers by focusing primarily on the data at the absolute boundaries of the interval, rather than calculating every single point in the middle.

When applied to the Riemann Zeta function, this formula acts as a practical tool to bypass the limitations of standard addition. By breaking the function down into a continuous integral and a set of boundary corrections, we can separate the terms that head toward infinity from the stable numbers left behind. This provides a clear path to analytic continuation, allowing us to evaluate the function for numbers that would otherwise cause the original sum to break down, such as proving that  $\zeta(-1) = -\frac{1}{12}$ .

Ultimately, the Euler-Maclaurin formula demonstrates how discrete sequences can be translated into continuous frameworks. By connecting algebra, calculus, and complex numbers, it remains a reliable tool for approximating values, evaluating infinite series, and understanding how functions behave across different domains.

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