A COMPREHENSIVE TREATMENT

Regularity and the Fourier Series

How the smoothness of a periodic function governs coefficient decay, convergence, and the Gibbs phenomenon

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ROADMAP

Outline

1 **Foundations**
coefficients, orthonormal basis, Parseval

2 **Regularity ⇒ decay**
Riemann–Lebesgue, the differentiation rule

3 **Decay ⇒ convergence**
M-test, C^1 uniform convergence

4 **Jumps & the Dirichlet kernel**
pointwise convergence, Lebesgue constants

5 **The Gibbs phenomenon**
the persistent 8.95% overshoot

6 **Case studies**
square, triangle, analytic: $1/n$, $1/n^2$, e^{-an}

7 **The Basel problem**
Parseval gives $\sum 1/n^2 = \pi^2/6$

8 **Fejér & summability**
averaging cures the pathologies

The logical spine: **regularity ⇒ coefficient decay ⇒ convergence**

SECTION 01

Foundations

Functions on the circle, Fourier coefficients, and the L^2 geometry

The circle, function spaces, and coefficients

We study 2π-periodic functions on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, with the normalized measure $dx/2\pi$.

Nested regularity classes:

$C^k \subset \dots \subset C \subset L^2 \subset L^1$.

Every integrable f has Fourier coefficients; the partial sum $S_N f$ is its symmetric truncation.

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n)\, e^{inx}$$

An orthonormal basis of exponentials

The exponentials **$e_n(x) = e^{inx}$** are orthonormal and complete in $L^2(\mathbb{T})$.

Coefficients are simply projections onto this basis: $\hat{c}_n = \langle f, e_n \rangle$.

Completeness ⇒ the partial sums converge to f in the L^2 norm for every $f \in L^2$.

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, dx$$

$$\langle e_m, e_n \rangle = \delta_{mn}, \qquad e_n(x) = e^{inx}$$

Bessel, Parseval, and the L^2 isometry

Bessel's inequality holds for any orthonormal system; completeness upgrades it to **Parseval's identity**. The coefficient map is an isometry $\ell^2 \cong L^2$ — “energy” is conserved.

$$\sum_{n=-\infty}^{\infty} \left|\widehat{f}(n)\right|^2 = \|f\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

Consequence: $S_N f \to f$ in L^2 for every $f \in L^2$. We will spend this identity twice — for convergence (§3) and to solve Basel (§7).

SECTION 02

Regularity Controls Decay

The smoother the function, the faster its Fourier coefficients vanish

The Riemann–Lebesgue lemma

Theorem (Riemann–Lebesgue). If $f \in L^1(\mathbb{T})$, its Fourier coefficients vanish at infinity:

$$\left|\widehat{f}(n)\right| \to 0 \qquad (|n| \to \infty)$$

Proof idea. Trigonometric polynomials are dense in L^1, and a trig polynomial has only finitely many nonzero coefficients — so the claim is exact for them and passes to the L^1 limit. (For smooth f it also follows from a single integration by parts.)

The differentiation rule — the engine of decay

Integrate by parts once: a derivative becomes multiplication by **in**. Boundary terms cancel by periodicity.

Iterating k times is exactly what turns k derivatives into k powers of decay.

$$\widehat{f'}(n) = in\,\widehat{f}(n)$$
$$\widehat{f^{(k)}}(n) = (in)^k\,\widehat{f}(n)$$

this single identity drives every decay estimate that follows

Smoothness ⇒ decay, quantitatively

Theorem. If $f \in C^k(\mathbb{T})$ then $\hat{c}_n = o(|n|^{-k})$. Sharp form: if $f^{(k-1)}$ is absolutely continuous and $f^{(k)} \in L^1$, then

$$f \in C^k(\mathbb{T}) \;\Rightarrow\; \widehat{f}(n) = o(|n|^{-k})$$
$$\left|\widehat{f}(n)\right| \le \|f^{(k)}\|_1 \, |n|^{-k}$$

Proof. Apply the differentiation rule k times, then Riemann–Lebesgue to $f^{(k)}$. The sharp hypothesis is precisely membership in the Sobolev space $W^{k,1}$ — absolute continuity is what makes the boundary terms legitimate.

The analytic endpoint & the decay dictionary

$$f \text{ analytic} \quad \Leftrightarrow \quad \left|\widehat{f}(n)\right| \le C e^{-a|n|} \ (a > 0)$$

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = \frac{1 - r^2}{1 - 2r\cos x + r^2}$$

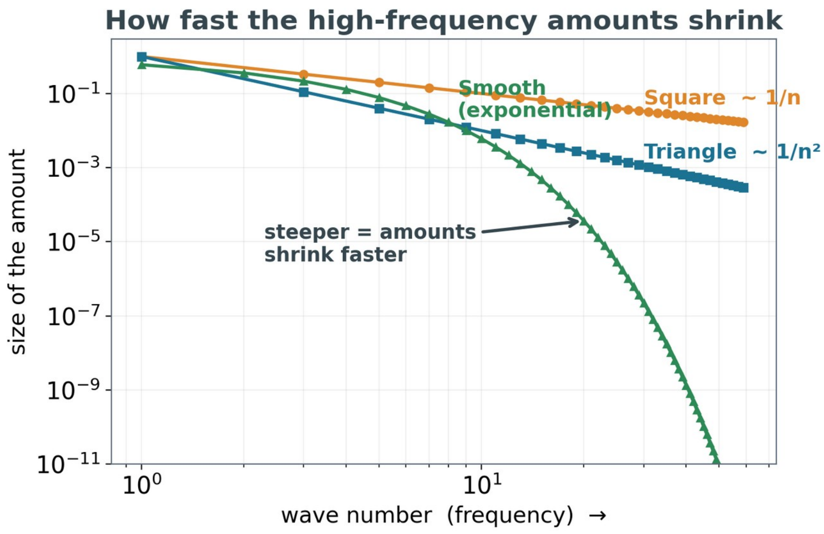
Decay dictionary

jump → $1/n$

corner → $1/n^2$

C^k → $1/n^k$

analytic → e^{-an}



steeper slope = faster decay

SECTION 03

Decay Controls Convergence

From the size of the coefficients to uniform and absolute convergence

Modes of convergence & the M-test

Convergence comes in flavors — **pointwise, uniform, L^2, absolute** — forming a strict hierarchy.

The Weierstrass M-test gives the cleanest sufficient condition: if the coefficients are absolutely summable, the series converges absolutely and uniformly to a continuous function.

$$\sum_n \left|\widehat{f}(n)\right| < \infty \quad \Rightarrow \quad S_N f \to f$$

A uniform limit of continuous partial sums is continuous — so summable coefficients force a continuous sum.

C^1 ⇒ absolute & uniform convergence

Theorem. If $f \in C^1(\mathbb{T})$, its Fourier series converges to f both absolutely and uniformly.

$$\sum_{n\neq 0} |\hat{f}(n)| \leq \Big(\sum_{n\neq 0} \frac{1}{n^2}\Big)^{1/2} \Big(\sum_{n\neq 0} n^2 |\hat{f}(n)|^2\Big)^{1/2} = \Big(\frac{\pi^2}{3}\Big)^{1/2} \|f'\|_2 < \infty$$

Proof: Cauchy–Schwarz + Parseval applied to f′. A clean *sufficient* condition — not sharp. The constant $\sum 1/n^2 = \pi^2/3$ is pinned down exactly in §7 (Basel).

SECTION 04

Jumps and Pointwise Convergence

The Dirichlet kernel and what happens at a discontinuity

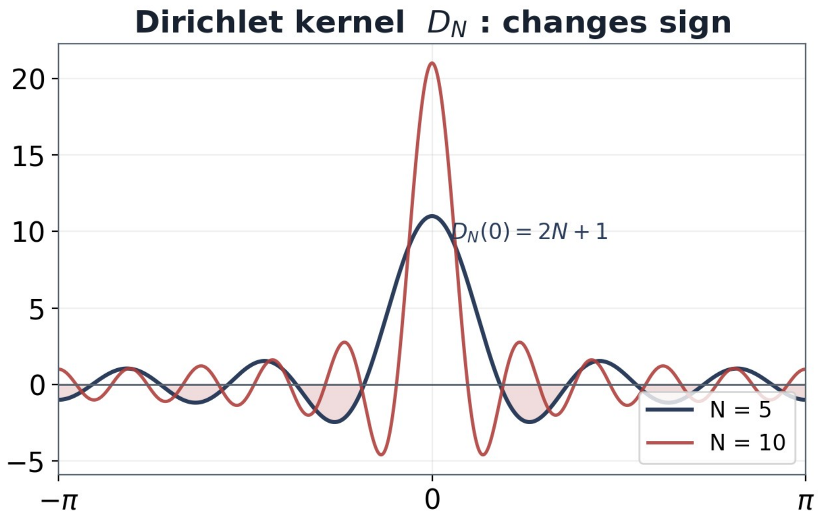
The Dirichlet kernel

Partial sums are convolution with the Dirichlet kernel D_N.

It **changes sign** and its L^1 norm grows — the source of all subtlety at jumps.

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin\left((N+\frac{1}{2})x\right)}{\sin(x/2)}$$

$$S_N f = D_N * f = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, D_N(t)\, dt$$



negative lobes drive the overshoot

Dirichlet's pointwise convergence theorem

Theorem (Dirichlet–Jordan). If f is piecewise C^1 (or of bounded variation), then at every point the partial sums converge to the average of the one-sided limits — and to f(x) wherever f is continuous:

$$S_N f(x) \to \frac{f(x^+) + f(x^-)}{2}$$

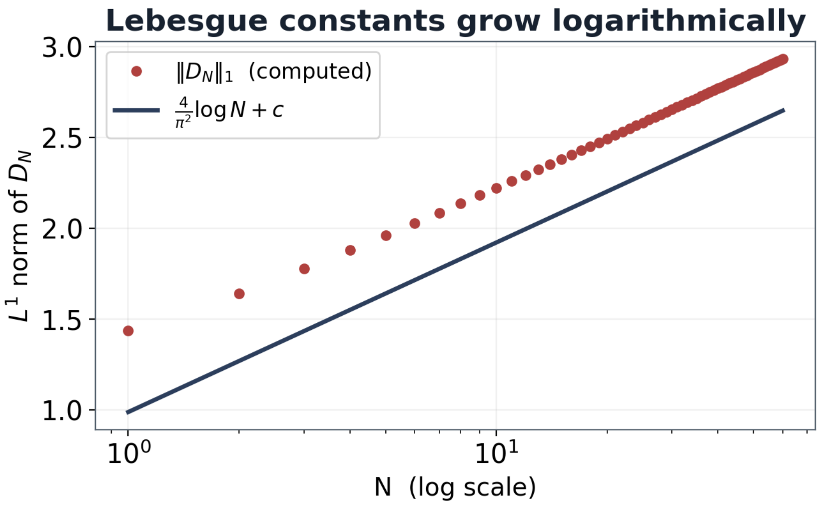
Proof idea. Localize with the kernel (its mass concentrates at the origin) and apply Riemann–Lebesgue to the difference quotient. The symmetric truncation is what produces the midpoint at a jump.

Lebesgue constants & failure of uniform convergence

The L^1 norms of the Dirichlet kernels — the **Lebesgue constants** — diverge logarithmically:

$$\|D_N\|_1 = \frac{4}{\pi^2}\log N + O(1) \;\to\; \infty$$

Consequence (du Bois-Reymond): there exist continuous functions whose Fourier series diverge at a point. Continuity alone gives neither uniform nor everywhere-pointwise convergence.



L^1 norm → ∞ as $(4/\pi^2)\log N$

SECTION 05

The Gibbs Phenomenon

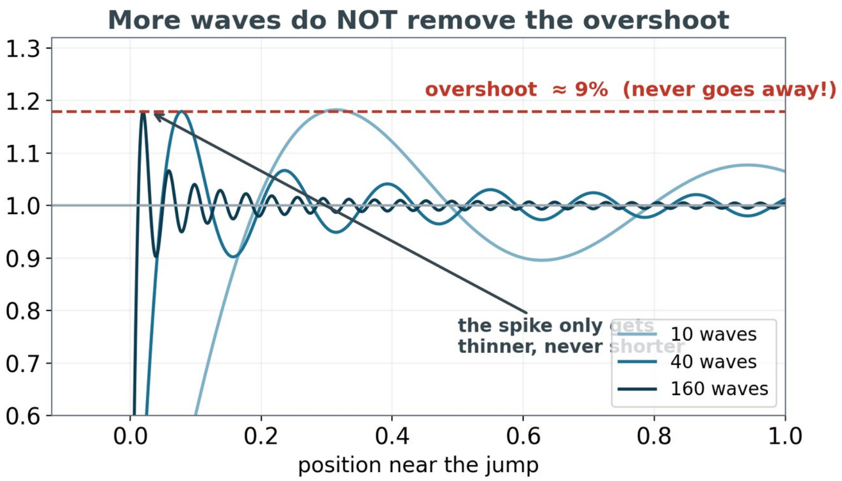
A persistent overshoot that refines, but never disappears

The model: the sawtooth

Model case: the sawtooth, with a jump of height π at the origin.

Watch the partial sums pile up near the discontinuity.

$$g(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2} \qquad (0 < x < 2\pi)$$



Computing the overshoot

Evaluate the partial sum at x = π/N. The sum is a **Riemann sum** for the sine-integral — giving a limiting peak and a fixed overshoot fraction, independent of N.

$$S_N g\left(\tfrac{\pi}{N}\right) \to \int_0^{\pi} \frac{\sin t}{t}\, dt = \mathrm{Si}(\pi) \approx 1.8519$$

$$\frac{\mathrm{Si}(\pi)}{\pi} - \tfrac{1}{2} \approx 0.0895 \quad (8.95\%)$$

$$\text{peak} = \tfrac{2}{\pi}\, \mathrm{Si}(\pi) \approx 1.1790$$

Universality & the L^2 picture

Universal overshoot

8.95%

By localization, every jump of a piecewise-smooth function overshoots by this same fraction of the jump height. It narrows with N — but never shortens.

Yet convergence in energy

In L^2, $S_N f \to f$ for every finite-energy f: the spike's width $\to 0$, so its energy $\to 0$.

Gibbs is a pointwise / uniform effect — never an L^2 one.

SECTION 06

Case Studies

Square, triangle, and analytic — the trichotomy $1/n$, $1/n^2$, exponential

Case 1 — Square wave (a jump: 1/n)

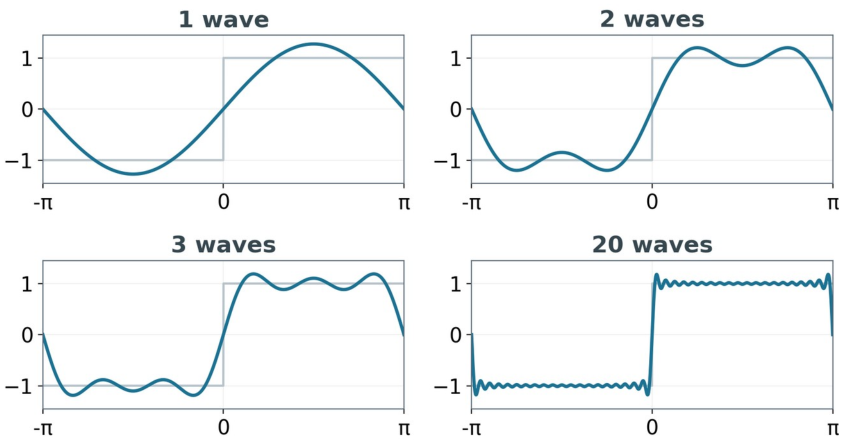
f = sgn(sin x). Odd ⇒ a pure sine series, computed once.

Coefficients **~ 1/n**: not absolutely summable, Gibbs present, midpoint convergence at the jumps.

$$f = \operatorname{sgn}(\sin x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{2k+1}$$

$$\widehat{f}(n) = \frac{2}{\pi i n}(1-(-1)^n) = O(n^{-1})$$

Adding simple sine waves builds the shape



partial sums pile up near the jump

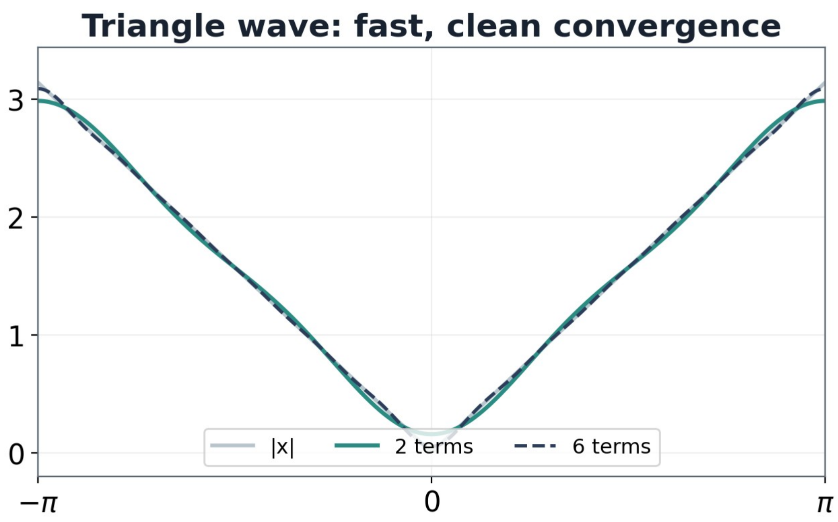
Case 2 — Triangle wave (a corner: $1/n^2$)

f = |x|. Continuous, with a corner at 0.

f′ = sgn is bounded-variation with O(1/n) coefficients, so by the differentiation rule $\hat{c}_n$ = **$O(1/n^2)$**. Absolutely summable ⇒ uniform convergence, no Gibbs.

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}$$

$$\widehat{f}(n) = O(n^{-2}) \quad (\text{via} \ \ \widehat{f'}(n) = in\,\widehat{f}(n))$$



6 terms already fit almost perfectly

Case 3 — Analytic (Poisson kernel: exponential)

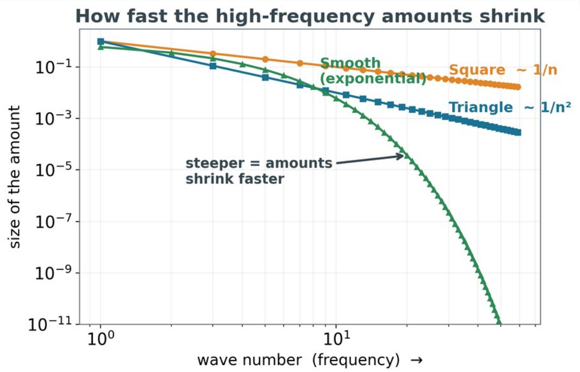
The Poisson kernel P_r is real-analytic; its coefficients $r^{|n|}$ decay exponentially. The three cases together realize the decay dictionary.

$$\widehat{P_r}(n) = r^{|n|} \quad \Rightarrow \quad \left|\widehat{P_r}(n)\right| \le e^{-a|n|}, \quad a = \log(1/r)$$

Square · jump · **1/n**

Triangle · corner · **$1/n^2$**

Analytic · smooth · **e^{-an}**



SECTION 07

Application: The Basel Problem

Parseval's identity evaluates $\sum 1/n^2 = \pi^2/6$

Basel via Parseval

Apply Parseval to the sawtooth f(x) = x on (−π, π).

Its coefficients and its energy are both elementary:

$$\widehat{f}(n) = \frac{(-1)^{n+1}}{in}, \qquad \left|\widehat{f}(n)\right|^2 = \frac{1}{n^2}$$

$$\|f\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

Result (Euler, 1735)

$$2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3} \implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Same method, triangle wave:

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

SECTION 08

Cesàro Summability & Fejér's Theorem

Averaging the partial sums cures the pathologies

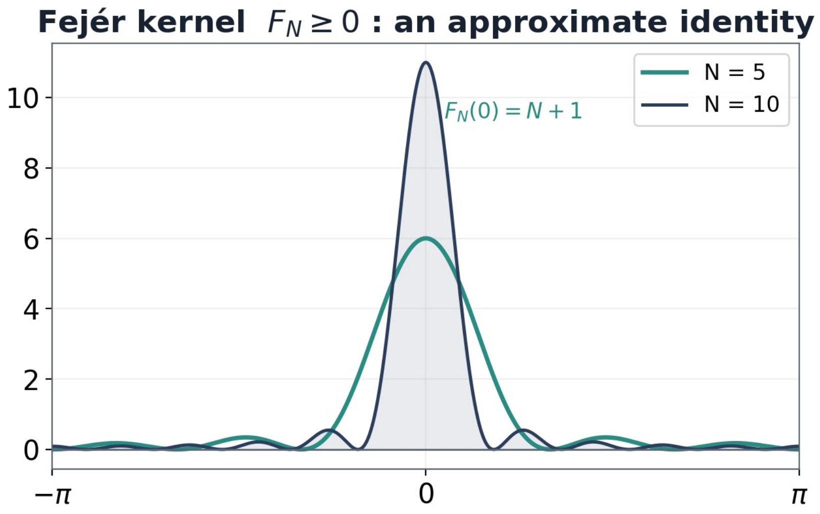
The Fejér kernel

Cesàro means average the partial sums: $\sigma_N f = F_N * f$.

The Fejér kernel is a **nonnegative** approximate identity — the crucial contrast with Dirichlet.

$$\sigma_N f = \frac{S_0 f + S_1 f + \cdots + S_N f}{N+1} = F_N * f$$

$$F_N(x) = \frac{1}{N+1}\left(\frac{\sin((N+1)x/2)}{\sin(x/2)}\right)^2 \geq 0$$



always ≥ 0, mass concentrates at 0

Fejér's theorem & its consequences

Theorem (Fejér). If f is continuous on $\mathbb{T}$, then $\sigma_N f \to f$ uniformly.

$$f \in C(\mathbb{T}) \quad \Rightarrow \quad \sigma_N f \to f \quad (\text{uniformly})$$

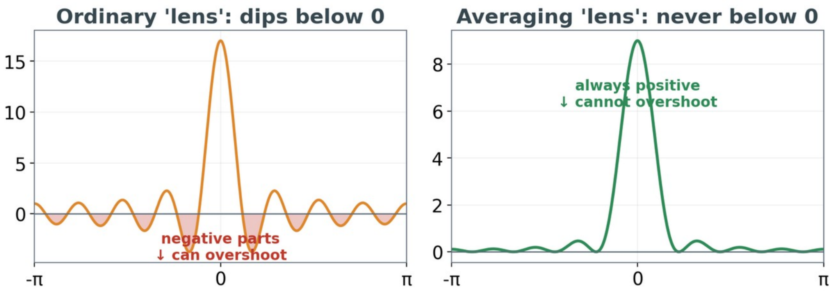
Consequences. (i) Weierstrass trigonometric approximation — continuous functions are uniform limits of trig polynomials. (ii) Uniqueness: if all $\hat{c}_n = 0$ then f = 0. (iii) Recovery exactly where the Dirichlet kernel fails.

Proof idea: a nonnegative, concentrating kernel of unit mass (a “good kernel”) makes convolution converge to the identity.

Why averaging cures Gibbs

Because $F_N \geq 0$, the Cesàro mean is a genuine weighted average — it cannot exceed the extremes of f, so no overshoot. The **sign of the kernel is the whole story**: Dirichlet changes sign; Fejér does not. (At a jump, σ_N f still → the midpoint.)

$$F_N \geq 0 \;\;\Rightarrow\;\; \inf f \;\leq\; \sigma_N f \;\leq\; \sup f$$



SYNTHESIS

The big picture

Regularity
C^k, H^s, analytic

Coefficient decay
$1/n^k$, e^{-an}

→

Convergence
uniform, L^2, a.e.

Dirichlet kernel — changes sign
Gibbs overshoot · log-divergent Lebesgue constants

Fejér kernel — nonnegative
no overshoot · uniform Cesàro convergence

Exact scale:

$$f \in H^s(\mathbb{T}) \iff \sum_n (1+n^2)^s \left|\widehat{f}(n)\right|^2 < \infty$$

Beyond, & references

Where this goes next

- Carleson (1966): the Fourier series of an L^2 function converges a.e.
- Kolmogorov: an L^1 function with a.e. divergent series.
- Higher dimensions & multiplier theory.
- Wavelets & time–frequency analysis.
- The FFT — Fourier analysis made computational.

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Thank you

Questions & discussion

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