Matrix Tree Theorem

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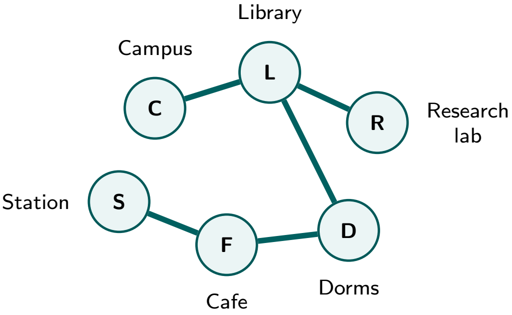
What are Spanning Trees, Why count them?



- It is a network that reaches every point using the minimum number of connections.
- Imagine building roads between buildings so every building can be reached, but there are no unnecessary circular routes.
- Spanning trees are widely used in optimization, circuits, and networks.
- Counting them shows how many minimal ways a network can stay connected, helping measure reliability and resilience.

What Is the Matrix Tree Theorem?

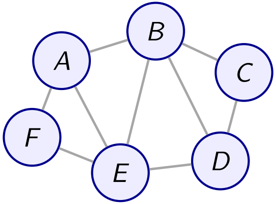
Core statement

The theorem counts spanning trees using a determinant from the graph Laplacian.

$$\tau(G) = \det\left(L^{(i)}\right)$$

- $L^{(i)}$ is formed by deleting one row and matching column from L.
- It was proved by Gustav Kirchhoff in 1847 while studying electrical circuits.
- It connects graph theory with linear algebra in a very efficient way.

Graphs, Vertices, and Edges

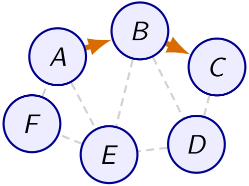


- A graph is $G = (V, E)$: a mathematical model for objects and their connections.
- Vertices are the objects or points in the network: $V = \{A, B, C, D, E, F\}$.
- Edges are the connections between pairs of vertices.
- Here $E = \{AB, BC, CD, DE, EF, FA, BE, BD, AE\}$.
- Connected means every vertex can be reached from every other vertex by following edges.
- Undirected means edges have no direction; for example, $AB = BA$.

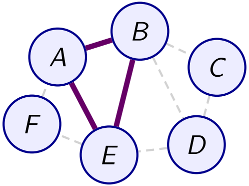
For the Matrix Tree Theorem, we focus on finite connected undirected graphs.

Degree, Path, and Cycle

Path $A \to B \to C$

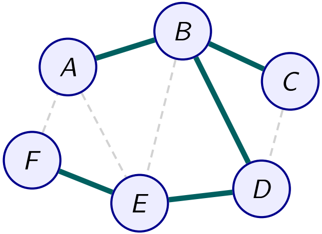


Cycle $A \to B \to E \to A$



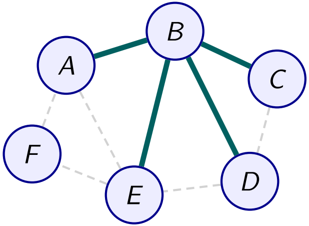
- Degree: number of edges touching a vertex.
- In our graph: $\deg(A) = 3$, $\deg(B) = 4$, $\deg(C) = 2$.
- All degrees: $(3, 4, 2, 3, 4, 2)$ for A, B, C, D, E, F.
- Path: a sequence of adjacent vertices with no repeated edges.
- Cycle: a closed path returning to its starting vertex.
- Acyclic means having no cycles.

Tree and Spanning Tree



- Tree: connected and acyclic.
- A tree on n vertices has $n-1$ edges.
- Spanning tree: keeps all vertices, selects tree edges from G.

Degree of a Vertex



- The degree of a vertex is the number of edges touching it.
- In the graph, vertex B touches A, C, D, and E.
- Therefore, $\deg(B) = 4$.
- The full degree list is

$$(3, 4, 2, 3, 4, 2)$$

for A, B, C, D, E, F.

Adjacency Matrix

Vertex order: A, B, C, D, E, F. The adjacency matrix records whether two vertices are connected: 1 means there is an edge, and 0 means there is no edge.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

For example, the entry in row B, column D is 1 because BD is an edge.

Degree Matrix

The degree matrix places each vertex degree on the diagonal and puts zeros everywhere else. With vertex order A, B, C, D, E, F, the degrees are $3, 4, 2, 3, 4, 2$.

$$D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Laplacian Matrix

The Laplacian combines degree information and adjacency information. It is defined by

$$L = D - A.$$

Diagonal entries are degrees; off-diagonal entries are -1 when two vertices are adjacent.

$$L = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 3 & -1 & 0 \\ -1 & -1 & 0 & -1 & 4 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Matrix Tree Theorem

Theorem (Kirchhoff)

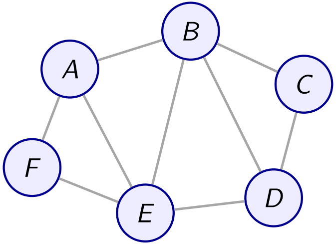
Let G be a finite connected undirected graph with Laplacian matrix L. Delete any one row and the corresponding column from L to form a smaller matrix $L^{(i)}$. Then

$$\tau(G) = \det\left(L^{(i)}\right),$$

where $\tau(G)$ is the number of spanning trees of G.

- The deleted row and column can be chosen freely.
- The determinant is always the same for every choice.
- This turns counting trees into a linear algebra calculation.

Result for the Reference Graph



$$\det\left(L^{(F)}\right) = 55.$$

Conclusion

The reference graph has exactly 55 spanning trees.

Proof Ideas and Applications

Proof viewpoints

- Cauchy–Binet theorem
- Random walks / Markov chains
- Generating functions
- Homology methods

Applications

- Circuits: effective resistance and current flow.
- Reliability: count backup connection patterns.
- Probability: random spanning trees and random walks.
- Chemistry: molecular stability through graph structure.
- Networks: measure redundancy without creating cycles.

Key Takeaways

- Spanning trees are minimal connected subgraphs.
- The Laplacian matrix is $L = D - A$.
- Deleting one row and column gives a reduced Laplacian.
- Its determinant counts spanning trees.
- For the reference graph, $\tau(G) = 55$.

Thank you!

Questions?