Block Designs

Theory, Constructions, and Algebraic Connections

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Guest Lecture Series

"How can we arrange experiments so that every pair of treatments is compared equally often?"

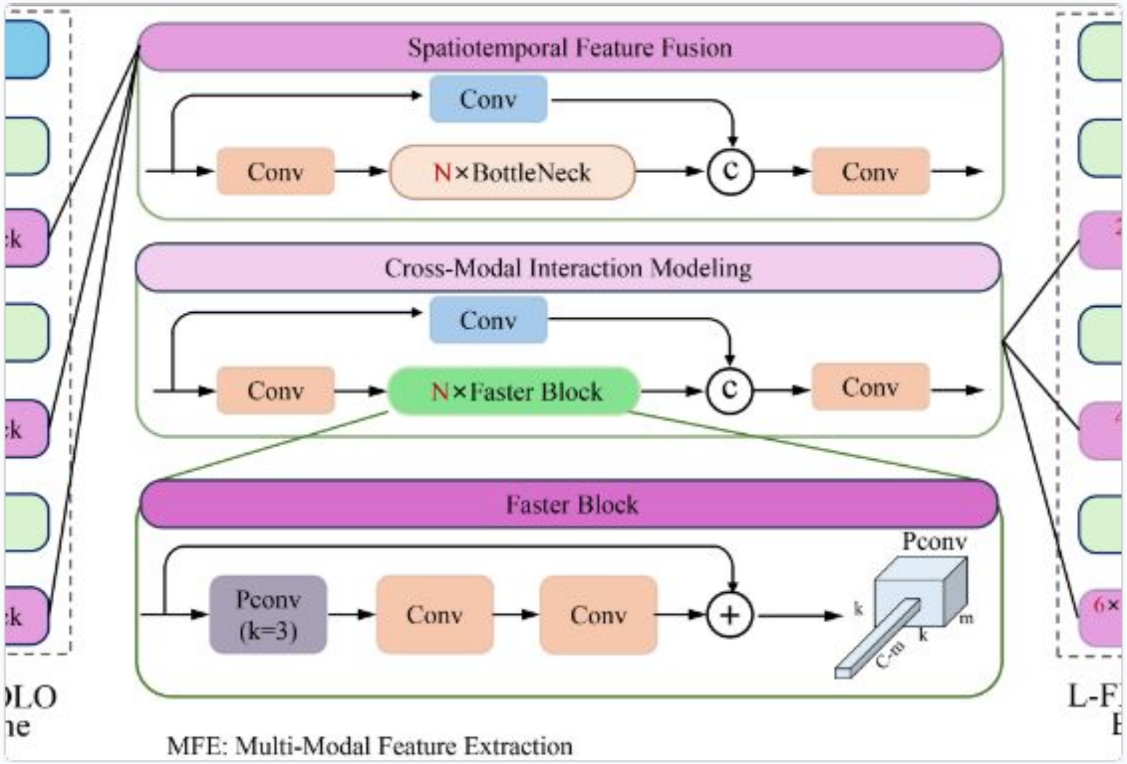
ORIGINS AND MOTIVATION

Block designs originated from the statistical **Theory of Experimental Design**, seeking optimal ways to compare treatments while minimizing systematic bias.

Agriculture: Testing fertilizers across varying soil plots.

Coding Theory: Construction of error-correcting codes.

Scheduling: Tournament layouts and network design.



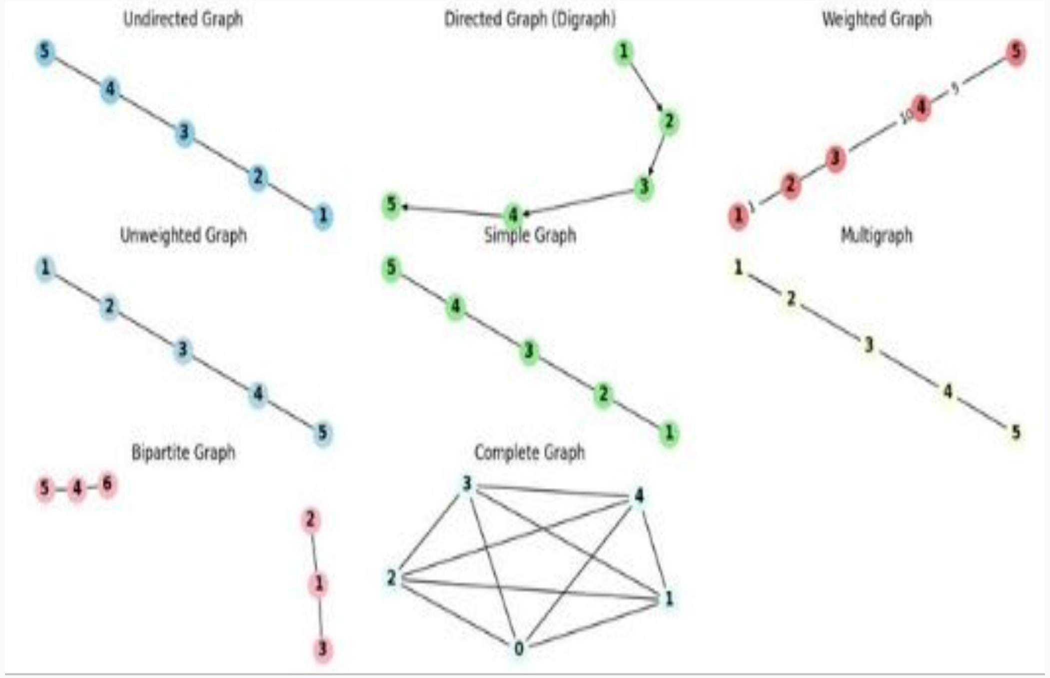
THE INCIDENCE FRAMEWORK

An **Incidence Structure** is a triple where: (P, B, I)

- **P:** Finite set of Points.
- **B:** Finite collection of subsets (Blocks).
- **I:** The relation defining Point-Block inclusion.

Duality Principle:

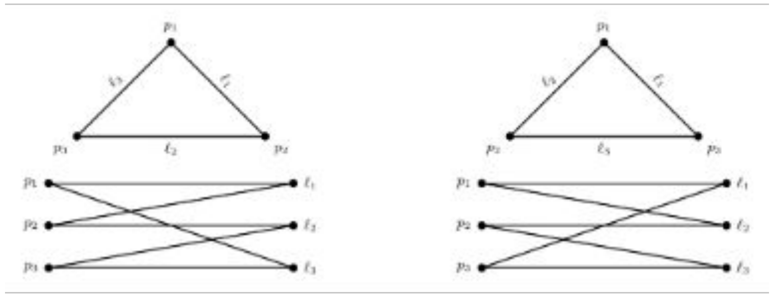
Interchanging points and blocks produces a dual structure D^*, often preserving key structural parameters.



Understanding t-Designs

Notation: t-(v, k, λ) **Design**

- **Generalizing Balance:** A t-design consists of a set of *v* points and *k*-sized blocks where every subset of *t* points appears in exactly λ blocks.
- **Structural Hierarchy:** They are the most general form of a block design, also denoted as $\mathcal{D} = (V, \mathcal{B})$
- **Cryptographic Utility:** Crucial for secure key distribution and threshold schemes, ensuring uniform representation of sub-keys.
- **Coding Theory:** Used to construct optimal error-correcting codes through the study of weight distributions in linear codes.



The Fano Plane: A classic Steiner Triple System or 2-(7, 3, 1) design.

THE FUNDAMENTAL PARAMETERS

Symbol	Definition	Combinatorial Meaning
v	Points Count	Total number of unique treatments or elements
b	Blocks Count	Total number of trials or subsets
r	Replication	Number of blocks containing each point
k	Block Size	Number of points within each block
λ	Index	Times every pair of distinct points occurs together

Comprehensive Classification Table

A unified view of combinatorial conditions and design categories

Summary of Design Types

Category	Conditions
BIBDS	A $2-(v,k,\lambda)$ design with $\lambda < v$.
STS	A $2-(v,3,1)$ design, denoted by $\mathrm{STS}(v)$.
Witt	The design $S(5,8,24)$, notable for its automorphism group M_{24}.
Symmetric Designs	A 2-design where $v = b$.
Simple / Complete	Distinct blocks / containing all $\binom{v}{k}$ possible blocks.
Decomposable	$\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2$ where both parts are designs.
Hadamard	A $2-(4n-1, 2n-1, n-1)$ design for $n \geq 2$.

Primary Design Families

Defining the fundamental structures of combinatorial design theory

Simple vs. Complete Designs

Simple Designs

A design structure where every block in the collection is unique.

Condition: Each block $B_i \in B$ *is distinct.*

Complete Designs

The exhaustive case of a simple design where all possible k-subsets are present.

Condition: Contains $\binom{v}{k}$ *blocks.*

Hadamard Designs

$4n-1$

Point Parameter

Large-Scale Families

These designs arise from **Hadamard Matrices** and form an infinite family of symmetric BIBDs.

Parameters: $2\text{-}(4n-1, 2n-1, n-1)$

Defined for all integers $n \geq 2$.

Decomposability & Hierarchy

Decomposable

The block set B is a disjoint union $B_1 \sqcup B_2$ where each part forms its own design.

Derived

Generated from a symmetric design by isolating block intersections.

Subdesign

A structural subset where $V' \subseteq V$ and $B' \subseteq B$ form a valid design.

ESSENTIAL IDENTITIES

Regularity Identity

Counting total incidences between points and blocks in two different ways:

$$vr = bk$$

Every point (v) in r blocks = Every block (b) has k points.

Pairwise Balance

Fixing a point and counting pairings with all other (v-1) points:

$$r(k-1) = \lambda(v-1)$$

This determines that r is uniquely fixed by the other parameters.

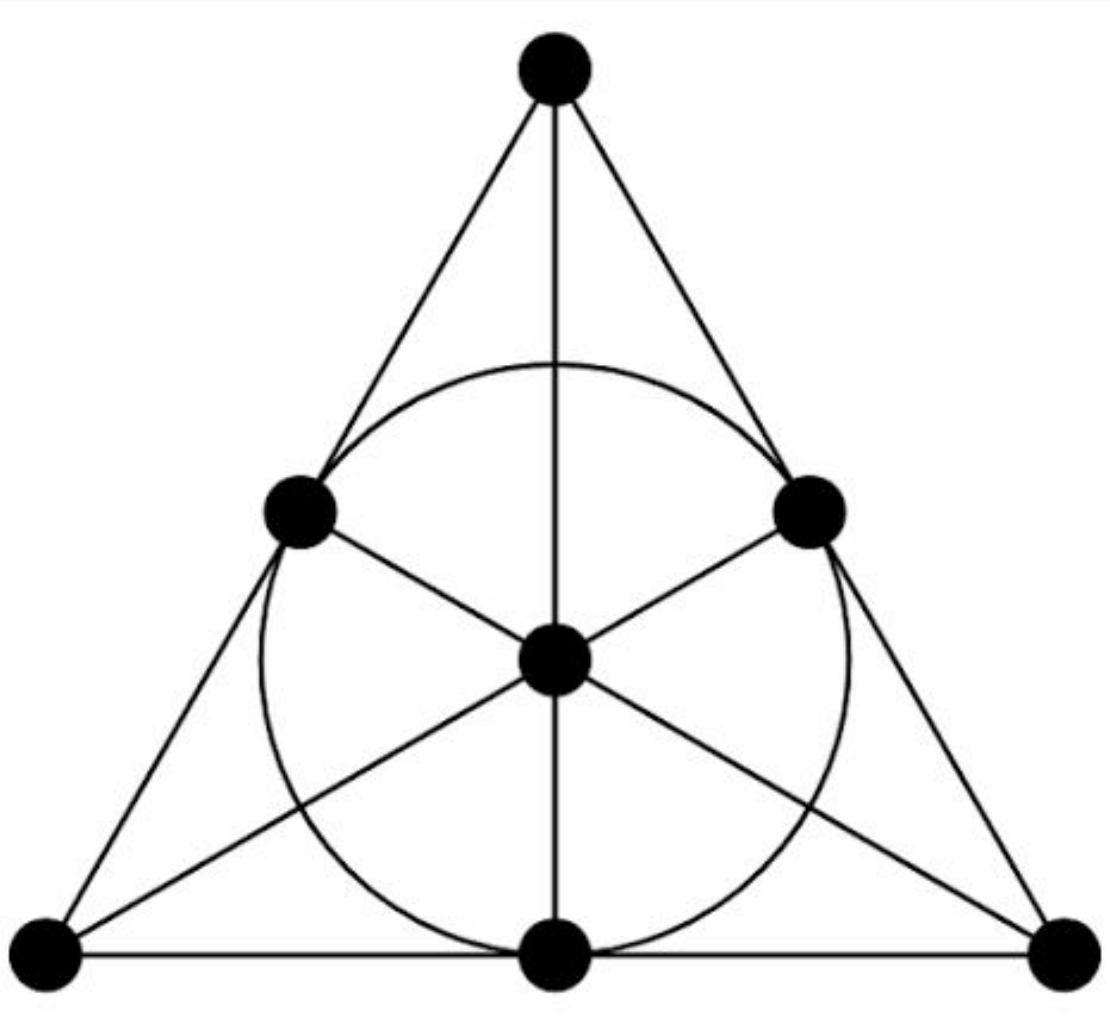
Fano Plane Geometry

THE FANO PLANE

The smallest non-trivial BIBD and archetypal **Finite Projective Plane**.

- Parameters: $(7,7,3,3,1)$
- 7 points, 7 lines, 3 points per line.
- Every point belongs to 3 lines.
- Every pair of points defines exactly 1 line.

It is a symmetric design since v = b.



STEINER SYSTEMS & TRIPLETS

t-Designs
Generalizes balance: every subset of t points occurs in λ blocks.

Steiner Triple Systems
2-(v, 3, 1) designs. STS(v) exists iff $v \equiv 1, 3 \bmod 6$.

Witt Designs
Specialized S(5, 8, 24) linked to the Mathieu group M_{24}.

Kirkman's Schoolgirl Problem: A classic puzzle leading to Steiner Triple Systems.

15 schoolgirls walk out daily in 5 rows of 3 for 7 days. How can they be arranged so that no two girls walk in the same row together more than once?

STEINER TRIPLE SYSTEMS

STS(v) Existence
A Steiner Triple System exists if and only if $v \equiv 1$ or $3 \pmod 6$.

Moore Construction
Combining STS(u) and STS(v) to form STS(u(v-w)+w).

2v + 1 Method
A recursive method to generate larger triple systems from base designs.

SYMMETRY AND DUALITY

CASE EQUALITY

b = v

Forcing $r = k$

Symmetric designs occupy a central position in theory due to **Fisher's Inequality** ($b \geq v$).

> **Bruck-Ryser-Chowla Theorem:** Imposes Diophantine restrictions on existence for odd v:
>
> $$x^2 = (k - \lambda) y^2 + (-1)^{\frac{v-1}{2}} \lambda z^2$$

ALGEBRA OF INCIDENCE

The combinatorial balance of a BIBD is reflected in the algebraic structure of its **Incidence Matrix (M)**.

$$MM^T = (r - \lambda) I + \lambda J$$

- **Rank:** $\text{rank}(M) = v$
- **Eigenvalues:** r^2 and $r - \lambda$

This spectrum provides a complete characterization: a symmetric structure is a BIBD iff it possesses this specific matrix signature.

ALGEBRAIC CONSTRUCTIONS

Algebra provides systematic methods for constructing designs using **Group Theory** and **Finite Fields**.

Automorphism Groups: Symmetries that preserve incidence.

Difference Sets: Constructing designs from subsets of additive groups.

Singer's Theorem: Every finite projective plane has a cyclic automorphism group.

Finite Fields: GF(q) provides coordinate systems for affine and projective spaces.

The Recipe

1. Select a Base Block (subset of a group).

2. Apply Group Action (translation/addition).

3. Generate Orbit of Blocks.

Result: A symmetric design.

FINITE GEOMETRIES

Projective Planes

Parameters:

$$(q^2 + q + 1,\ q^2 + q + 1,\ q + 1,\ q + 1,\ 1)$$

Every two distinct points determine a unique line; every two lines intersect in a unique point.

Affine Planes

Order q yields:

$$v = q^2,\quad k = q,\quad b = q^2 + q$$

Obtained by removing the "line at infinity" from a projective plane.

Summary & Research

- ✓ **Wilson's Theorem & Keevash (2014):** Proving existence for large v.
- ✓ **Spectral Methods:** Design signatures via eigenvalues.
- ✓ **Core Takeaway:** Local constraints create deep connections between algebra, geometry, and engineering.

Thank you for your attention.

Questions?

IMAGE SOURCES

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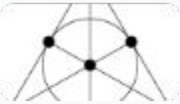

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