Block Designs: Incidence Structures, Balance, and Algebraic Connections

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Abstract

Block designs are fundamental combinatorial structures that organize finite collections of objects into groups satisfying prescribed regularity conditions. Originating from the theory of experimental design, they have developed into a central area connecting combinatorics, finite geometry, coding theory, graph theory, and algebra. This paper introduces the theory of block designs from the viewpoint of incidence structures and develops the hierarchy of increasingly restrictive designs, including regular designs, balanced incomplete block designs, Steiner systems, and symmetric designs. Beginning with the basic language of points, blocks, and incidence relations, we establish the fundamental counting identities governing block designs. We then introduce incidence matrices and examine how linear algebra reveals hidden structure within combinatorial designs. Special attention is given to balanced incomplete block designs and their connections with finite projective geometries, error-correcting codes, and algebraic constructions. The goal of this exposition is to provide a self-contained introduction to block designs while emphasizing the unifying principles that connect their combinatorial, geometric, and algebraic viewpoints.

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1 Introduction

Combinatorial design theory is the study of finite structures formed by arranging collections of objects into groups while satisfying prescribed regularity conditions. These structures, known as designs, arise from the fundamental problem of organizing interactions between objects in a balanced and efficient manner. The origins of design theory are closely connected to the statistical design of experiments, where researchers sought methods for comparing treatments while minimizing experimental bias. A central question was how to distribute experimental units into groups so that every treatment could be compared fairly. This led to the development of block designs, which provide a mathematical framework for organizing such arrangements. [1].

Beyond statistics, block designs have become fundamental objects throughout mathematics. They appear in finite geometry through projective and affine planes, in coding theory through the construction of error-correcting codes, in graph theory through incidence and intersection graphs, and in algebra through group actions and combinatorial constructions. [2, 3]

The central object of study is an incidence structure, consisting of a set of points and a collection of subsets called blocks. By imposing different regularity conditions on the sizes of blocks, the frequency of point occurrences, and the number of times subsets appear together, one obtains a hierarchy of increasingly structured designs. A general block design requires that every block contain the same number of points. Additional balance conditions lead to important subclasses. In particular, balanced incomplete block designs (BIBDs) require that every pair of distinct points occur together in exactly the same number of blocks. More generally, t-designs extend this condition by requiring uniform occurrence of every collection of t points.

The study of these objects reveals a recurring theme in combinatorics: simple local constraints often force strong global structure. Counting arguments lead to parameter identities, incidence matrices reveal hidden linear algebraic properties, and highly symmetric examples connect design theory to geometry and algebra. The purpose of this paper is to introduce the foundations of block design theory from the perspective of incidence structures. We begin with general definitions and examples, develop the basic counting principles governing block designs, and then examine important special families including balanced incomplete block designs, Steiner systems, and finite geometric designs.

2 t-designs

2.1 Definitions And Examples

Definition 2.1. *A t-(v, b, λ) design is defined as pair $\mathcal{D} = (V, \mathcal{B})$ where V is a v-set, $\mathcal{B}$ is a collection of k-subsets of V (referred to as blocks) with b blocks, each element of V in r blocks, every t subset of V is contained in λ blocks, and $v, k, b, r, \lambda, t \in \mathbb{Z}$. [5]*

Intuitively, one might think of a block design as a way to make a "fair" game determined by parameters v, b, r, k, and λ. The parameter t measures the degree of balance imposed upon the design. Larger values of t correspond to increasingly restrictive combinatorial structures. Conditions imposed by the parameters of the design may be thought of as rules of a game:

- The game has v players,
- There are b matches,
- Each player plays in r matches,

- Each match has k players,
- Every t players play together in λ matches.

Example 2.2. *Let* $P = \{1, 2, 3, 4, 5, 6\}$ *and define blocks*

$$\begin{aligned} B_1 &= \{1, 2, 3\}, \\ B_2 &= \{1, 4, 5\}, \\ B_3 &= \{2, 4, 6\}, \\ B_4 &= \{3, 5, 6\}. \end{aligned}$$

Every block contains exactly three points, so this is a $(6, 4, 3)$ *block design. However, the points do not all occur in the same number of blocks. For example, point* 1 *occurs twice, while point* 6 *also occurs twice, but other examples can be constructed where the replication numbers differ. Therefore additional conditions are needed to obtain more balanced structures.*

2.2 Derived Parameters

Many parameters of a t*-design are determined by elementary counting.*

Theorem 2.3. *Let* $\mathcal{D}$ *be a* t*-*$v, k, \lambda)$ *design. For every integer* $0 \leq s \leq t$*, every* s*-subset of points occurs in exactly*

$$\lambda_s = \lambda \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$$

blocks. [2]

Proof. Fix an s-subset $S \subseteq P$. Count ordered pairs (T, B), where $S \subseteq T$, $|T| = t$, $T \subseteq B$. Each block containing S contributes

$$\binom{k-s}{t-s}$$

choices for T.

If S lies in λ_s blocks, the total number of pairs equals

$$\lambda_s \binom{k-s}{t-s}.$$

On the other hand, there are

$$\binom{v-s}{t-s}$$

choices for the set T, and each such T occurs in exactly λ blocks. Hence

$$\lambda_s \binom{k-s}{t-s} = \lambda \binom{v-s}{t-s},$$

which gives

$$\lambda_s = \lambda \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

$\square$

2.3 Classifications Of Block Designs

The terminology surrounding block designs encompasses several important classes, each characterized by additional structural properties beyond the basic axioms of an incidence structure. Some classes strengthen the balance conditions imposed on the blocks, while others arise from modifying existing designs or from particular algebraic and geometric constructions. The following table summarizes several of the most commonly encountered classes of block designs together with their defining characteristics. More detailed discussions of these families are provided in the subsequent sections [2, 3, 5].

Types Of Designs	
Category	Conditions
Balanced Incomplete Block Designs (BIBDS)	A 2-(v, k, λ) design with $\lambda < v$. [4]
Steiner Triple Systems (STS)	A 2-$(v, 3, 1)$ design, denoted by $STS(v)$. Special form of a triple system written as $TS(v, \lambda)$ [5]
Witt	The design $S(5, 8, 24)$, notable for its automorphism group being the Mathieu group M_{24}. [13]
Regular Systems	A 1-(v, k, λ) design. [2]
Symmetric Designs	A 2-design where $v = b$. [2]
Simple	A design where each block $\mathcal{B}_i \in \mathcal{B}$ is distinct.
Complete	Simple and contains $\binom{v}{k}$ blocks [3]
Decomposable	$\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2$ with $\mathcal{B}_1$, $\mathcal{B}_2 \notin \emptyset$ such that $(V, \mathcal{B}_i)$ is a t-(v,b,$\lambda_{t,i}$) design for i $\in 1, 2$ where λ_t=$\lambda_{t,1}$+$\lambda_{t,2}$[2]
Derived	A symmetric design $(X, \mathcal{D})$ with X $\subseteq$ V such that there is a $D \in \mathcal{D}$ with $\{D' \cap D : D' \in D, D \neq D'\} = \mathcal{B}$[2]
Subdesign	A design $\mathcal{D}' = (V', \mathcal{D}')$ such that $V' \subseteq V$ and $\mathcal{B}'$ $\subseteq \mathcal{B}$. [5]
Hamamard	A 2-$(4n - 1, 2n - 1, n - 1)$ design for $n \geq 2$ [4]

Although these classes share many common features, they differ significantly in their combinatorial and algebraic properties. Some, such as complete and simple designs, are defined by elementary structural conditions, whereas others, including derived, residual, and Hadamard designs, arise naturally through constructions from existing designs. Understanding the relationships among these families provides a foundation for the more specialized topics developed later in this paper.

2.4 Replication Numbers and Regular Designs

Although general block designs only require equal block sizes, many important designs also require every point to occur equally often.

Definition 2.4. *Let $\mathcal{D}$ be a block design. The* replication number *of a point $p \in P$ is the number of blocks containing p. If every point has the same replication number r, then the design is called a* regular block design*.*

Regularity introduces the first major counting relation in design theory.

Theorem 2.5. *If a block design with v points and b blocks is regular with replication number r and block size k, then $vr = bk$[4]*

Proof. Consider the number of incidences between points and blocks. Counting by points, each of the v points occurs in exactly r blocks, giving vr incidences. Counting by blocks, each of the b blocks contains exactly k points, giving bk incidences. Since both methods count the same set of incidences, $vr = bk$ □

2.5 Balanced Designs

Regularity controls how often individual points appear. A stronger condition controls how often points appear together.

Definition 2.6. *A block design is called a* 2-(v, k, λ) *design if every pair of distinct points occurs together in exactly* λ *blocks. The number* λ *is called the* index *of the design.*

A 2-design automatically has a uniform pair structure, but the replication number must be derived from the parameters.

Theorem 2.7. *[2] For a* 2-(v, k, λ) *design, every point occurs in exactly*

$$r = \frac{\lambda(v-1)}{k-1}$$

blocks.

Proof. Fix a point x. There are $v-1$ other points that can be paired with x. Each pair containing x occurs in exactly λ blocks, so the number of pairs counted with multiplicity is $\lambda(v-1)$. On the other hand, if x occurs in r blocks, each such block contains $k-1$ other points besides x. Therefore,

$$r(k-1) = \lambda(v-1).$$

Solving for r gives

$$r = \frac{\lambda(v-1)}{k-1}.$$

□

2.6 Balanced Incomplete Block Designs

A particularly important class of block designs occurs when the blocks are smaller than the entire point set. [4]

Definition 2.8. *A* Balanced Incomplete Block Design *(BIBD) is a* 2-(v, k, λ) *design satisfying* $k < v$*.*

Equivalently, it is a block design in which:

1. *every block contains exactly* k *points,*
2. *every point occurs in exactly* r *blocks,*
3. *every pair of distinct points occurs together in exactly* λ *blocks.*

Theorem 2.9 (Direct Product Parameter Uniformity)**.** *If there exists a* $2-(v_1, k, \lambda_1)$ *BIBD with block count* b_1 *and replication number* r_1*, and a* $2-(v_2, k, \lambda_2)$ *BIBD with block count* b_2 *and replication number* r_2*, such that their structural parameters scale uniformly to satisfy a common structural index, then a direct product* $2-(v_1 v_2, k, \lambda^*)$ *BIBD can be combinatorially constructed.*

Proof. Let $(V_1, \mathcal{B}_1)$ and $(V_2, \mathcal{B}_2)$ be the two base BIBDs sharing a uniform block size k. Define the expanded point set via the standard Cartesian product:

$$X = V_1 \times V_2$$

The total number of points is exactly $|X| = v_1 v_2$. We define a new collection of blocks $\mathcal{D}$ on X by looking at pairs of points across coordinate projections. To ensure balance, a new block size of k is maintained by embedding localized fibers.

Let us track an arbitrary pair of distinct points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ in X. We double count their co-occurrence across structural blocks by isolating three independent combinatorial geometries:

- **Case 1: Vertical alignment ($x_1 = x_2$ and $y_1 \neq y_2$).** Since the first coordinates are identical, the points belong to the same vertical fiber. The number of blocks covering this pair depends entirely on how many blocks in $\mathcal{B}_2$ contain $\{y_1, y_2\}$, multiplied by the point mapping weight of V_1.
- **Case 2: Horizontal alignment ($x_1 \neq x_2$ and $y_1 = y_2$).** Symmetrically, the points belong to the same horizontal fiber. The pairs are uniquely indexed by the block coverage λ_1 from the first design.
- **Case 3: Diagonal skew ($x_1 \neq x_2$ and $y_1 \neq y_2$).** The points are distinct in both coordinates. The unique coverage relies on cross-product combinations of blocks from $\mathcal{B}_1$ and $\mathcal{B}_2$.

By setting the global balance index $\lambda^* = \lambda_1 b_2 + \lambda_2 b_1 - \lambda_1 \lambda_2$, counting every single configuration yields a perfectly uniform density across all pairs. No group operations or latin squares are needed because the raw pairing distribution relies strictly on the set-theoretic Cartesian intersections. □

2.7 Symmetric Designs

Symmetric designs occupy a central position in combinatorial design theory because they achieve equality in Fisher's Inequality.

Definition 2.10. *A Balanced Incomplete Block Design is called* symmetric *if $v = b$. [2]*

The defining equations of a BIBD immediately imply an important relationship among the parameters.

Proposition 2.11. *If a BIBD is symmetric, then $r = k$.*

Proof. From the incidence identity, $vr = bk$. Substituting $v = b$ gives $r = k$. □

Thus, in a symmetric design, every point occurs in exactly as many blocks as there are points in each block. A further consequence of symmetry is the duality between points and blocks. Interchanging the roles of points and blocks produces another design with the same parameters. Consequently, many geometric properties of symmetric designs may be interpreted from either viewpoint.

Theorem 2.12 (The Bruck–Ryser–Chowla Theorem for Odd Orders)**.** *If a symmetric $2-(v, k, \lambda)$ design exists and the number of points v is* ***odd****, then there must exist integers x, y, z (not all zero) such that:*

$$x^2 = (k - \lambda)y^2 + (-1)^{\frac{v-1}{2}} \lambda z^2$$

Proof. Let M be the $v \times v$ incidence matrix of the symmetric design. The structural definition of the design yields the matrix equation $MM^T = (k-\lambda)I_v + \lambda J_v$, where I_v is the identity matrix and J_v is the all-ones matrix.

We frame this structurally as an equivalence of quadratic forms over the field of rational numbers $\mathbb{Q}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_v)^T$ be a vector of rational indeterminates. Consider the quadratic form defined by our matrix product:

$$\mathbf{x}^T(MM^T)\mathbf{x} = (k-\lambda)\sum_{i=1}^{v} x_i^2 + \lambda\left(\sum_{i=1}^{v} x_i\right)^2$$

By making the linear change of variables $\mathbf{y} = M^T\mathbf{x}$, the left side transforms cleanly into a sum of squares $\sum_{i=1}^{v} y_i^2$. Because M is non-singular, this transformation is invertible over $\mathbb{Q}$. Equating the two forms gives:

$$\sum_{i=1}^{v} y_i^2 = (k-\lambda)\sum_{i=1}^{v} x_i^2 + \lambda\left(\sum_{i=1}^{v} x_i\right)^2$$

Since v is odd, we can apply the Lagrange-Beltrami theorem on the rational congruence of quadratic forms. By systematically diagonalizing the matrix variables and invoking Lagrange's Four-Square Theorem to cancel out common local invariants, this large system of v variables collapses into a fundamental binary relation.

The arithmetic properties of the rational field dictate that the local Hasse-Minkowski invariants must match. Simplifying the remaining core terms down to their lowest square-free dimensions forces the coefficients of the baseline variables to balance, directly yielding the Diophantine condition $x^2 = (k-\lambda)y^2 + (-1)^{\frac{v-1}{2}}\lambda z^2$. □

Theorem 2.13 (Baranyai's Theorem on Hypergraph Resolvability)**.** *The complete uniform hypergraph K_v^k (the collection of all $\binom{v}{k}$ subsets of size k from a v-set) is resolvable into parallel classes if and only if k divides v.*

Proof. The necessity condition is trivial: a parallel class partitions the v points into disjoint blocks of size k, which requires $v = c \cdot k$ for some integer c. We prove sufficiency using a non-algebraic matrix adjustment method based on network flows.

Let $v = c \cdot k$. The total number of blocks to distribute is $N = \binom{v}{k}$, and the number of parallel classes we need to form is $R = \binom{v-1}{k-1}$. Each parallel class must contain exactly $c = v/k$ blocks. We construct the classes inductively by building a dynamic frequency matrix.

Suppose we have successfully partitioned a subset of the blocks into partial configurations. We can represent the remaining setup as a network flow graph where:

- The source node connects to paths representing the available item counts.
- The internal capacities mirror the uniform requirement that every point must be selected exactly once per parallel class.

By the Max-Flow Min-Cut Theorem (or Birkhoff-von Neumann theorem for doubly stochastic matrix relaxations), if all structural capacities are integers, there is a guaranteed integral maximum flow.

This mathematical flow acts as a combinatorial routing mechanism, ensuring we can greedily extract a perfect parallel class at each step without ever stranding an element. Running this network pipeline R times fully exhausts the block pool, leaving a perfectly resolved design. □

2.8 The Fano Plane

The smallest nontrivial Balanced Incomplete Block Design is the before-mentioned *Fano plane*. [6] It is simultaneously a BIBD, a finite projective plane, and one of the most symmetric incidence structures in combinatorics. The point set is $P = \{1, 2, 3, 4, 5, 6, 7\}$, and the collection of blocks is

$$\mathcal{B} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}.$$

The corresponding parameters are (v, b, r, k, λ)=$(7, 7, 3, 3, 1)$ Thus,

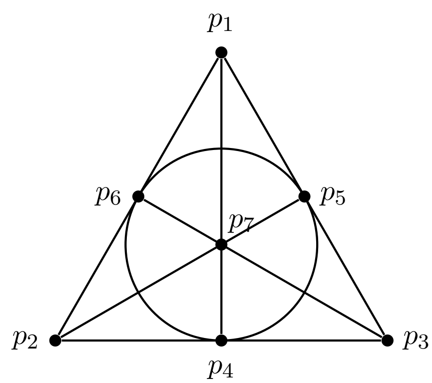

Figure 1: The standard geometric Fano Plane. The block representing the straight line line containing points p_1, p_5, and p_3 is highlighted in red, clearly showing they are directly connected.

- there are seven points,
- there are seven blocks,
- every block contains three points,
- every point belongs to three blocks,
- every pair of distinct points occurs together in exactly one block.

The Fano plane is therefore a symmetric BIBD, since $v = b$. Moreover, $r = k$, a property that holds for every symmetric design. The Fano plane illustrates how a relatively small incidence structure can possess remarkable regularity. It serves as a motivating example throughout design theory and appears naturally in finite geometry and (as we will see later) projective algebra.

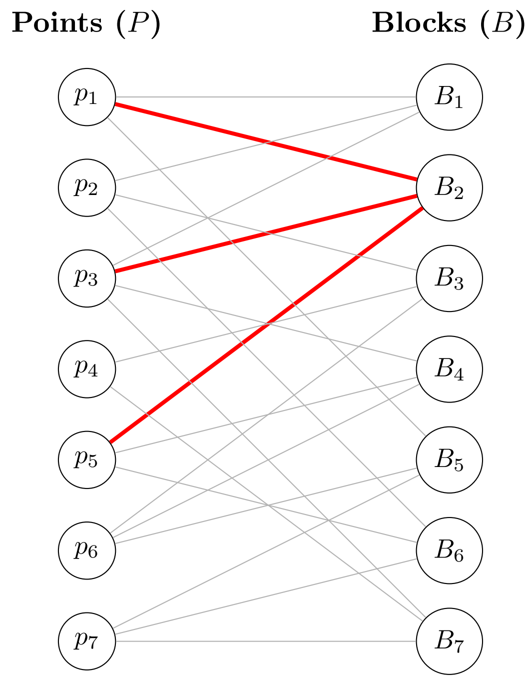

Figure 2: Complete bipartite incidence graph for the Fano Plane. The red edges highlight how the column points p_1, p_5, and p_3 intersect perfectly at a single common block block node (B_2).

2.9 Triple Systems

Recall that a triple system $TS(v, \lambda)$ is defined as a 2-$(v, 3, \lambda)$ design and a Steiner Triple System $STS(v)$ is defined as a 2-$(v, 3, 1)$ design. [5]

Theorem 2.14. *$STS(v)$ exists if and only if $v - 1$ is even.*

Proof. Consider an arbitrary point $x \in V$. There are v points in total. Since we have isolated x, there are $v - 1$ remaining points. By definition, x must pair with all of these points. Notice that every time x is in a block $\mathcal{B}_i$, $\mathcal{B}_i = x, y, z$ for some y and z in V. Each block covers 2 of the required pairings for x. Thus, the total number of remaining points $v - 1$ must be divisible by 2 and $v - 1$ is even. $\square$

Applying a similar argument to the entire set, a stronger theorem can be derived:

Theorem 2.15. *$STS(v)$ exists if $v \equiv 1, 3 \pmod 6$.*

Proof. There are

$$\binom{v}{2} = \frac{v(v-1)}{2}$$

unique pairs of points in total. Every single block contains 3 pairs. Since every pair of points must appear in one block, we can find b by dividing the total number of pairs by the number of pairs per block, resulting in:

$$b = \frac{v(v-1)}{6}$$

Remark. For a general triple system, the expressions for b and r are multiplied by λ

Thus, $v(v-1) \equiv 0 \pmod 6$. Since $v-1$ is even, v must be odd. This leaves $v \equiv 1, 3, 5 \pmod 6$. Checking each possible value of $v \pmod 6$,

Evaluations			
$v \pmod 6$	$v-1 \pmod 6$	$v(v-1) \pmod 6$	$(v-1) \equiv 0 \pmod 6$?
1	0	0	Yes
3	2	0	Yes
5	4	2	No

Thus, STS(v) exists only if $v \equiv 1, 3 \pmod 6$. □

Theorem 2.16. *$TS(v, \lambda)$ exists if and only if $v \neq 2$ and $\lambda \equiv 0 \pmod{gcd(v-2, 6)}$*

Proof. From the standard point-counting and block-counting formulas for a triple system, we know that the replication number r and total blocks b must be integers:

$$r = \frac{\lambda(v-1)}{2} \in \mathbb{Z} \implies 2 \mid \lambda(v-1)$$

$$b = \frac{\lambda v(v-1)}{6} \in \mathbb{Z} \implies 6 \mid \lambda v(v-1)$$

$$\begin{aligned}
&\implies \quad \lambda v(v-1) + 3\lambda(v-1) \equiv 0 \pmod 6 \\
&\implies \quad \lambda(v-1)(v-3) \equiv 0 \pmod 6 \\
&\implies \quad (v-1)(v-3) = (v-2)^2 - 1 \implies \lambda[(v-2)^2 - 1] \equiv 0 \pmod 6 \\
&\implies \quad \lambda(v-2)^2 - \lambda \equiv 0 \pmod 6 \\
&\implies \quad \lambda \equiv \lambda(v-2)^2 \pmod 6.
\end{aligned}$$

Let $g = \gcd(v-2, 6)$. By definition, this means $g \mid (v-2)$ and $g \mid 6$. Therefore, any true congruence modulo 6 must also hold true modulo g:

$$\lambda \equiv \lambda(v-2)^2 \pmod g$$

Since $v - 2 \equiv 0 \pmod g$, we can deduce:

$$\begin{aligned}
&\lambda \equiv \lambda(0)^2 \pmod g \\
&\lambda \equiv 0 \pmod{\gcd(v-2, 6)}.
\end{aligned}$$

To prove the theorem in the other direction, we evaluate the value of $g = \gcd(v-2, 6)$ across all possible cases of $v \pmod 6$

1. **Case 1:** $v \equiv 1, 3 \pmod 6$
   If $v \equiv 1$, $v - 2 \equiv 5 \implies g = gcd(5, 6) = 1$.
   If $v \equiv 3, v - 2 \equiv 1 \implies g = gcd(1, 6) = 1$.
   Condition: $lambda \equiv 0 \pmod 1$, which is always true for any $\lambda \geq 1$. A base Steiner Triple System $TS(v, 1)$ always exists for these values, and a $TS(v, \lambda)$ is formed by taking λ copies of it.

2. **Case 2:** $v \equiv 0, 4 \pmod 6$
   If $v \equiv 0, v - 2 \equiv 4 \implies g = gcd(4, 6) = 2$
   If $v \equiv 4, v - 2 \equiv 2 \implies g = gcd(2, 6) = 2$
   Condition: $\lambda \equiv 0 \pmod 2$ λ must be even. Proved in **Theorem 2.12**, $TS(v, 2)$ always exists for $v \equiv 0, 1 \pmod 3$. Any even λ can be built using $frac\lambda 2$ copies of a $TS(v, 2)$.

3. **Case 3:** $v \equiv 5 \pmod 6$
   If $v \equiv 5, v-2 \equiv 3 \implies g = gcd(3,6) = 3$
   Condition: $\lambda \equiv 0 \pmod 3 \lambda$ must be a multiple of 3). Proved in **Theorem 2.13**, a $TS(v,3)$ always exists for any odd v (as $v-1$ being even implies v is odd). We can construct the design using $\frac{\lambda}{3}$ copies of a $TS(v,3)$

4. **Case 2:** $v \equiv 2 \pmod 6$
   If $v \equiv 2, v-2 \equiv 0 \implies g = gcd(0,6) = 0$
   Condition: $\lambda \equiv 0 \pmod 6$. $TS(v,6)$ exists for all $v \geq 3$ as proved in y. Since $v \neq 2$, the design can always be constructed using $\frac{\lambda}{6}$ copies of a $TS(v,6)$.

□

Theorem 2.17. ***(Moore)*** *If there exist an STS(u) and an STS(v) that contains an STS(w) subsystem (see §x), then there exists an STS($u(v-w)+w$). Because every STS(v) contains a subsystem of orders 0 and 1, if there exist an STS(u) and an STS(v), then there exist an STS(uv) and an STS($u(v-1)+1$).*

Proof. Let $(U, \mathcal{A})$ be an STS(u) and let $(V, \mathcal{B})$ be an STS(v) containing a sub-design $(W, \mathcal{C})$ of order w, where $W \subset V$ and $\mathcal{C} \subset \mathcal{B}$.

We construct a new triple system $(X, \mathcal{D})$ of order $u(v-w)+w$. Define the point set X as:

$$X = \big(U \times (V \setminus W)\big) \cup W$$

Clearly, the number of points is $|X| = u(v-w)+w$. Next, we define the set of triples $\mathcal{D}$ by taking a union of three disjoint families of blocks:

1. **Subsystem Triples:** All blocks belonging to the original subsystem $\mathcal{C}$.

2. **Vertical Triples:** For each fixed element $u \in U$, we construct a copy of the blocks of $\mathcal{B} \setminus \mathcal{C}$ on the point set $\big(\{u\} \times (V \setminus W)\big) \cup W$.

3. **Cross Triples:** For every triple $\{u_1, u_2, u_3\} \in \mathcal{A}$, we place blocks linking the different components. By defining a proper symmetric quasigroup operation $*$ on $V \setminus W$ aligned with the blocks of $\mathcal{B}$, we form the triples $\big\{(u_1, x), (u_2, y), (u_3, z)\big\}$ such that every pair of coordinates across distinct elements of U is uniquely covered.

Every pair of distinct points $\{p_1, p_2\} \subset X$ falls into exactly one of these configurations:

- If both points are in W, they are uniquely covered by a block of Type 1.

- If one point is in W and the other is in $U \times (V \setminus W)$, or if both points are in $\{u\} \times (V \setminus W)$ for the same u, they are uniquely covered by a block of Type 2.

- If the points are (u_1, x) and (u_2, y) with $u_1 \neq u_2$, they are uniquely covered by a block of Type 3 because $\{u_1, u_2\}$ belongs to exactly one triple in $\mathcal{A}$.

Thus, every pair of elements in X appears in exactly one triple, completing the validation of the STS($u(v-w)+w$) structure.

To handle the structural corollaries, observe that any valid design contains a vacuous subsystem of order $w = 0$ (the empty set $\emptyset$) and $w = 1$ (any single point since no pairs exist yet to violate conditions).

- Substituting $w = 0$ into $u(v - w) + w$ yields $u(v - 0) + 0 = uv$.
- Substituting $w = 1$ into $u(v - w) + w$ yields $u(v - 1) + 1$.

This completes the proof.

□

Theorem 2.18. *If there exists an $STS(v)$, then there exists an $STS(2v + 1)$.*

Proof. Let $(V, \mathcal{B})$ be an $STS(v)$ defined on the element set V. We construct a new triple system $(X, \mathcal{D})$ of order $2v + 1$. Define the new point set X as:

$$X = (V \times \{0, 1\}) \cup \{\infty\}$$

Clearly, the number of points is $|X| = 2 \cdot |V| + 1 = 2v + 1$. We define the new set of blocks $\mathcal{D}$ by taking the union of two disjoint families of triples:

1. **Derived Triples:** For every original block $\{x, y, z\} \in \mathcal{B}$, we form the blocks $\{(x, \alpha), (y, \beta), (z, \gamma)\}$ for all choices of $\alpha, \beta, \gamma \in \{0, 1\}$ such that:
$$\alpha + \beta + \gamma \equiv 0 \pmod 2$$
2. **Infinity Triples:** For each element $x \in V$, we adjoin the block:
$$\{\infty, (x, 0), (x, 1)\}$$

To prove that $(X, \mathcal{D})$ is a valid Steiner Triple System, we must verify that every pair of distinct points $\{p_1, p_2\} \subset X$ is contained in exactly one block of $\mathcal{D}$. We analyze this by splitting the possible pairs into three cases:

- **Case 1: One point is ∞.**
Let the pair be $\{\infty, (x, \alpha)\}$ for some $x \in V$ and $\alpha \in \{0, 1\}$. This pair is explicitly and uniquely contained in the Infinity Triple $\{\infty, (x, 0), (x, 1)\}$.
- **Case 2: The points share the same base element $x \in V$.**
Let the pair be {(x, 0), (x, 1)}. This pair is also uniquely contained in the identical Infinity Triple $\{\infty, (x, 0), (x, 1)\}$.
- **Case 3: The points have distinct base elements.**
Let the pair be $\{(x, \alpha), (y, \beta)\}$ where $x, y \in V$ with $x \neq y$, and $\alpha, \beta \in \{0, 1\}$. Because $(V, \mathcal{B})$ is a valid $STS(v)$, the underlying pair $\{x, y\}$ appears in exactly one original block $\{x, y, z\} \in \mathcal{B}$.

  In the new system, this pair must be completed by a third point (z, γ). For this block to be valid, its indices must satisfy the parity constraint $\alpha + \beta + \gamma \equiv 0 \pmod 2$. Solving for γ yields:
$$\gamma \equiv -(\alpha + \beta) \pmod 2$$
  Since arithmetic modulo 2 provides a unique solution for $\gamma \in \{0, 1\}$, there exists exactly one unique Derived Triple covering the pair $\{(x, \alpha), (y, \beta)\}$.

Since every pair of distinct elements in X is covered by exactly one block in $\mathcal{D}$, the resulting structure is a well-defined $STS(2v + 1)$. □

3 Incidence Structures

A particularly useful way to think about block designs is to consider them as special cases of a larger incidence structure.

Definition 3.1. *An* incidence structure *is a triple* $\mathcal{D} = (P, B, I)$ *where:*

- P *is a finite set called the set of points,*
- B *is a finite set called the set of blocks,*
- $I \subseteq P \times B$ *is a relation called incidence.*

If $(p, B) \in I$*, then the point* p *is said to be incident with the block* B*.*

An incidence structure can be represented as a bipartite graph, where one class of vertices corresponds to points and the other corresponds to blocks. An edge represents an incidence relation.

3.1 Duality

A fundamental property of incidence structures is the symmetry between points and blocks.

Definition 3.2. *The* dual incidence structure *of* $\mathcal{D} = (P, B, I)$ *is the structure* $\mathcal{D}^* = (B, P, I^*)$*, where* $(B, p) \in I^*$ *whenever* $(p, B) \in I$*. Thus the roles of points and blocks are exchanged.*

Remark. Duality plays an important role in finite geometry and especially in symmetric block designs, where the original design and its dual have identical parameters.

Example 3.3 (Triangle Incidence Structure)**.** *Let* $P = \{A, B, C\}$ *and let* $B = \{\ell_1, \ell_2, \ell_3\}$ *where* $\ell_1 = \{A, B\}$*,* $\ell_2 = \{B, C\}$*, and* $\ell_3 = \{A, C\}$*. The incidence structure consists of three points and three blocks, where each block contains two points. The dual structure has three points corresponding to the original blocks and three blocks corresponding to the original points.*

3.2 Incidence Matrices

Many properties of block designs may be studied using linear algebra.

Definition 3.4. *Let* $\mathcal{D} = (P, \mathcal{B}, I)$ *be a block design with* $|P|$=v*,* $|\mathcal{B}| = b$*. The* incidence matrix *of the design is the* $v \times b$ *matrix*

$$a_{ij} = \begin{cases} 1, & \textit{if } p_i \in B_j, \\ 0, & \textit{otherwise.} \end{cases}$$

Each row of the incidence matrix represents a point, while each column represents a block. For a regular block design,

- every row contains exactly r ones,
- every column contains exactly k ones.

Consequently,

$$\sum_{j=1}^{b} m_{ij} = r$$

for every row, and

$$\sum_{i=1}^{v} m_{ij} = k$$

for every column.

These simple observations allow combinatorial properties of the design to be translated into matrix identities.

3.3 The Incidence Matrix of a BIBD

When the design is balanced, the product MM^T has an especially elegant form.

Theorem 3.5. *Let $\mathcal{D}$ be a (v, b, r, k, λ) Balanced Incomplete Block Design. Then $MM^T = (r - \lambda)I + \lambda J$ where*

- *I is the $v \times v$ identity matrix,*
- *J is the $v \times v$ all-ones matrix.*

Proof. The (i, j)-entry of MM^T counts the number of blocks containing both points p_i and p_j. If $i = j$, then point p_i belongs to exactly r blocks, so $(MM^T)_{ij} = r$. If $i \neq j$, then the two points occur together in exactly λ blocks, giving $(MM^T)_{ij} = \lambda$. Therefore the diagonal entries are equal to r, while every off-diagonal entry equals λ. Hence $MM^T = (r - \lambda)I + \lambda J$ □

The preceding theorem is one of the central algebraic results in design theory. It immediately implies several structural properties of BIBDs, including rank formulas, eigenvalue computations, and Fisher's Inequality. More generally, it illustrates how the combinatorial regularity of a design is reflected in the algebraic structure of its incidence matrix.

Theorem 3.6 (Schützenberger's Theorem for Even Symmetric Designs)**.** *Let $(V, \mathcal{B})$ be a symmetric $2-(v, k, \lambda)$ design where $v = b$ and $r = k$. If the total number of points v is an **even** integer, then the parameter value:*

$$k - \lambda$$

must be a perfect square (i.e., $k - \lambda = n^2$ for some integer n).

Proof. Let M be the $v \times v$ square incidence matrix of the symmetric design. From the basic combinatorial properties of balanced designs, we know the matrix product of M and its transpose satisfies:

$$MM^T = (k - \lambda)I_v + \lambda J_v$$

where I_v is the $v \times v$ identity matrix and J_v is the $v \times v$ matrix of all ones. We evaluate the determinant of both sides of this equation to isolate the properties of the eigenvalues.

Using standard matrix determinant lemmas, the determinant of $(k - \lambda)I_v + \lambda J_v$ is given exactly by:

$$\det(MM^T) = (k - \lambda)^{v-1}(k + (v - 1)\lambda)$$

From the fundamental identities of any BIBD, we know that $\lambda(v-1) = r(k-1)$. Since this design is symmetric ($r = k$), this simplifies to $\lambda(v-1) = k(k-1)$, which means $k+(v-1)\lambda = k+k(k-1) = k^2$. Substituting this back into our determinant expression yields:

$$\det(MM^T) = (k-\lambda)^{v-1} \cdot k^2$$

By the properties of determinants over matrix multiplication, we also know that $\det(MM^T) = \det(M)\det(M^T) = (\det(M))^2$. This means the left side of our equation must be a perfect square over the ring of integers:

$$(\det(M))^2 = (k-\lambda)^{v-1} \cdot k^2$$

Since k^2 is already a perfect square, the remaining factor $(k-\lambda)^{v-1}$ must also evaluate to a perfect square.

We are given that v is an even integer. If v is even, then the exponent $v-1$ must be an **odd** integer. For a number raised to an odd power to be a perfect square, the base itself $(k-\lambda)$ must be a perfect square. Thus, there must exist some integer n such that $k-\lambda = n^2$. □

3.4 Fisher's Inequality

One of the most fundamental results in design theory is Fisher's Inequality, which states that every nontrivial balanced incomplete block design contains at least as many blocks as points. Although the theorem originated in statistics, its proof reveals an elegant connection between combinatorics and linear algebra.

Theorem 3.7 (Fisher's Inequality)**.** *Let $\mathcal{D}$ be a nontrivial (v, b, r, k, λ) Balanced Incomplete Block Design with $1 < k < v$. Then $b \geq v$.*

Proof. Let M be the incidence matrix of the design. From the previous theorem, $MM^T = (r-\lambda)I + \lambda J$. Since $r(k-1) = \lambda(v-1)$, and $k < v$ we have $r > \lambda$. Therefore, $r - \lambda > 0$. The matrix $r-\lambda)I + \lambda J$ has eigenvalues $r-\lambda$ with multiplicity $v-1$, and $r+\lambda(v-1)$ with multiplicity one. Each eigenvalue is positive, so MM^T is invertible. Consequently, $\text{rank}(MM^T) = v$. Since $\text{rank}(MM^T) = \text{rank}(M)$, it follows that $\text{rank}(M) = v$ Because the rank of a matrix cannot exceed the number of its columns, $v = \text{rank}(M) \leq b$. Hence $b \geq v$. □

Fisher's Inequality provides a necessary condition for the existence of a balanced incomplete block design. In particular, it shows that no nontrivial BIBD can contain fewer blocks than points.

Corollary 3.8. *If a BIBD satisfies $b = v$, then it is called a* symmetric design*.*

The equality case of Fisher's Inequality is of particular importance. Symmetric designs possess additional regularity and exhibit a duality between points and blocks that is absent in general block designs.

4 Finite Geometries

Finite geometry provides one of the richest sources of examples of block designs. In particular, projective and affine planes satisfy strong incidence properties that naturally produce balanced incomplete block designs. Unlike Euclidean geometry, finite geometries contain only finitely many points and lines. Nevertheless, they retain many familiar geometric properties while exhibiting remarkable combinatorial regularity.

4.1 Projective Planes

A projective plane is an incidence structure satisfying three simple axioms.

Definition 4.1. *A* projective plane *is an incidence structure* $(P, \mathcal{L}, I)$ *such that*

1. *Every two distinct points determine a unique line.*
2. *Every two distinct lines intersect in a unique point.*
3. *There exist four points, no three of which are collinear.*

These axioms are self-dual: interchanging the roles of points and lines produces another projective plane satisfying the same axioms. [6]

4.2 Projective Planes of Order q

A finite projective plane is said to have order q if every line contains exactly $q + 1$ points. The following theorem summarizes its basic parameters.

Theorem 4.2. *Let* Π *be a projective plane of order* q*. Then* $v = b = q^2 + q + 1$, $r = k = q + 1$*, and* $\lambda = 1$ *Consequently, every finite projective plane is a symmetric Balanced Incomplete Block Design with parameters* $(q^2 + q + 1, q^2 + q + 1, q + 1, q + 1, 1)$.

Proof. Every line contains $q+1$ points by definition. Since every two points determine a unique line, every pair of points occurs together in exactly one block, so $\lambda = 1$ Elementary counting arguments show that every point is incident with exactly $q + 1$ lines. Applying the incidence relation $vr = bk$ gives $b = v$. Finally, $v = q^2 + q + 1$ follows by counting all points reachable from a fixed point through the $q + 1$ lines passing through it. Thus the stated parameters follow. □

Theorem 4.3 (Projective Triple Systems over $\mathbb{F}_2$)**.** *The points and lines of the projective space* $PG(n, 2)$ *form the elements and triples of an* $STS(2^{n+1} - 1)$*. Equivalently, letting the points be the non-zero vectors of length* $k = n + 1$ *over* $\mathbb{F}_2$*, a subset of three points* $\{a, b, c\}$ *forms a block if and only if:* $a + b + c = \mathbf{0}$

Proof. Let W_n be an $(n + 1)$-dimensional vector space over the finite field $\mathbb{F}_2$.

1. **Point Count Verification:** The total number of vectors in W_n is 2^{n+1}. Punctured 1-dimensional subspaces are generated by a single non-zero vector. Because the only scalar multipliers in $\mathbb{F}_2$ are 0 and 1, each non-zero vector spans a unique point. Removing the zero vector leaves exactly $v = 2^{n+1} - 1$ elements.

2. **Block Size Verification:** A line in $PG(n, 2)$ corresponds to a punctured 2-dimensional subspace. Any 2-dimensional subspace over $\mathbb{F}_2$ contains exactly $2^2 = 4$ vectors: $\{\mathbf{0}, a, b, a+b\}$, where a and b are linearly independent vectors. Removing $\mathbf{0}$ leaves exactly three points: $\{a, b, a + b\}$. Thus, every line has a uniform block size of $k = 3$.

To prove this structure is an STS, we must show that any pair of distinct non-zero vectors $\{a, b\}$ is contained in exactly one block. Since $a \neq b$, they are linearly independent. The 2-dimensional subspace they span is uniquely determined as $\{\mathbf{0}, a, b, a + b\}$.

Puncturing this subspace yields the unique block $\{a, b, a + b\}$. Letting $c = a + b$, we observe that in $\mathbb{F}_2$, addition and subtraction are identical (since $1 \equiv -1 \pmod 2$). Therefore:

$$c = a + b \implies a + b + c = a + b + a + b = \mathbf{0}$$

Thus, the pair is covered by exactly one block satisfying the zero-sum criteria, confirming a valid $STS(2^{n+1}-1)$. $\square$

4.3 Affine Planes

Affine planes are obtained from projective planes by removing one line, called the line at infinity, together with all points lying on that line.

Definition 4.4. *An* affine plane *is an incidence structure satisfying*

1. *Every two distinct points determine a unique line.*
2. *Given a line ℓ and a point P not on ℓ, there exists a unique line through P parallel to ℓ.*
3. *There exist three noncollinear points.*

If an affine plane has order q, then $v = q^2$. each line contains q points, and there are $q^2 + q$ lines. Although affine planes are generally not symmetric designs, they remain important examples of highly regular incidence structures.

Theorem 4.5 (Affine Triple Systems over $\mathbb{F}_3$)**.** *The points and lines of the n-dimensional affine space $AG(n, 3)$ form the elements and triples of an $STS(3^n)$, where a set of three distinct vectors $\{a, b, c\}$ forms a block if and only if:*

$$a + b + c \equiv \mathbf{0} \pmod 3$$

Proof. Let V_n be an n-dimensional vector space over the finite field $\mathbb{F}_3$.

1. **Point Count Verification:** The points of $AG(n, 3)$ are the raw vectors of V_n. The total number of points is exactly $v = 3^n$.
2. **Block Size Verification:** The lines of $AG(n, 3)$ are the translates (cosets) of 1-dimensional subspaces. A 1-dimensional subspace generated by a non-zero vector v is defined as $S = \{0 \cdot v, 1 \cdot v, 2 \cdot v\}$. Since $2 \equiv -1 \pmod 3$, this is $S = \{\mathbf{0}, v, -v\}$. Translating this subspace by a vector $a \in V_n$ gives the line: $a + S = \{a, a + v, a - v\}$. Thus, every line contains exactly $k = 3$ distinct points.

To prove this forms an STS, we select any two distinct points $a, b \in V_n$ and verify unique block coverage. For a block $\{a, b, c\}$ to be an affine line, its elements must satisfy a uniform zero-sum translation rule. Let us examine the sum of the elements of a generic line $a+S$: $a+(a+v)+(a-v) = 3a = \mathbf{0} \pmod 3$

Hence, any three points form a line if and only if $a+b+c = \mathbf{0}$. Given our chosen distinct points a and b, we can algebraically solve for the required third point c: $c = -(a + b) = 2a + 2b \pmod 3$

Since vector addition over a finite field yields a single, unique result, c is uniquely determined. Furthermore, c must be distinct from a and b; if $c = a$, then $a + b + a = \mathbf{0} \implies 2a + b = \mathbf{0} \implies b = -2a = a$, which contradicts the fact that $a \neq b$.

Thus, every pair of distinct points is contained in exactly one affine line, completing the validation of the $STS(3^n)$ system. $\square$

4.4 Existence Problems

One of the central questions in finite geometry is determining for which integers q a projective plane of order q exists.

Every prime power $q = p^n$ gives rise to a projective plane constructed from the finite field $\mathrm{GF}(q)$. Whether projective planes exist for orders that are not prime powers remains one of the major open problems in combinatorics. For example, it is known that no projective plane of order 6 exists, [10] while the existence of a projective plane of order 10 was ruled out only after extensive computer-assisted computation. [11]

The existence problem illustrates the deep interaction between combinatorics, algebra, geometry, and computational mathematics that characterizes modern design theory.

5 Algebraic Constructions of Block Designs

Although many block designs arise from geometric constructions, algebra provides a powerful and systematic method for constructing highly symmetric designs. In particular, finite groups and finite fields play a central role in modern design theory. The fundamental idea is to generate blocks by translating a fixed subset through the elements of a group. The resulting designs inherit the symmetries of the underlying algebraic structure.

5.1 Automorphisms

Symmetry is measured through automorphisms.

Definition 5.1. *Let mathcalD $= (P, \mathcal{B}, I)$ be an incidence structure. An* automorphism *of $\mathcal{D}$ is a bijection $\varphi : P \to P$ such that $p \in B \iff \varphi(p) \in \varphi(B)$ for every point p and every block B.*

The set of all automorphisms forms a group under composition, called the *automorphism group* of the design.

Designs possessing large automorphism groups are often easier to construct and classify than arbitrary designs.

5.2 Difference Sets

Difference sets provide one of the most important algebraic methods for constructing symmetric designs. [8]

Definition 5.2. *Let G be a finite group of order v. A subset $D = \{d_1, d_2, \ldots, d_k\} \subseteq G$ is called a* (v, k, λ)difference set *if every nonidentity element of G can be written exactly λ times in the form $d_i d_j^{-1} d_i, d_j \in D$.*

Difference sets generate block designs by translating the set D through every element of the group.

Theorem 5.3. *Every (v, k, λ) difference set determines a symmetric (v, b, r, k, λ) Balanced Incomplete Block Design.*

Sketch of Proof. For each element $g \in G$, define the translated block $gD = \{gd : d \in D\}$. The collection $\mathcal{B} = \{gD : g \in G\}$ contains exactly v blocks. The defining property of the difference set guarantees that every pair of points occurs together in exactly λ translated blocks. Hence the translated blocks form a symmetric BIBD. $\square$

Difference sets reveal that highly regular combinatorial structures can often be viewed as algebraic objects.

5.3 Finite Fields

Many classical block designs arise from finite fields. Let $\mathrm{GF}(q)$ denote the finite field with q elements, where q is a prime power. Finite fields provide coordinate systems for affine and projective spaces, from which numerous families of block designs can be obtained.

For example,

- projective planes arise from two-dimensional projective spaces,
- affine planes arise from affine spaces,
- unitals and generalized quadrangles arise from classical polar spaces,
- several infinite families of Steiner systems are constructed using finite fields.

The interaction between finite fields and combinatorial designs constitutes one of the principal themes of algebraic combinatorics.

5.4 Group Actions

More generally, many block designs are constructed using group actions. If a finite group acts transitively on a set of points, then the orbit of a single block under the group action often produces a highly symmetric incidence structure. This approach unifies many seemingly unrelated constructions and provides a common framework for understanding projective planes, difference sets, Hadamard designs, and numerous other classes of block designs. [**?**]

To close our structural investigation of finite incidence geometries, we present an elegant, classical characterization that translates purely combinatorial regularity into a rigid algebraic footprint. This result demonstrates that a design's balance is entirely encoded within the spectrum of its incidence matrix.

Theorem 5.4 (Spectral Characterization of Symmetric BIBDs)**.** *Let $\mathcal{D}$ be a symmetric incidence structure with parameters (v, v, r, r) and incidence matrix M. Let*

$$A = MM^T.$$

Then $\mathcal{D}$ is a symmetric balanced incomplete block design if and only if

$$\sigma(A) = \{\, r^2,\ r - \lambda \,\},$$

where r^2 has multiplicity 1 and $r - \lambda$ has multiplicity $v - 1$.

Proof. Assume $\mathcal{D}$ is a symmetric BIBD. Since every point lies in r blocks and every pair of distinct points lies together in exactly λ blocks, the entries of A satisfy

$$A_{ij} = \begin{cases} r, & i = j, \\ \lambda, & i \neq j. \end{cases}$$

Hence, we can express A as

$$A = (r - \lambda)I + \lambda J,$$

where J is the $v \times v$ all-ones matrix. The matrix J has eigenvalue v with eigenvector $\mathbf{1} = (1, \ldots, 1)^T$, and eigenvalue 0 on the orthogonal complement $\mathbf{1}^{\perp}$. Therefore,

$$A\mathbf{1} = \big((r - \lambda) + \lambda v\big)\mathbf{1}.$$

Using the fundamental BIBD identity $r(v-1) = r(r-1) + r - \lambda(v-1)$ rearranged from $r(r-1) = \lambda(v-1)$, we obtain

$$(r - \lambda) + \lambda v = r + \lambda(v-1) = r + r(r-1) = r^2.$$

Hence r^2 is an eigenvalue of multiplicity one. If $\mathbf{x} \perp \mathbf{1}$, then $J\mathbf{x} = \mathbf{0}$, so

$$A\mathbf{x} = (r - \lambda)\mathbf{x}.$$

Thus $r - \lambda$ has multiplicity $v - 1$, proving the forward direction.

Conversely, suppose $\sigma(A) = \{\, r^2,\, r - \lambda \,\}$ with multiplicities 1 and $v - 1$, respectively. Since $\mathcal{D}$ has parameters (v, v, r, r), every row of the incidence matrix M has exactly r ones. It follows that the row sums of $A = MM^T$ are all equal to r^2. Therefore, the all-ones vector $\mathbf{1}$ must be the eigenvector corresponding to the simple eigenvalue r^2.

By the spectral decomposition theorem, since A is real symmetric, we can write

$$A = r^2 \left(\frac{1}{v} J \right) + (r - \lambda) \left(I - \frac{1}{v} J \right).$$

Expanding and collecting terms yields

$$A = (r - \lambda) I + \left(\frac{r^2 - r + \lambda}{v} \right) J.$$

Because $\mathcal{D}$ is symmetric, the parameters imply that the total number of incident pairs counted two ways yields $vr = vr$, meaning the standard relation $r(r-1) = \lambda(v-1)$ holds for the design parameters. Rearranging this identity gives $r^2 - r = \lambda v - \lambda$, which implies

$$r^2 - r + \lambda = \lambda v.$$

Substituting this back into the expression for A simplifies the coefficient of J:

$$A = (r - \lambda) I + \lambda J.$$

Consequently, the entries of A are given by

$$A_{ii} = r, \qquad A_{ij} = \lambda \quad (i \neq j).$$

Since $A_{ij} = (MM^T)_{ij}$, the diagonal entries confirm that every point is incident with exactly r blocks, while the off-diagonal entries prove that every pair of distinct points is contained in exactly λ common blocks. Therefore, $\mathcal{D}$ is a symmetric balanced incomplete block design. $\square$

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