# Influence of Paradoxical Decomposition on the Banach-Tarski Paradox

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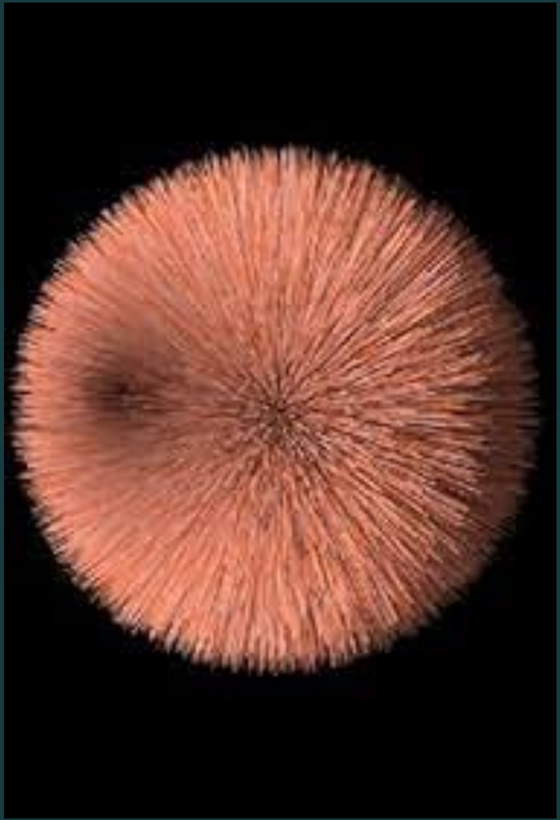




SPOTLIGHT

Theorem: *The solid unit ball in R3 can be partitioned into finitely many pieces, which can then be reassembled using only rotations and translations into two solid balls, each identical in size to the original.*

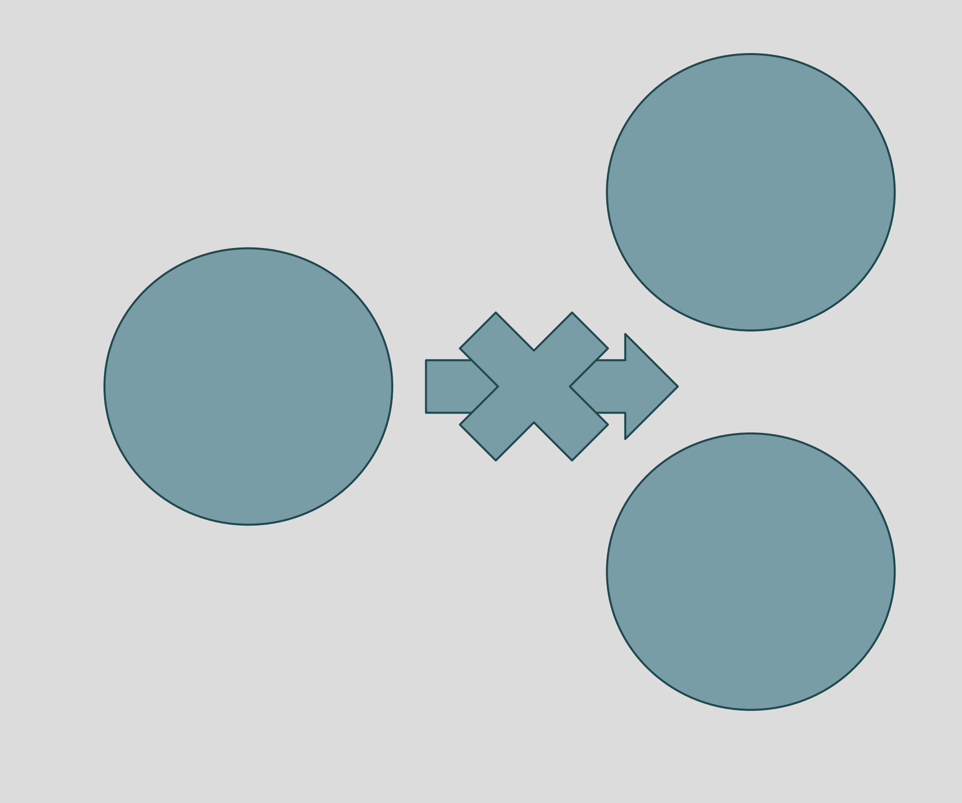
Basic Ideas

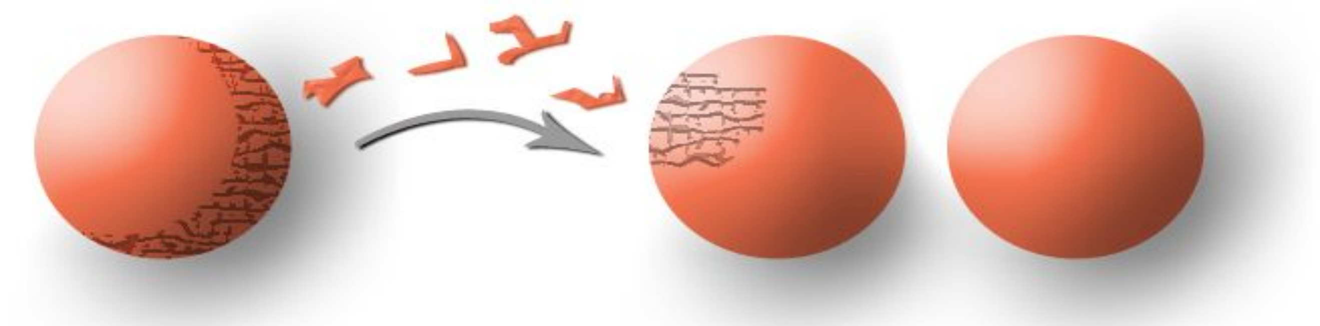
# Firstly, volume is preserved under rigid motions.

Ball with r=1: Volume = $4\pi/3$
2 Balls with r=1: Volume = $8\pi/3$

But $4\pi/3 \neq 8\pi/3$

What's going on?





- The pieces have no volume
- They are non-Lebesgue measurable
- Volume doubling isn't a contradiction if the pieces can't be assigned a volume

Infinite geometric complexity within the ball is what allows the Banach Tarski Paradox to hold shape.

# The Free Group $F_2$

- A Free Group is a set of actions that can be combined, undone, and can do nothing.
- $F_2$ is the free group on the generators {a, b}.
- We can partition $F_2$ into five different buckets.
  - $S(a)$, $S(b)$, $S(a^{-1})$, $S(b^{-1})$, and $\{e\}$
- What happens when we add $a^{-1}$ to the beginning of $S(a)$?
- What about $b^{-1}$ and $S(b)$?

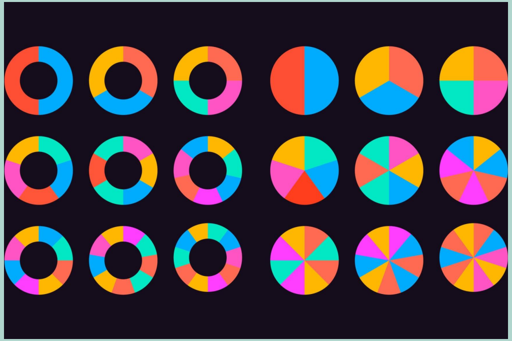
# The Free Group of Rotations SO(3)

- Let $t = \arccos(1/3)$, A be the rotation by t across the x-axis, and B be the rotation by t across the z-axis.
- No sequence will ever return back to the identity.
- Track where any word sends (0, 1, 0). The output will always have the form:

$$\frac{1}{3^n}(a\sqrt{2}, b, c\sqrt{2}) \text{ for integers } a,b,c.$$

- The identity requires $b = 3^n$ and $a = c = 0$...
- Which will never happen.

# Decomposing the Ball



- L is the unit ball, and $L^0$ doesn't have a center.
- A and B partition $L^0$ into orbits O(p), collections of points reachable from each other - and they never overlap.
- The axiom of choice allows us to pick one point from each orbit and put them in a set M.
- M has no volume, as when the volume of M is zero, an infinite amount yields zero which makes no sense, and when the volume of M is positive, an infinite volume also makes no sense.
- Points lying exactly on a rotation axis are fixed. Such a point could be reached from M by multiple different rotations simultaneously, which would put it in multiple pieces of our partition at once. We remove these points called D for now.

# Decomposing the Ball

- Each point in $L^0 \setminus D$ from M is a unique rotation, meaning that every point has a label from their rotations from M.
- We result with 4 pieces:
  - $P_1$ - rotations starting with A, $P_2$ - rotations starting with $A^{-1}$, $P_3$ - rotations starting with B, $P_4$ - rotations starting with $B^{-1}$.
- Applying $F_2$ cloning geometrically:
  - Rotating $P_2$ over A covers everything not in $P_1$, so $P_1$ and $AP_2$ yields the whole ball, or $P_3$ and $BP_4$ as well.
  - So $F_2$ that was once S(a), S(b), S($a^{-1}$) and S($b^{-1}$) is now $L^0 \setminus D = P_1 \cup A \cdot P_2$, likewise for B rotation in $P_3$ and P.

# Set D and the Center

- Using a rotation p, we can add repeated rotations to D {D, pD, $p^2$D,...}, and shifting the entire chain forward allows D to absorb into the set, and D disappears. This means that $L^0 \setminus D$ is now moved into the same place as $L^0$
- On the sphere create a slice of a circle where the origin is. Rotating an infinite orbit of points through the irrational arccos(⅓) fills in all holes.
- We can now finally say that the full ball L is *equidecomposable*, or able to be shifted, into two unique copies.

# Volume Further Defined

- Volume (Lebesgue Measures) assign a consistent size to objects like tetrahedrons, meaning that volume will stay the same during rotations and translations.
- If the pieces had well defined volumes, then shifting the area wouldn't change it from $4\pi/3$ to $8\pi/3$ regardless.
- The paradox stands as set M was provided easily due to the Axiom of Choice, just picking one from each orbit.
- Any assignment of volume in M yields a contradiction.
- Every element within M - $P_1$, $P_2$, $P_3$, $P_4$ has no volume at all.

# Implications

- Physically impossible: matter is only small enough for atoms in the real world, not infinitely complex slices.
- The Axiom of Choice: the non-measurable set M cannot be constructed without it as then it cannot even be proven non-measurable sets exist.
- Why it fails in lower dimensions: In $R^1$ and $R^2$, it fails due to less rotational freedom, and with axis only perpendicular to the plane, they cannot be independent rotations like A and B are. Without two rotational axis, a free group cannot be made within the rotational group, and so no actions can occur.

# Thank you!

“Such beauty only exists in the realm of mathematics.” – Katie Bucchorn