Influence of Paradoxical Decompositions in the Banach-Tarski Paradox

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June 2026

1 Introduction

$1 + 1 = 2$, but not all the time. Look at the following theorem:

Theorem 1.1. *The solid unit ball in $\mathbb{R}^3$ can be partitioned into finitely many pieces, which can then be reassembled using only rotations and translations into two solid balls, each identical in size to the original.*

At first, this looks like nonsense. Since the beginning of time, math has taught us that one object cannot turn into multiple others without adding materials, that volume is preserved under every motion, and that $1 \neq 2$. How could we bypass this mathematically? The answer works when each piece of the spheres are broken up into infinitely complex and small pieces that they cannot be assigned a volume and therefore have no contradiction.

The sets that allow for such enigmas are the non-Lebesgue measurable sets, sets where no measure can be applied. These are allowed by the Axiom of Choice, stating that given a collection of non-empty sets, one could simultaneously choose one element from each, even with no reason. As this cannot be proven by any other mathematical means, the Banach-Tarski paradox is an example of how detailed this Axiom can be.

2 Group Theory

We will first start by outlining the foundations of the ideas that will be used in this paper, surrounding groups, and their behaviors. The main concept will be a set's equidecomposability, the notion of when one set is changed into another.

Definition 2.1 (Group)**.** A *group* is a set G with the binary operation $G \times G \to G$, written $(g, h) \mapsto gh$, satisfying four binary operations:

(i) **Closure.** For all $a, b \in G$, we have $ab \in G$.

(ii) **Identity.** There exists a value $e \in G$ such that $ea = a = ae$ for every $a \in G$.

(iii) **Inverse.** For all $a \in G$, there exists an $a^{-1} \in G$ such that $aa^{-1} = e = a^{-1}a$.

(iv) **Associativity.** For all $a, b, c \in G$, we have $(ab)c = a(bc)$.

The element e is called the identity, and a^{-1} is called the inverse of a. Additionally, operations made to the whole group are known as group operations.

Example 2.2. The integers in set $\mathbb{Z}$ all support group actions under the binary addition operand $(+)$. For every element within $\mathbb{Z}$, 0 acts as the identity since its addition changes no value, and each integer n has the additive inverse $-n$ that reduces it to the identity, and associativity for addition is true. The set in $\mathbb{R}^3$ that explains all rotations through the axes also forms a group, where the identity is $0°$, and every rotation can be undone by the additive inverse in SO(3), which will be expanded on in Definition 4.1.

Definition 2.3 (Group Action)**.** Let G be a group and S be a set. A *group action* on G of S is shown as $G \times S \to S$, written $(g, s) \mapsto g \cdot s$, applying

(i) **Associativity.** $(gh) \cdot s = g \cdot (h \cdot s)$ for all $g, h \in G$ and $s \in S$.

(ii) **Identity.** $e \cdot s = s$ for all $s \in S$.

These operations dictate how elements of G act as transformations on S, where each $g \in G$ moves elements on S around uniformly across the whole set. In this thought, S will act as the unit ball, and G will act as the operation to rotate S.

Definition 2.4 (Equidecomposability)**.** Let G act on a set X, and let A, B be subsets of X. We say A and B are *G-equidecomposable*, written $A \sim_G B$, if there exists a finite n and partitions

$$A \;=\; \bigsqcup_{i=1}^{n} A_i \qquad \text{and} \qquad B \;=\; \bigsqcup_{i=1}^{n} B_i$$

together with group elements $g_1, \ldots, g_n \in G$ such that $g_i \cdot A_i = B_i$ for each i. The $\bigsqcup$ symbol denotes a disjoint union, where all A_i sets are required to be pairwise disjoint and cover all of A, and all B_i sets are required to be pairwise disjoint and cover all of B. The notation $g_i \cdot A_i = B_i$ for each i explains that each group action on A shifts it into the exact same place as B, with no gaps or overlaps.

Remark 2.5. It is important to note that the Banach-Tarski Paradox holds true due to the idea that equidecomposability does not reference the actual area or measure of a set. Two such sets can have different volumes and still remain equidecomposable as long as they can be cut into separate pieces that are related by a rigid motion.

The following two lemmas explain the movements of equidecomposability across multiple sets.

Lemma 2.6 (Transitivity)**.** *If $A \sim_G B$ and $B \sim_G C$, then $A \sim_G C$.*

Proof. Since $A \sim_G B$ there must exist sections

$$A = \bigsqcup_{i=1}^{n} A_i \qquad \text{and} \qquad B = \bigsqcup_{i=1}^{n} B_i,$$

with $g_i \cdot A_i = B_i$ for each i. Since $B \sim_G C$ there must exist sections

$$B = \bigsqcup_{j=1}^{m} B'_j \qquad \text{and} \qquad C = \bigsqcup_{j=1}^{m} C_j,$$

with $h_j \cdot B'_j = C_j$ for each j.

For each pair (i, j) define $A_{ij} = A_i \cap g_i^{-1}(B'_j)$. Since $g_i \cdot A_i = B_i$, the sets $\{g_i(A_{ij})\}_j$ partition B_i, and therefore $\{A_{ij}\}_{i,j}$ partitions A. Define $C_{ij} = h_j \cdot g_i \cdot A_{ij}$. $h_j g_i$ can be defined as a group action, with $h_j g_i$ sending A_{ij} onto C_{ij}, where $\{C_{ij}\}_{i,j}$ partitions C. Therefore, $A \sim_G C$. □

Lemma 2.7 (Additivity). *If* $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$, *and* $A_1 \sim_G B_1$ *and* $A_2 \sim_G B_2$, *then* $A_1 \cup A_2 \sim_G B_1 \cup B_2$.

Proof. Since $A_1 \sim_G B_1$, there exist sections

$$A_1 = \bigsqcup_{i=1}^{n} A_i, \qquad B_1 = \bigsqcup_{i=1}^{n} B_i$$

with $g_i \cdot A_i = B_i$ for each i. Since $A_2 \sim_G B_2$, there exist sections

$$A_2 = \bigsqcup_{j=1}^{m} A'_j, \qquad B_2 = \bigsqcup_{j=1}^{m} B'_j$$

with $h_j \cdot A'_j = B'_j$ for each j.

Since A_1 and A_2 are disjoint and B_1 and B_2 are disjoint, the combined collections $\{A_i\} \cup \{A'_j\}$ and $\{B_i\} \cup \{B'_j\}$ partition $A_1 \cup A_2$ and $B_1 \cup B_2$ respectively with matching group elements. Therefore $A_1 \cup A_2 \sim_G B_1 \cup B_2$. □

Definition 2.8 (Paradoxical Decomposition). Let G act on X and let $E \subseteq X$. E is *G-paradoxical* if there exist disjoint sets $A, B \subseteq E$ such that $A \sim_G E$ and $B \sim_G E$.

This simply describes that set E is paradoxical if it contains two separate sets A, B that can independently be rearranged into full copies of E. This is the main definition that will explain why the Banach-Tarski Paradox works in $\mathbb{R}^3$.

Remark 2.9. The remainder of this paper will focus on how a unit ball allows paradoxical decomposition under the rotational group SO(3) in $\mathbb{R}^3$. By Definition 2.7 and relating back to the main theorem, this intuitively is interpreted as a ball containing two disjoint pieces, each equidecomposable to the entire ball.

3 The Free Group F_2

The main component of this proof is the free group on two generators (movements right-left and up-down). We will decompose the ball into pieces that can reconstruct the whole group twice.

Definition 3.1 (Free Group)**.** A *free group* on a set of generators is defined as a group where two distinct reduced words using those generators are not related between the generators beyond the group axioms. Denoting the free group on two generators as a and b in F_2, all elements are reduced words built from $\{a, a^{-1}, b, b^{-1}\}$, and the identity and empty word e. The operation would be concatenation and then reduction.

Definition 3.2 (Reduced Word)**.** A *reduced word* is any word that does not contain any adjacent pair of ss^{-1} or $s^{-1}s$ for any generator s. These consecutive pairs will cancel to the identity and thereby be removed. Every word can be reduced to some unique form.

For $\rho \in \{a, a^{-1}, b, b^{-1}\}$, let $S(\rho)$ denote the set of all reduced words in the free group F_2 whose first symbol is ρ.

Remark 3.3. Every element of F_2 either begins with a, a^{-1}, b, b^{-1}, or is just the empty word. Therefore:

$$F_2 = \{e\} \sqcup S(a) \sqcup S(a^{-1}) \sqcup S(b) \sqcup S(b^{-1}).$$

Note that all five sets are disjointly unioned and completely exhaust F_2.

Theorem 3.4 (Decomposition of F_2)**.** *The following two reconstructions of F_2 are:*

$$F_2 = S(a) \sqcup aS(a^{-1}) \qquad \text{and} \qquad F_2 = S(b) \sqcup bS(b^{-1}),$$

where $aS(a^{-1})$ denotes the set of all words in $S(a^{-1})$ with a concatenated on the left, same for $bS(b^{-1})$.

Proof. Proving the first equation allows the second to be followed by similar reasoning with b instead of a.

Take a word $w \in F_2$. If w begins with a, then $w \in S(a)$. If w does not begin with a, then $w \in \{e\} \sqcup S(a^{-1}) \sqcup S(b) \sqcup S(b^{-1})$. Adding a to the beginning of w creates the reduced word aw beginning with a, therefore $w = a^{-1}(aw) \in a^{-1}S(a)$. In other words, $aS(a^{-1}) = F_2 \setminus S(a)$.

Take a word $w \notin S(a)$. This means that w will not begin with a, and so adding a^{-1} to the beginning of w causes no cancellation, where $a^{-1}w \in S(a^{-1})$. Therefore, $w = a(a^{-1}w) \in aS(a^{-1})$.

Conversely, any word in $aS(a^{-1})$ begins with a consecutively followed by a^{-1} immediately reduces. If $w' \in S(a^{-1})$, aw' cancels the front. This cancellation means that the first term of the reduced word is a, so $aw' \notin S(a)$. This confirms that $aS(a^{-1}) \subseteq F_2 \setminus S(a)$.

Together, $aS(a^{-1}) = F_2 \setminus S(a)$, follows $S(a) \sqcup aS(a^{-1}) = F_2$. $\square$

Remark 3.5. $\{e\}$ does not appear in either reconstruction, since it is absorbed in $aS(a^{-1})$ and $bS(b^{-1})$, since the empty word does not begin with a or b, respectively. We have partitioned F_2 into five pieces and used four of them in the two reconstructions. $\{e\}$ will be expanded on in Section 5, as the identity will require special handling.

4 The Rotational Free Group in $\mathrm{SO}(3)$

Now we will find two specific rotations in $\mathbb{R}^3$ that can generate a subgroup isomorphic to F_2. This takes the algebra in Section 3 and applies it to the geometry of the ball.

Definition 4.1 (SO(3))**.** The "Special Orthogonal" group SO(3) is defined as the group of all 3×3 real matrices R that satisfy $R^\top R = I$ and $\det(R) = 1$. These matrices are chosen as all rotations through the axes will fix at the origin. The group operation would be matrix multiplication.

Let $\theta = \arccos\left(\frac{1}{3}\right)$. Since $\cos\theta = \frac{1}{3}$, we have

$$\sin\theta = \sqrt{1 - \tfrac{1}{9}} = \tfrac{2\sqrt{2}}{3}.$$

A is defined as the rotation by θ about the x-axis, and B is defined as the rotation by θ about the z-axis. Both have matrix representations:

$$A = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix}, \qquad A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 1 \end{pmatrix},$$

$$B = \frac{1}{3}\begin{pmatrix} 1 & -2\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Remark 4.2 (Why $\arccos\left(\frac{1}{3}\right)$?). This theorem depends on points not being able to overlap at all, and to do this, each action must be irrational. $\arccos\left(\frac{1}{3}\right)$ is an irrational multiple of π, meaning that no integer power of these rotations returns to the identity, and the orbit of any point under repeated application of A and B never repeats. The value $\frac{1}{3}$ is used when these matrices are applied repeatedly, causing the image point's coordinates always to carry a factor of $\frac{1}{3^n}$ in the denominator, and the numerator will always satisfy an arithmetic condition mod 3 that prevents the identity from ever being obtained. See more in Lemma 4.3 and Theorem 4.4.

Lemma 4.3 (The Form Lemma)**.** *Let G be the group generated by A and B. If $\rho \in G$ is any reduced word of length $n \geq 0$ in $\{A, A^{-1}, B, B^{-1}\}$, then*

$$\rho(0,1,0) = \frac{1}{3^n}\begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix}$$

for some integers a, b, c.

Proof. Inductive on n.

Base case ($n = 0$): $\rho = e$ (identity), so $\rho(0,1,0) = (0,1,0) = \frac{1}{3^0}(0 \cdot \sqrt{2},\ 1,\ 0 \cdot \sqrt{2})$. Taking $a = 0$, $b = 1$, $c = 0$ gives the required form.

Inductive step: Let $n > 0$ and assume the lemma holds for all reduced words of length $n-1$. Any reduced word ρ of length n can be written as $\rho = \phi\rho'$ where $\phi \in \{A, A^{-1}, B, B^{-1}\}$ and ρ' is a reduced word of length $n-1$. By this hypothesis $\rho'(0,1,0) = \frac{1}{3^{n-1}}(a\sqrt{2},\, b,\, c\sqrt{2})$ for some integers a, b, c. Now we compute $\phi \cdot \rho'(0,1,0)$ for each of the four possible choices of ϕ.

Case $\phi = A$:

$$A \cdot \frac{1}{3^{n-1}} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix} = \frac{1}{3^{n-1}} \cdot \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix} = \frac{1}{3^n} \begin{pmatrix} 3a\sqrt{2} \\ b - 4c \\ (2b + c)\sqrt{2} \end{pmatrix}.$$

This has the form $\frac{1}{3^n}(a'\sqrt{2},\, b',\, c'\sqrt{2})$ with $a' = 3a$, $b' = b - 4c$, $c' = 2b + c$, all integers.

Case $\phi = A^{-1}$:

$$A^{-1} \cdot \frac{1}{3^{n-1}} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix} = \frac{1}{3^n} \begin{pmatrix} 3a\sqrt{2} \\ b + 4c \\ (-2b + c)\sqrt{2} \end{pmatrix}.$$

With $a' = 3a$, $b' = b + 4c$, $c' = -2b + c$, all integers.

Case $\phi = B$:

$$B \cdot \frac{1}{3^{n-1}} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix} = \frac{1}{3^n} \begin{pmatrix} (a - 2b)\sqrt{2} \\ 4a + b \\ 3c\sqrt{2} \end{pmatrix}.$$

With $a' = a - 2b$, $b' = 4a + b$, $c' = 3c$, all integers.

Case $\phi = B^{-1}$:

$$B^{-1} \cdot \frac{1}{3^{n-1}} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix} = \frac{1}{3^n} \begin{pmatrix} (a + 2b)\sqrt{2} \\ -4a + b \\ 3c\sqrt{2} \end{pmatrix}.$$

With $a' = a + 2b$, $b' = -4a + b$, $c' = 3c$, all integers.

Because all four cases yield the required output form with integer coefficients, by induction the lemma holds for all $n \geq 0$. $\square$

Theorem 4.4 (G is a Free Group)**.** *No nontrivial reduced word in* $\{A, A^{-1}, B, B^{-1}\}$ *equals the identity rotation. Therefore* $G \cong F_2$*.*

Proof. Suppose for contradiction that ρ is a nontrivial reduced word of length $n > 0$ with $\rho = e$ (identity element). Then $\rho(0,1,0) = (0,1,0)$.

By Lemma 4.3:

$$\frac{1}{3^n}\begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

This forces $a\sqrt{2} = 0$, $b = 3^n$, and $c\sqrt{2} = 0$, so $a = c = 0$ and $b = 3^n$. Therefore $a \equiv b \equiv c \equiv 0 \pmod 3$.

This we can claim as impossible: for any reduced word ρ of length $n > 0$, when $\rho(0,1,0) = \frac{1}{3^n}(a\sqrt{2},\, b,\, c\sqrt{2})$, then $(a,b,c) \not\equiv (0,0,0) \pmod 3$. Inspecting the four update rules from Lemma 4.3 modulo 3:

- Under A: $(a,b,c) \mapsto (0,\, b-c,\, 2b+c) \pmod 3$. If $b \not\equiv 0$ or $c \not\equiv 0$, then output is nonzero modulo 3.
- Under A^{-1}: $(a,b,c) \mapsto (0,\, b+c,\, -2b+c) \pmod 3$. The output is nonzero modulo 3.
- Under B: $(a,b,c) \mapsto (a-2b,\, a+b,\, 0) \pmod 3$. If $a \not\equiv 0$ or $b \not\equiv 0$, the output is nonzero modulo 3.
- Under B^{-1}: $(a,b,c) \mapsto (a+2b,\, -a+b,\, 0) \pmod 3$. The output is nonzero modulo 3.

The full case analysis is systematic but lengthy, and we don't fully write it here. Confirming the claim, no sequence of these updates from $(a,b,c) = (0,1,0)$ (corresponding to $n = 0$, where $b = 1 \not\equiv 0$) will ever reach $(0,0,0) \pmod 3$.

Since the initial assumption that ρ is a nontrivial identity causes $a \equiv b \equiv c \equiv 0 \pmod 3$, which does not hold true, no such ρ can exist, meaning G must be a free group. □

Corollary 4.5. *A corresponding to a and B corresponding to b means that the group G generated by A and B is isomorphic to F_2. Therefore, Theorem 3.4 applies to G. Where*

$S(A)$, $S(A^{-1})$, $S(B)$, $S(B^{-1})$ *are subsets of* G, *we have*

$$G = S(A) \sqcup AS(A^{-1}) = S(B) \sqcup BS(B^{-1}).$$

5 Decomposing the Ball(s)

Now that G is defined as a free group, we can decompose almost all of the unit ball into two copies.

Let $L = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ denote the closed unit ball, and let $L^0 = L \setminus \{(0, 0, 0)\}$ denote the ball with its center removed.

Definition 5.1 (Orbit)**.** Two points $p, q \in L^0$ belong to the same G-orbit if there exists $\rho \in G$ such that $\rho(p) = q$. The orbit for a point p is the set $\mathcal{O}(p) = \{\rho(p) : \rho \in G\}$.

Remark 5.2. The orbits of G acting on L^0 partition L^0, as every point belongs to exactly one orbit, and two orbits are either identical or disjoint, as a standard consequence of the group action axioms, holding for any group action.

Definition 5.3 (Choice Set M)**.** By the Axiom of Choice, there exists a set $M \subset L^0$ containing only one point from each G-orbit.

Remark 5.4. The set M cannot be assigned a Lebesgue measure. Due to the countably many disjoint related copies $\{\rho(M) : \rho \in G\}$ of M together covering L^0, which has finite positive measure, if $\mu(M) = 0$, then L^0 would have a measure of zero as well, showing a contradiction. If $\mu(M) > 0$, then L^0, which is a countable union of sets with positive measure $\mu(M)$, would have infinite measure, also showing a contradiction. This essentially implies that there can not be any consistent value of $\mu(M)$, and more importantly, that M is non-measurable.

Definition 5.5 (Fixed Points D)**.** Let D be the set of all points in L^0 fixed by a nontrivial element of G, or in other words, all $p \in L^0$ such that $\rho \in G$, $\rho \neq e$, and $\rho(p) = p$ must exist. This defines the points lying on an axis of rotation of an element from G.

Remark 5.6. Because G is countable and a nontrivial rotation has one rotation each through the origin, then D is contained in a countable union of the lines through the origin. Due to this nature, D must have a Lebesgue measure of zero, so D is a null set.

Remark 5.7. For any $p \in L^0 \setminus D$ and any $m \in M$, there is at most one $\rho \in G$ with $\rho(m) = p$. This is because if $\rho_1(m) = \rho_2(m) = p$, it means that $\rho_2^{-1}\rho_1(m) = m$, and so m must be fixed by $\rho_2^{-1}\rho_1$. Since m is not in D, $\rho_2^{-1} = e$ must be true, implying $\rho_1 = \rho_2$. This means that every point in $L^0 \setminus D$ can be reached from M by some unique element of G, making the partition well-defined.

Definition 5.8 (Set X)**.** Let

$$X = \bigcup_{i=1}^{\infty} A^{-i} M,$$

where X is the set of all points reachable from M by applying A^{-1} one or more times consecutively.

Definition 5.9 (The Four Pieces)**.** Denote four subsets of $L^0 \setminus D$:

$$\begin{aligned} P_1 &= S(A)M \cup M \cup X, \\ P_2 &= S(A^{-1})M \setminus X, \\ P_3 &= S(B)M, \\ P_4 &= S(B^{-1})M, \end{aligned}$$

where $S(\phi)M = \{\rho(m) : \rho \in S(\phi),\, m \in M\}$ is the set of all points reachable from M by words that begin with ϕ.

Theorem 5.10 (Paradoxical Decomposition of $L^0 \setminus D$)**.** *The sets P_1, P_2, P_3, P_4 partition $L^0 \setminus D$, such that*

$$L^0 \setminus D = P_1 \cup AP_2 \qquad \textit{and} \qquad L^0 \setminus D = P_3 \cup BP_4.$$

$L^0 \setminus D$ must be equidecomposable with two copies of itself, hence G-paradoxical.

Proof. Since every point in $L^0 \setminus D$ is reachable from a unique element of M by a unique element of G, the sets $S(A)M$, M, X, $S(A^{-1})M \setminus X$, $S(B)M$, $S(B^{-1})M$ all account for every point in $L^0 \setminus D$ exactly once (corresponding to the partition of G, where M corresponds to $\{e\}$, and X deals with the A^{-1} chains). This implies that P_1, P_2, P_3, P_4 partition $L^0 \setminus D$.

Now we must verify the two reconstructions.

First ($L^0 \setminus D = P_1 \cup AP_2$): $AP_2 = A(S(A^{-1})M \setminus X)$. Since adding A to the beginning of an A^{-1} string cancels the leading A^{-1}, caused words to not start with A (by $F_2 = S(a) \sqcup aS(a^{-1})$), so $S(A^{-1})M \setminus X$, $S(B)M$, $S(B^{-1})M$, and final areas of M and X not in P_1. Together, $P_1 \cup AP_2 = S(A)M \cup M \cup X \cup A(S(A^{-1})M \setminus X) = L^0 \setminus D$.

Second ($L^0 \setminus D = P_3 \cup BP_4$): $BP_4 = B(S(B^{-1})M)$. Similar to A, adding B to strings starting with B^{-1} cancels out the B, creating words that don't start with it. Together, $P_3 \cup BP_4 = S(B)M \cup B(S(B^{-1})M) = L^0 \setminus D$.

Since P_1, P_2 are disjoint subsets of $L^0 \setminus D$ that cover all of $L^0 \setminus D$ after moving P_2 by A, and similar reasoning with P_3, P_4 (moving P_4 by B), the set $L^0 \setminus D$ is G-paradoxical by Definition 2.7. $\square$

6 All Exceptions

We've created two representations of the paradoxical decomposition, and now we extend from $L^0 \setminus D$ to L by removing the restriction to disregard fixed points, and bringing back the center.

Theorem 6.1 ($L^0 \setminus D \sim L^0$)**.** *The ball without its center and fixed points is equidecomposable with the ball without its center.*

Proof. We know D is contained in countably many axes through the origin, and since there are uncountably many lines and countably many bad axes, almost every line avoids D. Find a line ℓ through the origin that does not intersect D, and find an angle α for which the rotation ρ by α about ℓ. These rotated copies $\rho^n(D)$ are pairwise disjoint for integers $n \geq 0$. Because the set of angles for which $\rho^n(D) \cap \rho^m(D) \neq \emptyset$ for some $n \neq m$ requires a point from

D to map onto another one from D, and D is countable, it restricts ρ to a countable set of angles, leaving an uncountable set of available choices.

Define $E = D \cup \rho(D) \cup \rho^2(D) \cup \rho^3(D) \cup \cdots$. This implies that $L^0 = E \cup (L^0 \setminus E)$ is equidecomposable to $\rho(E) \cup (L^0 \setminus E)$ by one ρ applied to E, and also implies that $\rho(E) = \rho(D) \cup \rho^2(D) \cup \cdots = E \setminus D$, meaning that in totality $\rho(E) \cup (L^0 \setminus E) = (E \setminus D) \cup (L^0 \setminus E) = L^0 \setminus D$. So $L^0 \sim L^0 \setminus D$, and by symmetry $L^0 \setminus D \sim L^0$. $\square$

Lemma 6.2 (Circle $\sim$ Circle Minus a Point)**.** *A unit circle S^1 is equidecomposable with $S^1 \setminus \{(1,0)\}$.*

Proof. Let $A = \{(\cos n, \sin n) : n \in \mathbb{N}\}$, the set of points when $(1,0)$ is rotated counterclockwise by a radian at a time. Since π is irrational, a radian is an irrational multiple of 2π, which means that no two rotations by different positive integers will meet at the same point. This means that A is countably infinite, and since $\cos n = 1$ for some positive integer would never work (since n must be a multiple of 2π and 2π is irrational), none of its elements equal $(1,0)$.

Define a circle not in A and not equal to $(1,0)$. Let $B = S^1 \setminus (\{(1,0)\} \cup A)$. Rotating A by -1 radian creates $A' = \{(\cos(n-1), \sin(n-1)) : n \in \mathbb{N}\} = \{(\cos m, \sin m) : m \geq 0\} = A \cup \{(1,0)\}$, because $m = 0$ implies $(\cos 0, \sin 0) = (1,0)$.

Putting it all together, $S^1 \setminus \{(1,0)\} = A \cup B \sim A' \cup B = (A \cup \{(1,0)\}) \cup B = S^1$, meaning that A (rotated by -1 radian) and B are the two pieces. $\square$

Theorem 6.3 ($L^0 \sim L$)**.** *The unit ball without its center is equidecomposable with the full unit ball.*

Proof. Create a slice in the unit sphere for a circle in L that passes through the origin $(0,0,0)$. For example, the intersection of L with the xy-plane is a disk, and its boundary circle passes through the origin if it's a circle with radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0, 0)$ passing through the origin. From Lemma 6.2, the circle is equidecomposable to itself without the intersection with the origin. This means that $L \sim L^0$, or in other words $L^0 \sim L$ by symmetry of equidecomposability. $\square$

7 Final Proof

We now will combine all prior results into the satisfying theorem.

Theorem 7.1 (The Banach-Tarski Paradox)**.** *The unit ball L is* SO(3)*-paradoxical, and can be partitioned into finitely many pieces which can be reassembled using rotations and translations into two copies of L.*

Proof. By Theorem 5.8, $L^0 \setminus D$ is equidecomposable with two copies $P_1 \cup AP_2$ and $P_3 \cup BP_4$. By Theorem 6.1, $L^0 \setminus D \sim L^0$, and by Theorem 6.3, $L^0 \sim L$, so by transitivity L^0 is equidecomposable to two copies of itself. Therefore L is SO(3)-paradoxical. □

8 Implications

The Banach-Tarski paradox is a theorem, not a trick. It follows rigorously from standard axioms of mathematics. But it raises several natural questions worth addressing.

8.1 — Physically Possible? Physical matter is made of a group of finitely many atoms, which are not complex enough to make the required cuts. This paradox works as the decomposition requires pieces of infinite complexity as scattered points with no geometric definition. Because of this quality, the pieces cannot be assigned a size, and therefore a volume.

8.2 — The Axiom of Choice. M being non-measurable couldn't have been constructed without the Axiom of Choice, which allows a point from each orbit to be chosen. In frameworks where the Axiom of Choice is not allowed, the Banach-Tarski Paradox consistently fails, as well as massive groups of math in analysis, topology, and algebra.

8.3 — Minimum Pieces. In 1947, Raphael Robinson proved that five pieces are enough for the paradox, and also that four pieces stop it from working. In this paper, more than enough pieces are used for simplicity's sake.

8.4 — What Happens in Lower Dimensions? Banach and Tarski's original paper showed that the paradox doesn't work in $\mathbb{R}^1$ or $\mathbb{R}^2$. The reason for this is because of amenability, or admitting a finitely additive invariant measure defined on all subsets which prevents paradoxical decompositions. SO(3) is non-amenable because it contains F_2, and F_2 is non-amenable due to its paradoxical decomposition. Amenability is the final boundary between allowing this paradox to work or not.