Fast integer and matrix multiplication algorithms

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Goals

1. Intuition for theoretical computer science

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2. Appreciation for depth of algorithms

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2. Appreciation for depth of algorithms

3. Maybe an algorithm or two!

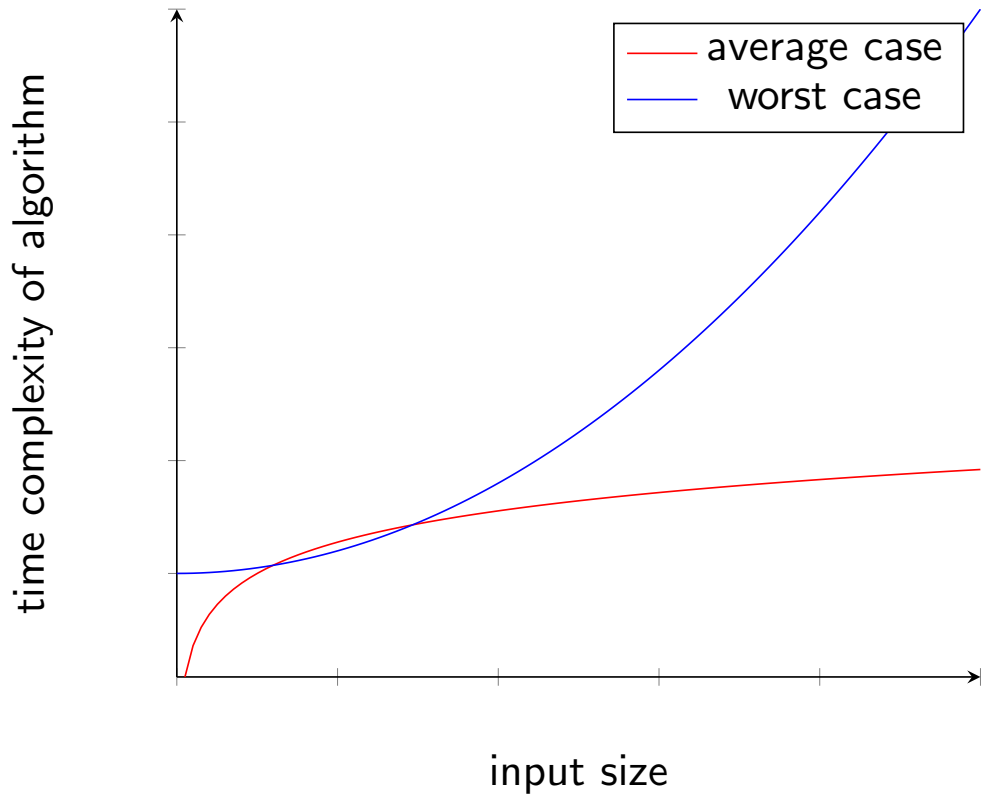
Thinking in theoretical CS

For our purposes, theoretical computer science is the field of finding and analyzing algorithms.

Thinking in theoretical CS

When thinking about the speed of algorithms, we have two major "axioms":

1. We always think in terms of the worst case.

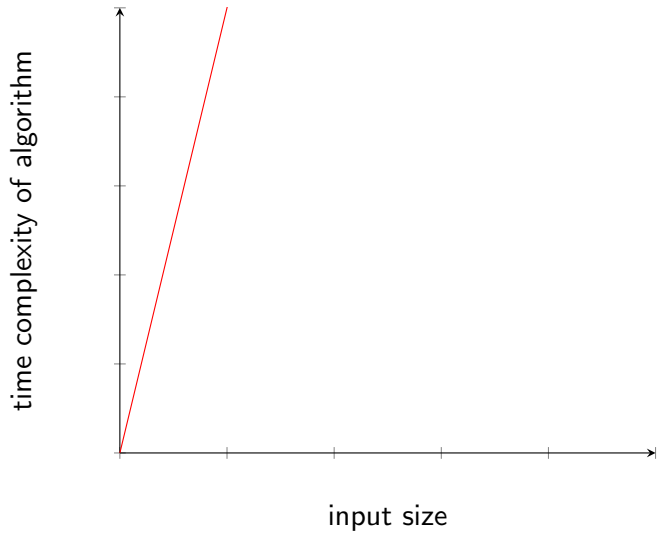


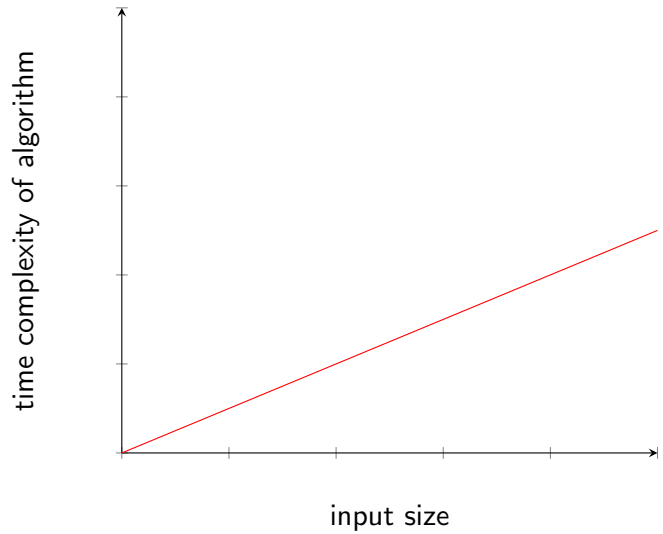
Thinking in theoretical CS

When thinking about the speed of algorithms, we have two major "axioms":

2. We only care about asymptotic behavior.

For example, the following two complexities are equivalent for our purposes:





Thinking in theoretical CS

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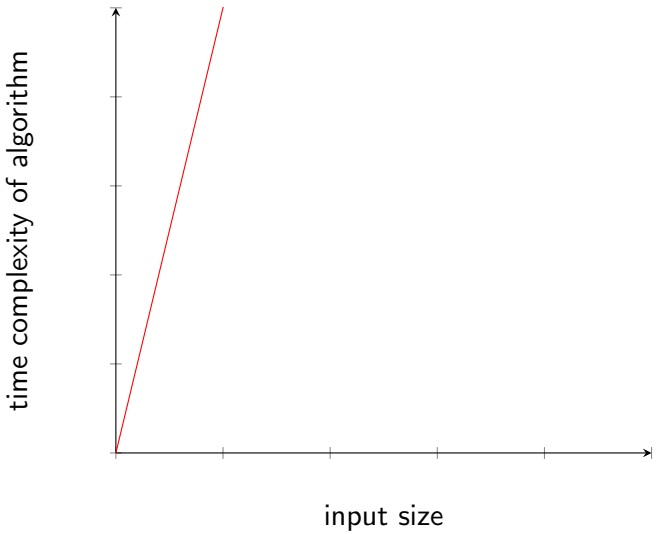
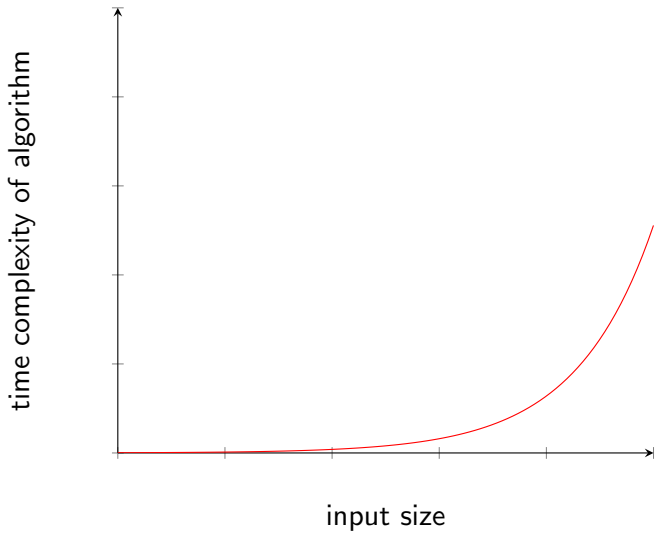
2. We only care about asymptotic behavior.

And, counterintuitively, we treat the graph on the left as faster than that on the right:





My area

1. Integer multiplication

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2. Matrix multiplication

My area

1. Integer multiplication ($O(n^2)$ naively)

2. Matrix multiplication ($O(n^3)$ naively)

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1. Integer multiplication ($O(n^2)$ naively) **(now $O(n \log n)$)**

2. Matrix multiplication ($O(n^3)$ naively) **(now $O(n^{2.371339})$)**

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   However, both of these theoretical upper bounds result in *galactic algorithms*.

Example: Toom-Cook

1. Express inputs X and Y as polynomials where each block of digits is a coefficient

$$X = 123, Y = 456$$
$$p(b) = 1b^2 + 2b + 3$$
$$q(b) = 4b^2 + 5b + 6.$$

Note

$$p(10) = 123 = X, q(10) = 456 = Y.$$

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The big idea:

$$r(b) = p(b)q(b)$$
$$r(10) = p(10)q(10) = XY.$$

If we find the coefficients of r using interpolation, we can just plug in 10 to compute the output.

Example: Toom-Cook

2. Find input-output pairs for p & q, then r

$$p(0) = 3, p(1) = 6, p(-1) = 2, p(2) = 11, p(-2) = 3$$

$$q(0) = 6, q(1) = 15, q(-1) = 5, q(2) = 32, q(-2) = 12.$$

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Divide-and-conquer: use this *same algorithm* to compute the products of p and q.

Example: Toom-Cook

3. Use multiplication by inverse matrix to compute coefficients of r

   Define the coefficients

$$r(b) = c_4 b^4 + c_3 b^3 + c_2 b^2 + c_1 b + c_0.$$

   Then:

$$\begin{bmatrix} i_1^4 & i_1^3 & i_1^2 & i_1^1 & i_1^0 \\ i_2^4 & i_2^3 & i_2^2 & i_2^1 & i_2^0 \\ i_3^4 & i_3^3 & i_3^2 & i_3^1 & i_3^0 \\ i_4^4 & i_4^3 & i_4^2 & i_4^1 & i_4^0 \\ i_5^4 & i_5^3 & i_5^2 & i_5^1 & i_5^0 \end{bmatrix} \begin{bmatrix} c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} r(i_1) \\ r(i_2) \\ r(i_3) \\ r(i_4) \\ r(i_5) \end{bmatrix}.$$

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5. Toom-Cook multiplication has an time complexity of

$$O(n^{\log_k(2k-1)})$$

($\approx O(n^{1.465})$ for our example).

Example: Matrix multiplication

1. Multiplication of two $n \times n$ matrices can be represented with a (usually) sparse $n^2 \times n^2 \times n^2$ tensor. Multiplication becomes equivalent to tensor contraction.

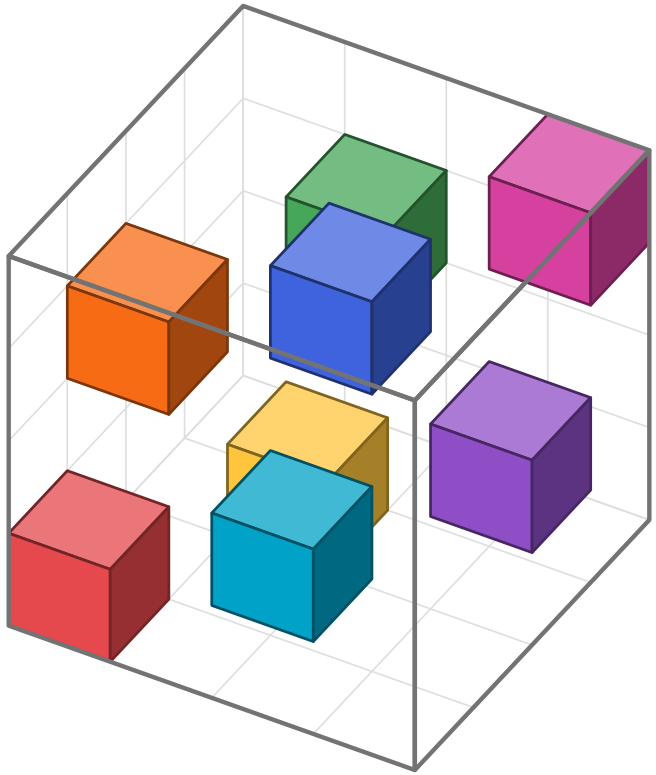

Figure: 2×2 matrix multiplication tensor; each colored cube represents an entry of 1, while all other entries are 0.

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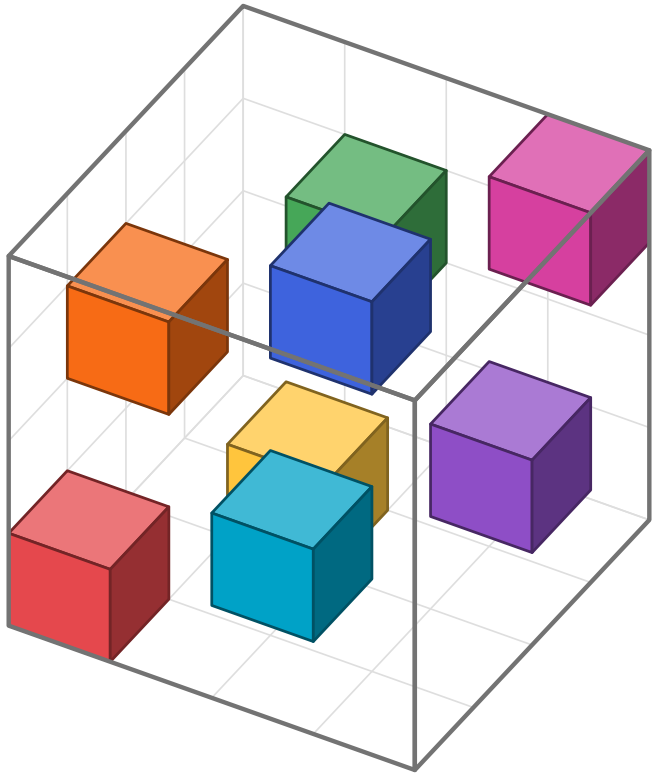
2. Decomposition of a matmul tensor into R rank-1 tensors is equivalent to a matrix multiplication algorithm with R bilinear multiplications.

Remark

Why do we only care about bilinear multiplications?



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3. More efficient decompositions (with lower R) lead to faster matmul algorithms.

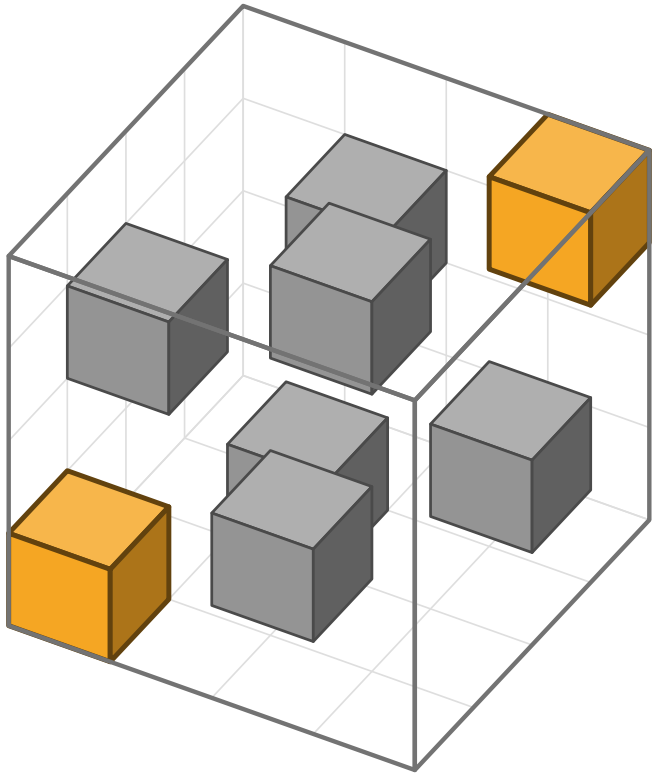

Figure: Strassen's 1969 decomposition of the same 2×2 matrix multiplication tensor, this time into only 7 rank-1 subtensors.

Example: Matrix multiplication

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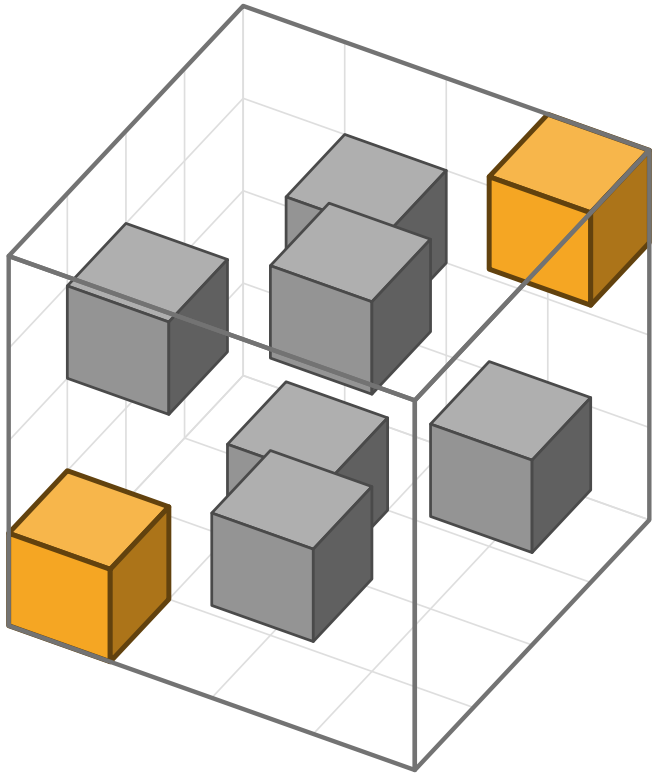
4. Strassen's laser method (not pictured): same general idea, way more complex.



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2. Algorithms often work with abstract mathematical representations of a problem
3. Best time complexity $\not\Rightarrow$ actually useful
4. Shared techniques (e.g., divide-and-conquer)