On Bessel Functions

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Section 1

Introduction

Introduction

Introduction

This talk is about Bessel Functions, which are the solutions to the Bessel Equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) y = 0.$$

This equation arises in a variety of problems, especially those to do with cylindrical symmetry such as vibrations and electromagnetic waves.

Section 2

Emergence of Bessel Functions

Drumhead Example

Here, we will take the example of a drumhead that has been struck to show how the equation emerges. We use $u(x, y, t)$ to denote the height of the drum at position (x, y) at time t. First, we define the drumhead: Its a thin elastic membrane clamped at a circular rim stretched under uniform tension. Clamped meaning that $u(x, y, t) = 0$ at that rim. There are two important physical properties:

1. **Tension** (T): The tension is the force pulling it outwards at every edge. Note that T specifically, however, is not referring to a force. If you cut a line through the drumhead of length L, the force would be $T \times L$. T has units Nm^{-1}.
2. **Surface density** (ρ): This is how heavy the membrane is per unit area. Its units are kgm^{-2}.

Drumhead Example

When it's at rest, the tension cancels out as they are in opposite directions. i. e. $u = 0$ everywhere when it's at rest.

Derivation

Note that in this section, we will use $\nabla^2 u$ to denote

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Newton's Second Law

For a rectangular patch of membrane (as shown in Figure 1) at (x_0, y_0) with width Δx and height Δy, its mass is

$$m = \rho \cdot \Delta x \cdot \Delta y.$$

Its vertical acceleration is $\frac{\partial^2 u}{\partial t^2}$. Newton's second law states that the force is the product of mass and acceleration, so we get

$$F_{\text{net}} = \rho \cdot \Delta x \cdot \Delta y \cdot \frac{\partial^2 u}{\partial t^2}.$$

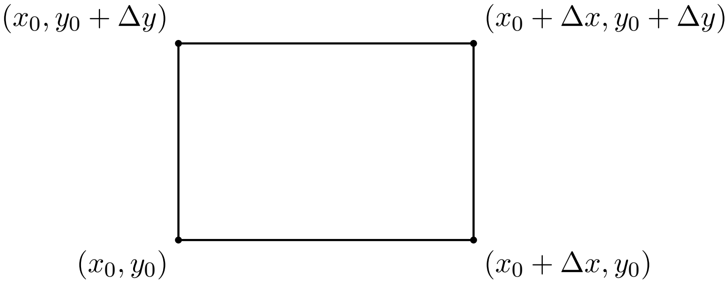

Figure: A rectangle in the Cartesian plane.

Restoring Force

The rectangular patch has four edges. What needs to be found is the net vertical component of all these pulls. First, consider only one direction (x). The right edge of the patch ($x_0 + \Delta x$) is pulled by force $T \times \Delta y$ since Δy is the length:

$$F_{\text{right}} = T \cdot \Delta y \cdot \left. \frac{\partial u}{\partial x} \right|_{x = x_0 + \Delta x}.$$

Restoring Force

The left edge of the patch (x) is pulled by force $-T \cdot \Delta y$. This is negative because this force is in the opposite direction:

$$F_{\text{left}} = -T \cdot \Delta y \cdot \left.\frac{\partial u}{\partial x}\right|_{x=x_0}.$$

This means that the net force on the x axis is

$$F_x = T \cdot \Delta y \cdot \left[\left.\frac{\partial u}{\partial x}\right|_{x=x_0+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x=x_0} \right] = T \cdot \Delta y \cdot \Delta x \cdot \frac{\partial^2 u}{\partial x^2}.$$

Restoring Force

This is because the change in slope over distance Δx is the rate of change of slope with respect to x times Δx. The same is used for F_y:

$$F_y = T \cdot \Delta y \cdot \Delta x \cdot \frac{\partial^2 u}{\partial y^2},$$

$$F_{\text{net}} = T \cdot \Delta x \Delta y \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Restoring Force

Putting the previous sections together, we get

$$\rho \cdot \Delta x \Delta y \cdot \frac{\partial^2 u}{\partial t^2} = T \cdot \Delta x \Delta y \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

$$\rho \cdot \frac{\partial^2 u}{\partial t^2} = T \cdot \nabla^2 u.$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \nabla^2 u. \tag{1}$$

where $c^2 = T/\rho$.

We can now switch to polar coordinates because it is easier to express the condition of the rim. This is because in cartesian coordinates, the condition is $u = 0$ when $x^2 + y^2 = R$, whereas in polar coordinates, the condition is $u = 0$ when $r = R$. The chain rule is used to switch Equation (1) to polar coordinates, which gives

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right). \tag{2}$$

There are three variables: r, θ, t. Assume the solution has the form

$$u(r, \theta, t) = R(r) \cdot \Theta(\theta) \cdot T(t).$$

Equation

Using the separation of variables methods twice:

$$r^2 R'' + rR' + (k^2 r^2 - n^2)R = 0.$$

Define $x = kr$. $\frac{d}{dr} = k\frac{d}{dx}$. So:

$$R' = k\frac{dR}{dx}, \;\; R'' = k^2\frac{d^2R}{dx^2}.$$

Equation

Substituting,

$$\frac{x^2}{k^2}(k^2)\frac{d^2R}{dx^2} + \frac{x}{k}(k)\frac{dR}{dx} + (x^2 - n^2)R = 0,$$

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0.$$

This is a Bessel Equation of order n.

Section 3

Derivation

Derivation

This section will cover the derivation of the Bessel Function from the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0.$$

To find a solution to this equation, we use the Frobenius Method. This means that y will have a solution of the form

$$y(x) = \sum_{h=0}^{\infty} a_h x^{h+r}.$$

Therefore, the derivatives of y will be

$$y'(x) = \sum_{h=0}^{\infty} (h+r) a_h x^{h+r-1}, \;\; y''(x) = \sum_{h=0}^{\infty} (h+r-1)(h+r) a_h x^{h+r-2}.$$

Derivation

We can now substitute these into the Bessel Equation to obtain:

$$x^2 \sum_{h=0}^{\infty} (h+r-1)(h+r)a_h x^{h+r-2} + x \sum_{h=0}^{\infty} (h+r)a_h x^{h+r-1}$$

$$+ (x^2 - n^2) \sum_{h=0}^{\infty} a_h x^{h+r} = 0. \tag{3}$$

Bringing the x^2, x, and $x^2 - n^2$ into the sums, we are left with

$$\sum_{h=0}^{\infty}(h+r-1)(h+r)a_h x^{h+r} + \sum_{h=0}^{\infty}(h+r)a_h x^{h+r} + \sum_{h=0}^{\infty} a_h x^{h+r+2} - n^2 \sum_{h=0}^{\infty} a_h x^{h+r} = 0. \tag{4}$$

To be able to find an expression for a_h, we would like the powers of x to be the same. In the third series, the power of x is 2 higher than the others

To rectify this, we introduce a variable s. In the third series, we use $s = h + 2$ and in the others, $s = h$. Due to this, the beginning of the third series should be changed from $h = 0$ to $h = 2$ as $s = h + 2 = 0 + 2 = 2$. The others series would remain as $s = 0$:

$$\sum_{s=0}^{\infty}(s+r-1)(s+r)a_s x^{s+r} + \sum_{s=0}^{\infty}(s+r)a_s x^{s+r} + \sum_{s=2}^{\infty} a_{s-2}x^{s+r} - n^2 \sum_{s=0}^{\infty} a_s x^{s+r} = 0. \tag{5}$$

Derivation

For the series to converge to a value, the LHS has to be equal to 0 for each value of s. First, for $s = 0$, using Equation (5) (Note that the third series has no contribution to $s = 0$ and $s = 1$ because it begins at $s = 2$):

$$r(r-1)a_0x^r + ra_0x^r - n^2a_0x^r = 0,$$

$$r^2 - n^2 = 0, \implies r_1 = n, r_2 = -n. \tag{6}$$

For this function, we take $r = r_1 = n$. For $s = 1$, using Equation (5), this gives us

$$(1+n)(1+n-1)a_1 + (1+n)a_1 - n^2 a_1 = 0,$$

$$(n+1)^2 a_1 - n^2 a_1 = 0 \implies (n^2+2n+1)a_1 - n^2 a_1 = 0$$

$$(2n+1)a_1 = 0 \implies a_1 = 0.$$

Bessel Functions of the First Kind

Now, let's take the case $s = s$ where $s \neq 0, 1$:

$$(s+n)(s+n-1)a_s + (s+n)a_s + a_{s-2} - n^2 a_s = 0,$$

$$(s+n)^2 a_s - n^2 a_s + a_{s-2} = 0,$$

$$s(s+2n)a_s + a_{s-2} = 0,$$

$$a_s = -\frac{a_{s-2}}{s(s+2n)}. \tag{7}$$

Since a_1 is 0, Equation (7) shows that all odd numbered coefficients are 0. This means that there will be only even numbered coefficients, so we can use $s = 2m$:

$$a_{2m} = -\frac{a_{2m-2}}{2m(2m+2n)} = -\frac{a_{2m-2}}{2^2 m(m+n)}.$$

By using the recursive relation, we can see that

$$\begin{aligned} a_{2m} &= (-1)\frac{1}{2^2 m(m+n)}\left((-1)\frac{1}{2^2(m-1)(m-1+n)}(\cdots)\right) \\ &= \frac{(-1)^m a_0}{2^{2m} m!(m+n)\cdots(n+1)}. \end{aligned}$$

The coefficient a_0 is arbitrary because any scaled version of a solution is also a solution and a_0 is a constant. Since a_0 is arbitrary and we are trying to find a general solution, we can pick any value of a_0. Here, we pick a value to make the denominator simpler:

$$a_0 = \frac{1}{2^2 n!}.$$

Using this, the coefficient formula becomes

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!}.$$

Bessel Functions of the First Kind

Substituting this into the infinite sum, we get the Bessel Function of the First Kind ($J_n(x)$):

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}.$$

Bessel Functions of the First Kind

However, this only works for $n \in \mathbb{Z}$. In order to extend it to any nonnegative v, we need to use the Gamma Function. The Gamma function is an extension of the factorial function with $\Gamma(n+1) = n!$. The new formula using the gamma function is

$$J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} m! \Gamma(v+m+1)}. \tag{8}$$

Wronskian

To check whether two functions are linearly independent, we use the Wronskian. The Wronskian is equal to 0 when the functions are linearly dependent. The formula for the Wronskian is

$$W(f, g) = f \cdot g' - f' \cdot g.$$

Wronskian

Compute the Wronskian (the working has been omitted):

$$W\left(J_v(x), J_{-v}(x)\right) = J_v(x) \cdot J'_{-v}(x) - J'_v(x) \cdot J_{-v}(x),$$

$$= -\frac{2\sin\left(v\pi\right)}{\pi x}.$$

Finding Yn

Here are the conditions for the the second solution, Y_n:

1. Needs to be independent from J_v as that is a condition to form the full solution space.
2. Varies smoothly as $v \to n$, for all $n \in \mathbb{Z}$ to fulfill the condition set at the end of the previous section ($Y_n(x) = \lim_{v \to n} Y_v(x)$) to ensure that the function remains well-defined and provides a continuous extension to integer orders.
3. $W(J_v(x), Y_v(x)) = \frac{2}{\pi x}$. This is chosen merely for convenience and to simplify formulas that come out of the Bessel Functions.

Finding Yn

Any solution for the equation has to be a linear combination of J_v and J_{-v} (i.e. $aJ_v + bJ_{-v}$) and the Wronskian has to be $\frac{2}{\pi x}$:

$$W(J_v, aJ_v + bJ_{-v}) = a \cdot W(J_v, J_v) + b \cdot W(J_v, J_{-v}) = b \cdot \left(-\frac{2\sin(v\pi)}{\pi x} \right).$$

Finding Yn

The Wronskian of a function with itself is always 0 as they are always linearly dependent. The Wronskian can be split up as shown above because it is linear in its second argument due to it being calculated from the determinant which is linear.

Finding Yn

Equating this to $\frac{2}{\pi x}$ gives us

$$b \cdot \left(-\frac{2\sin\left(v\pi\right)}{\pi x}\right) = \frac{2}{\pi x} \implies b = -\frac{1}{\sin\left(v\pi\right)}.$$

At integer $v = n$, $J_{-n} = (-1)^n J_n$, so $aJ_v + bJ_{-v}$ approaches

$$aJ_n + (-1)^n \left(-\frac{1}{\sin(v\pi)}\right) J_n = \left(a - \frac{(-1)^n}{\sin(v\pi)}\right) J_n.$$

Finding Yn

$\frac{1}{\sin(v\pi)}$ diverges as $v \to n$. To cancel out this divergence, a also needs to diverge in the same manner. We choose $a = \frac{\cos(v\pi)}{\sin(v\pi)}$ because near $v = n$,

$$\frac{\cos(v\pi)}{\sin(v\pi)} \approx \frac{(-1)^n}{\sin(v\pi)}.$$

This produces a $\frac{0}{0}$ form with a finite L'Hôpital limit rather than diverging. This gives us

$$Y_v(x) = \frac{J_v(x)\cos(v\pi) - J_{-v}(x)}{\sin(v\pi)}.$$

Finding Yn

Now, L'Hôpital's rule computing it with (steps omitted), we get

$$Y_n\left(x\right) = \frac{\left(-1\right)^n \frac{\partial J_v}{\partial v}\Big|_{v=n} + \frac{\partial J_v}{\partial v}\Big|_{v=-n}}{\pi\left(-1\right)^n} = \frac{1}{\pi}\frac{\partial J_v}{\partial v}\Bigg|_{v=n} + \frac{\left(-1\right)^n}{\pi}\frac{\partial J_v}{\partial v}\Bigg|_{v=-n}. \tag{9}$$

Finding Yn

Evaluating Equation (9) (steps omitted), gives us

$$Y_n(x) = \frac{2}{\pi} J_n(x) \log \frac{x}{2} - \frac{1}{\pi} (-1)^{n-1} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-n}$$
$$- \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n} \left[\psi(m+n+1) + \psi(m+1)\right]. \tag{10}$$

Section 4

Gauss Circle Problem

Gauss Circle Problem

This is the first application of Bessel Functions that will be covered in this paper. The main question of the problem is, in a circle of radius R, how many points (m, n) with $m, n \in \mathbb{Z}$ land on or in the circle.

The number of points is denoted as $N(R)$. For example, $N(1) = 5$ with the points being $(0, 0), (\pm 1, 0), (0, \pm 1)$. The most obvious guess is that it is πR^2 as each unit of area would have roughly one point inside it. This gives an error term of

$$E(R) = N(R) - \pi R^2.$$

Gauss proved that $|E(R)| \leq C \cdot R$ for some constant C. This is because the error can only come from the boundary strip of width 1 of the circle, which is proportional to R. The problem asks about how much closer we can get.

Integral Representation of the Bessel Function

The problem involves the usage of the Bessel Function through an integral representation for which the proof is omitted:

$$J_0\left(x\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\cos(\theta)}\, d\theta. \tag{11}$$

Rephrasing

Here we use

$$r_2(n) = \#\left\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\right\}.$$

Using this, we can define $N(R)$ as

$$N(R) = \sum_{n=0}^{\lfloor R^2 \rfloor} r_2(n).$$

This formula is summing up each shell of radius $\sqrt{n}$.

The Poisson Summation formula says that

$$\sum_{(m,n)\in\mathbb{Z}^2} f(m,n) = \sum_{(j,k)\in\mathbb{Z}^2} \hat{f}(j,k)$$

where $\hat{f}$ is the Fourier Transform of f. Let f be the indicator function of the disk, i. e. $f(x) = 1$ if $x^2 + y^2 \leq R^2$ and 0 otherwise. This means that the left side of the formula will be $N(R)$ and the right side will be a sum of Fourier Transforms of the disk.

Fourier Transform

The Fourier Transform of the indicator function of the disk of radius R is:

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} 1_{x^2+y^2 \leq R^2} e^{-2\pi i (x\xi_1 + y\xi_2)} \, dx \, dy.$$

We can now switch to polar coordinates to make the function simpler, with $\rho = |\xi|$:

$$\hat{f}(\xi) = \int_0^R r \left(\int_0^{2\pi} e^{-2\pi i r \rho \cos(\theta)} \, d\theta \right) \, dr.$$

The inner integral is a representation of the Bessel Function:

$$\int_0^{2\pi} e^{-2\pi i r\rho\cos(\theta)}\, d\theta = 2\pi J_0\left(2\pi r\rho\right),$$

$$\hat{f}\left(\xi\right) = 2\pi \int_0^R r J_0\left(2\pi r\rho\right)\, dr.$$

Using the Bessel Recurrence Identity $\frac{d}{dt}\left[tJ_1\left(t\right)\right] = tJ_0\left(t\right)$ and $t = 2\pi r\rho$,

$$\hat{f}\left(\xi\right) = \frac{R}{\rho} J_1\left(2\pi R\rho\right) = \frac{R}{|\xi|} J_1\left(2\pi R|\xi|\right).$$

Fourier Transform

Putting it together, we get:

$$E\left(R\right) = N\left(R\right) - \pi R^2,$$

$$E\left(R\right) = \sum_{(j,k)\neq(0,0)} \frac{R}{\sqrt{j^2+k^2}} J_1\left(2\pi R\sqrt{j^2+k^2}\right).$$

The $(0, 0)$ term omitted from the sum is the πR^2 and the rest of it forms the error term. Simplifying with $n = \sqrt{j^2 + k^2}$ and the number of pairs being $r_2(n)$:

$$E\left(R\right) = R \sum_{n=1}^{\infty} \frac{r_2\left(n\right)}{\sqrt{n}} J_1\left(2\pi\sqrt{n}R\right).$$

Section 5

Heat Equation on a Cylinder

Heat Equation on a Cylinder

Heat Equation on a Cylinder

This section is another application of the Bessel Functions. The main topic here is modelling the heat over time on a cylinder. The derivation will not be covered in the talk, just the results.

Heat Equation on a Cylinder

Consider a cylinder of radius a and height L. In this section, we will be working with cylindrical coordinates (r, θ, z). This is an extension of the polar coordinates with z as the height. The bounds for these are: $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq L$. The heat equation is:

$$u_t = \kappa \Delta u.$$

where κ is the thermal diffusivity and Δ is the Laplacian.

By using the Separation of Variables Method, we would get the final solution as, which uses the Bessel Functions:

$$\boxed{\begin{aligned} u(r,\theta,z,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} & J_m\left(\frac{j_{m,n}r}{a}\right) \\ & \left[A_{m,n,p}\cos(m\theta) + B_{m,n,p}\sin(m\theta)\right] \\ & \sin\left(\frac{p\pi z}{L}\right)\exp\left(-\kappa\lambda_{m,n,p}^{2}t\right). \end{aligned}} \tag{12}$$

The paper

Since this talk is 15 minutes, I have not been able to cover all the topics. Below is the list that are covered in the paper:

1. The detailed derivation of the Bessel Functions (main topic of the paper, roughly 10 pages)
2. The full explanation of the Heat Equation problem
3. Some important properties of the Bessel Functions I referenced during the talk
4. Modified Bessel Functions, which use a complex substitution to model exponential growth and decay
5. An explanation of the Frobenius Method
6. Rewriting with Harmonic Numbers