On Bessel Functions

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Abstract

Bessel functions are solutions to the Bessel Equation, which arise in a variety of fields such as Physics and Number Theory. This paper provides an introduction to the Bessel Functions by detailing the derivation of the Bessel Functions of the First and Second Kinds using the Frobenius Method. Their key properties such as when they are linearly dependent and some important identities that are used are explained and proved. It also provides a short introduction to Modified Bessel Functions, which have to do with exponential growth and decay rather than oscillations like the normal Bessel Functions. It concludes with two examples of the applications: the Gauss Circle Problem and the Heat Equation on a Cylinder.

1 Introduction

This paper will cover the Bessel Functions that arise from solving the Bessel Equation,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0,$$

which is an equation that arises in a variety of different fields, and especially in physics. They occur specifically in problems that involve cylindrical symmetry, such as heat diffusion in a cylinder, electromagnetic waves, vibrations, and quantum mechanics. This paper aims to show how they come up, how the functions are derived through the Frobenius Method, their properties, their applications, and a modified version of them.

Section 2 will cover the background knowledge required for this paper including the linear independence of functions, the Wronskian, the Gamma and Digamma functions, and the Laplacian and cylindrical coordinates.

Section 3 will show how the Bessel Equation arises from the example of a drumhead being struck by using the Laplacian and rearranging the equation by separating the variables.

Section 4 will cover how the Bessel Functions are derived from the equation using the Frobenius Method, including the construction of the second solution. It also details the process of rewriting the function in terms of Harmonic numbers.

Section 5 will cover some important properties of the Bessel Functions such as when they are linearly dependent, when they are elementary, and some key identities relating to the derivatives and recursive relations.

Section 6 will cover the derivation of the Modified Bessel Functions and the main difference between them and normal Bessel Functions.

Section 7 will cover an application of Bessel Functions in Number Theory: the Gauss Circle Problem, which is about the number of lattice points in a circle of radius R. It uses Bessel Functions along with the Fourier Transform, Fourier Series, and the Poisson Summation to come up with a formula for the error term.

Section 8 will cover an application of Bessel Functions in Thermal Physics: Heat Equation on a Cylinder, which models the diffusion of heat in a cylinder using Bessel Functions, Differential Equations, and the Laplacian.

2 Background

This section will cover the background information for this paper.

2.1 Linear Independence in Functions

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ defined on an interval I are linearly independent if none of the functions can be expressed as a linear combination of other functions in the set. That is, if the equation,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0,$$

for all $x \in I$ implies that

$$c_1 = c_2 = \cdots = c_n = 0.$$

Linearly dependent functions are the opposite, they can be expressed as a linear combination of the others.

2.2 Wronskian

To determine if a set of solutions are linearly dependent, the Wronskian is used. For two differentiable functions, it is defined as

$$W(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x).$$

If for all x in an interval, the Wronskian is equal to 0, then the functions are linearly independent. Specifically in the case of a second-order differential equation, if for any point x_0 in the interval, the Wronskian does not equal 0, the solutions are linearly independent. This is because of Abel's identity which shows that for a differential equation of the form

$$\frac{d^2y}{dx^2} + P\left(x\right)\frac{dy}{dx} + Q\left(x\right)y = 0,$$

the Wronskian is:

$$W(f,g)(x) = W(f,g)(x_0) \cdot \exp\left(-\int_{x_0}^{x} P(t)dt\right).$$

The exp function is never equal to 0, so the Wronskian is either always 0 or always not 0. The Wronskian is linear in the second argument because it uses the determinant which has linearity.

2.3 Gamma and Digamma functions

The Gamma function is a generalisation of the factorial, to extend to the negative numbers and is denoted by $\Gamma(x)$. It is defined as

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx.$$

Below are some important results of the Gamma Function.
The Gamma function evaluated at 1:

$$\Gamma(1) = \int_0^\infty e^{-x}x^0\,dx = \left[-e^{-x}\right]_0^\infty = 1.$$

The Gamma function evaluated at $\frac{1}{2}$ using the substitution $x = t^2$ and the Gaussian integral $\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-\frac{1}{2}}\,dx = 2\int_0^\infty e^{-t^2}\,dt = 2\left[\frac{\sqrt{\pi}}{2}\right] = \sqrt{\pi}.$$

The most important property of the Gamma function is that $\Gamma(n+1) = n\Gamma(n)$ as shown below:

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^n\,dx = -\left[x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1}e^{-x}\,dx = 0 + n\Gamma(n) = n\Gamma(n).$$

This property is used to extend the factorial for positive integers as it is the same property that the factorial has. Since we have $\Gamma(n+1) = n\Gamma(n)$,

$$\Gamma(n+1) = n!.$$

When it is a negative rational number,

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} = \frac{(n+1)\,\Gamma(n+1)}{n\,(n+1)} = \frac{\Gamma(n+2)}{n\,(n+1)} = \cdots = \frac{\Gamma(n+k+1)}{n\,(n+1)\cdots(n+k)},$$

where k is the least positive integer such that $(n+k+1) > 0$. The last property that needs to be discussed is that the Gamma Function is not defined for negative integers and 0 since

$$\Gamma(0) = \frac{\Gamma(0+k+1)}{0\,(1)\cdots(0+k)}, \quad \Gamma(-1) = \frac{\Gamma(1+k+1)}{(-1)\,(0)\cdots(-1+k)}.$$

Since there is a 0 in the denominator, it can be concluded that the function is undefined at these values.
The Digamma Function is defined as the logarithmic derivative of the Gamma Function and is denoted by $\psi(x)$:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

A useful value of the function is

$$\psi(1) = -\gamma.$$

where γ is the Euler-Mascheroni constant.

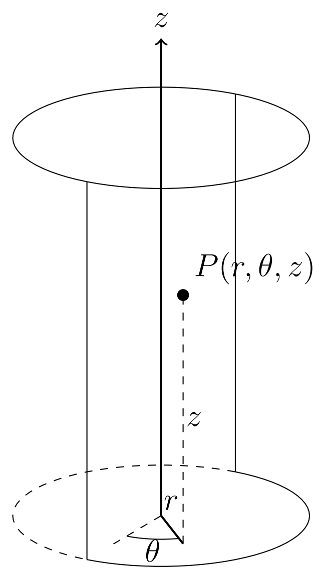

Figure 1: The cylindrical coordinate system.

2.4 Cylindrical Coordinates and the Laplacian

2.4.1 Cylindrical Coordinates

Cylindrical coordinates, which are an extension of polar coordinates, are useful in problems involving cylindrical symmetry. Rather than (x, y, z) in the cartesian coordinates, it uses

$$(r, \theta, z),$$

where r is the radial distance, θ is the angle, and z is the same as in cartesian coordinates. The transformation is:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Figure 1 shows how the cylindrical coordinates work.

2.4.2 The Laplacian

The Laplacian is a second-order differential operator that measures how different a point is from its surroundings. In cartesian coordinates with 3 dimensions, it is defined as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

In cylindrical coordinates, using the transformations shown above and the chain rule,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

3 Emergence of the Bessel Equation

Here, we will take the example of a drumhead that has been struck to show how the equation emerges. We use $u(x, y, t)$ to denote the height of the drum at position (x, y) at time t. First, we define the drumhead: It's a thin elastic membrane clamped at a circular rim stretched under uniform tension. Clamped meaning that $u(x, y, t) = 0$ at that rim. There are two important physical properties:

1. **Tension** (T): The tension is the force that pulls it outwards at every edge. Note that this quantity T, however, is not referring to the force. If you cut a line through the drumhead of length L, the force would be $T \times L$. T has units Nm^{-1}.

2. **Surface density** (ρ): This is how heavy the membrane is per unit area. Its units are kgm^{-2}.

When it's at rest, the tension cancels out as they are in opposite directions. i. e. $u = 0$ everywhere when it's at rest.

3.1 Derivation

Note that in this section, we will use $\nabla^2 u$ to denote

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

3.1.1 Newton's Second Law

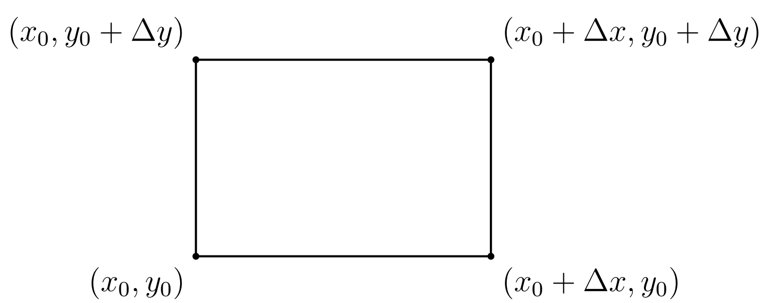

Figure 2: A rectangle in the Cartesian plane.

For a rectangular patch of membrane (as shown in Figure 2) at (x_0, y_0) with width Δx and height Δy, its mass is

$$m = \rho \cdot \Delta x \cdot \Delta y.$$

Its vertical acceleration is $\frac{\partial^2 u}{\partial t^2}$. Newton's second law states that the force is the product of mass and acceleration, so we get

$$F_{\text{net}} = \rho \cdot \Delta x \cdot \Delta y \cdot \frac{\partial^2 u}{\partial t^2}.$$

3.1.2 Restoring Force

The rectangular patch has four edges. What needs to be found is the net vertical component of all these pulls. First, consider only one direction (x). The right edge of the patch ($x_0 + \Delta x$) is pulled by force $T \times \Delta y$ since Δy is the length:

$$F_{\text{right}} = T \cdot \Delta y \cdot \left. \frac{\partial u}{\partial x} \right|_{x = x_0 + \Delta x}.$$

The left edge of the patch (x) is pulled by force $-T \cdot \Delta y$. This is negative because this force is in the opposite direction:

$$F_{\text{left}} = -T \cdot \Delta y \cdot \left. \frac{\partial u}{\partial x} \right|_{x = x_0}.$$

This means that the net force on the x axis is

$$F_x = T \cdot \Delta y \cdot \left[\left. \frac{\partial u}{\partial x} \right|_{x = x_0 + \Delta x} - \left. \frac{\partial u}{\partial x} \right|_{x = x_0} \right] = T \cdot \Delta y \cdot \Delta x \cdot \frac{\partial^2 u}{\partial x^2}.$$

This is because the change in slope over distance Δx is the rate of change of slope with respect to x times Δx. The same is used for F_y:

$$F_y = T \cdot \Delta y \cdot \Delta x \cdot \frac{\partial^2 u}{\partial y^2},$$

$$F_{\text{net}} = T \cdot \Delta x \Delta y \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Putting the previous sections together, we get

$$\rho \cdot \Delta x \Delta y \cdot \frac{\partial^2 u}{\partial t^2} = T \cdot \Delta x \Delta y \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

$$\rho \cdot \frac{\partial^2 u}{\partial t^2} = T \cdot \nabla^2 u.$$
$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \nabla^2 u. \tag{1}$$

where $c^2 = T/\rho$.

3.2 The Differential Equation

We can now switch to polar coordinates because it is easier to express the condition of the rim. This is because in cartesian coordinates, the condition is $u = 0$ when $x^2 + y^2 = R$, whereas in polar coordinates, the condition is $u = 0$ when $r = R$. The chain rule is used to switch Equation (1) to polar coordinates, which gives

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right). \tag{2}$$

3.2.1 Separation of Variables

There are three variables: r, θ, t. Assume the solution has the form

$$u(r, \theta, t) = R(r) \cdot \Theta(\theta) \cdot T(t).$$

Substitute into Equation (2). The left side is

$$R(r) \cdot \Theta(\theta) \cdot T''(t).$$

The right side is

$$c^2 \left(\Theta T R'' + \Theta T \frac{R'}{r} + \frac{RT}{r^2}\Theta'' \right).$$

Divide everything by $\Theta T R$ to get

$$\frac{T''}{T} = c^2 \left(\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2 \Theta} \right).$$

Since the sides are dependant on different variables, the only way they can always be equal is if both sides are equal to a constant. We pick $-\omega^2$ because if it were positive, T would grow exponentially, and so would u, which does not represent the situation. The reason it is squared is so that a cleaner solution is produced without a square root in it:

$$\frac{T''}{T} = -\omega^2 \implies T'' + \omega^2 T = 0.$$

Solving this, we get

$$T(t) = A\cos(\omega t) + B\sin(\omega t).$$

Now, for the right side, rearranging gives

$$\frac{r^2 R''}{R} + \frac{rR'}{R} + r^2k^2 = -\frac{\Theta''}{\Theta}.$$

Using the same method as last time, since both sides depend on different variables, they must be equal to a constant which we will call n^2:

$$-\frac{\Theta''}{\Theta} = n^2 \implies \Theta'' + n^2\Theta = 0.$$

Solving gives:

$$\Theta(\theta) = A\cos(n\theta) + B\sin(n\theta).$$

3.2.2 Equation

Rewriting:

$$r^2R'' + rR' + (k^2r^2 - n^2)R = 0.$$

Define $x = kr$. $\frac{d}{dr} = k\frac{d}{dx}$. So:

$$R' = k\frac{dR}{dx}, \quad R'' = k^2\frac{d^2R}{dx^2}.$$

Substituting,

$$\frac{x^2}{k^2}(k^2)\frac{d^2R}{dx^2} + \frac{x}{k}(k)\frac{dR}{dx} + (x^2 - n^2)R = 0,$$
$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0.$$

This is a Bessel Equation of order n.

4 Derivation

This section will cover the derivation of the Bessel Function (as shown by [3]) from the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0.$$

To find a solution to this equation, we use the Frobenius Method. This means that y will have a solution of the form

$$y(x) = \sum_{h=0}^{\infty} a_h x^{h+r}.$$

This means that the derivatives of y will be

$$y'(x) = \sum_{h=0}^{\infty}(h+r)a_h x^{h+r-1}, \quad y''(x) = \sum_{h=0}^{\infty}(h+r-1)(h+r)a_h x^{h+r-2}.$$

We can now substitute these into the Bessel Equation to obtain:

$$x^2\sum_{h=0}^{\infty}(h+r-1)(h+r)a_h x^{h+r-2} + x\sum_{h=0}^{\infty}(h+r)a_h x^{h+r-1} + (x^2-n^2)\sum_{h=0}^{\infty}a_h x^{h+r} = 0.$$

Bringing the x^2, x, and $x^2 - n^2$ into the sums, we are left with

$$\sum_{h=0}^{\infty}(h+r-1)(h+r)a_h x^{h+r} + \sum_{h=0}^{\infty}(h+r)a_h x^{h+r} + \sum_{h=0}^{\infty}a_h x^{h+r+2} - n^2\sum_{h=0}^{\infty}a_h x^{h+r} = 0.$$

To be able to find an expression for a_h, we would like the powers of x to be the same. In the third series, the power of x is 2 higher than the others. To rectify this, we introduce a variable s. In the third series, we use $s = h+2$ and in the others, $s = h$. Due to this, the beginning of the third series should be changed from $h = 0$ to $h = 2$ as $s = h+2 = 0+2 = 2$. The others series would remain as $s = 0$:

$$\sum_{s=0}^{\infty}(s+r-1)(s+r)a_s x^{s+r} + \sum_{s=0}^{\infty}(s+r)a_s x^{s+r} + \sum_{s=2}^{\infty}a_{s-2}x^{s+r} - n^2\sum_{s=0}^{\infty}a_s x^{s+r} = 0. \quad (3)$$

For the series to converge to a value, the LHS has to be equal to 0 for each value of s. First, for $s = 0$, using Equation (3) (Note that the third series has no contribution to $s = 0$ and $s = 1$ because it begins at $s = 2$):

$$r(r-1)a_0x^r + ra_0x^r - n^2a_0x^r = 0,$$

$$r^2 - n^2 = 0, \implies r_1 = n, r_2 = -n. \quad (4)$$

4.1 Bessel Functions of the First Kind

For this function, we take $r = r_1 = n$. For $s = 1$, using Equation (3), this gives us

$$(1+n)(1+n-1)a_1 + (1+n)a_1 - n^2a_1 = 0,$$

$$(n+1)^2a_1 - n^2a_1 = 0 \implies (n^2+2n+1)a_1 - n^2a_1 = 0 \implies (2n+1)a_1 = 0 \implies a_1 = 0.$$

Now, let's take the case $s = s$ where $s \neq 0, 1$:

$$(s+n)(s+n-1)a_s + (s+n)a_s + a_{s-2} - n^2a_s = 0,$$

$$
\begin{aligned}
(s+n)^2 a_s - n^2 a_s + a_{s-2} &= 0, \\
s(s+2n)a_s + a_{s-2} &= 0, \\
a_s &= -\frac{a_{s-2}}{s(s+2n)}.
\end{aligned} \tag{5}
$$

Since a_1 is 0, Equation (5) shows that all odd numbered coefficients are 0. This means that there will be only even numbered coefficients, so we can use $s = 2m$:

$$
a_{2m} = -\frac{a_{2m-2}}{2m(2m+2n)} = -\frac{a_{2m-2}}{2^2 m(m+n)}.
$$

By using the recursive relation, we can see that

$$
a_{2m} = (-1)\frac{1}{2^2 m(m+n)}\left((-1)\frac{1}{2^2(m-1)(m-1+n)}(\cdots)\right) = \frac{(-1)^m a_0}{2^{2m} m!(m+n)\cdots(n+1)}.
$$

The coefficient a_0 is arbitrary because any scaled version of a solution is also a solution and a_0 is a constant. Since a_0 is arbitrary and we are trying to find a general solution, we can pick any value of a_0. Here, we pick a value to make the denominator simpler:

$$
a_0 = \frac{1}{2^2 n!}.
$$

Using this, the coefficient formula becomes

$$
a_{2m} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!}.
$$

Substituting this into the infinite sum, we get the Bessel Function of the First Kind ($J_n(x)$):

$$
J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}.
$$

However, this only works for $n \in \mathbb{Z}$. In order to extend it to any nonnegative v, we need to use the Gamma Function. The Gamma function is an extension of the factorial function with $\Gamma(n+1) = n!$. The new formula using the gamma function is

$$
\boxed{J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} m!\Gamma(v+m+1)}.} \tag{6}
$$

4.2 Bessel Functions of the Second Kind

To create the general solution, we need both functions. These two functions must not be linearly dependent. Equation (4) gives us $r = -n$ as the other solution. However when n is an integer, the two are linearly dependent. The proof for this is explained in the Properties Section. Note that in this section we use v to refer to any solution and n to refer to integer solutions.

4.2.1 The Wronskian

To check whether two functions are linearly independent, we use the Wronskian. The Wronskian is equal to 0 when the functions are linearly dependent. The formula for the Wronskian is

$$W(f,g) = f \cdot g' - f' \cdot g.$$

The following formula is for second-order ODEs and is from Abel's Identity:

$$W(f,g)(x) = W(f,g)(x_0) \cdot \exp\left(-\int_{x_0}^{x} P(t)dt\right),$$

$$\frac{d^2y}{dx^2} + P\left(x\right)\frac{dy}{dx} + Q\left(x\right)y = 0.$$

This shows that the Wronskian of two solutions is zero everywhere or non-zero everywhere because it uses the exp function which is never equal to 0. Therefore, the 0 can only come from the other part which is always 0 or never 0. Since our two solutions were $r = v$ and $r = -v$, we first try to compute the Wronskian of $J_v(x)$ and $J_{-v}(x)$ near $x = 0$ by using the leading terms from Equation (6):

$$J_v(x) \approx \frac{1}{\Gamma(v+1)}\left(\frac{x}{2}\right)^v, \quad J_{-v}\left(x\right) \approx \frac{1}{\Gamma(1-v)}\left(\frac{x}{2}\right)^{-v}.$$

Differentiating each gives us

$$J_v'\left(x\right) \approx \frac{v}{x\Gamma(v+1)}\left(\frac{x}{2}\right)^v, \quad J_{-v}'\left(x\right) \approx \frac{-v}{x\Gamma(1-v)}\left(\frac{x}{2}\right)^{-v}.$$

Compute the Wronskian:

$$W\left(J_v(x), J_{-v}(x)\right) = J_v(x) \cdot J_{-v}'(x) - J_v'(x) \cdot J_{-v}(x),$$

$$= \frac{1}{\Gamma(v+1)}\left(\frac{x}{2}\right)^v \cdot \frac{-v}{x\Gamma(1-v)}\left(\frac{x}{2}\right)^{-v} - \frac{v}{x\Gamma(v+1)}\left(\frac{x}{2}\right)^v \cdot \frac{1}{\Gamma(1-v)}\left(\frac{x}{2}\right)^{-v}.$$

Since $(\frac{x}{2})^v \cdot (\frac{x}{2})^{-v} = 1$, the terms can be simplified to

$$W\left(J_v, J_{-v}\right) = \frac{-v}{x\Gamma(v+1)\Gamma(1-v)} - \frac{v}{x\Gamma(v+1)\Gamma(1-v)} = \frac{-2v}{x\Gamma(v+1)\Gamma(1-v)}.$$

Now, using the Gamma reflection formula $\Gamma(v)\Gamma(1-v) = \frac{\pi}{\sin(v\pi)}$ and the fact that $\Gamma(v+1) = v\Gamma(v)$:

$$\Gamma(v+1)\Gamma(1-v) = v\Gamma(v)\Gamma(1-v) = v \cdot \frac{\pi}{\sin\left(v\pi\right)} = \frac{v\pi}{\sin\left(v\pi\right)}.$$

Substituting it back in to the Wronskian:

$$W\left(J_v, J_{-v}\right) = \frac{-2v}{x} \cdot \frac{\sin\left(v\pi\right)}{v\pi} = -\frac{2\sin\left(v\pi\right)}{\pi x}.$$

Note that when v is an integer, $\sin(v\pi) = 0$ which shows that they are linearly dependent and $J_{-v}(x)$ would not work as a solution. So, instead we try to find a $Y_n(x)$ that is linearly independent and has the property:

$$Y_n(x) = \lim_{v \to n} Y_v\left(x\right).$$

4.2.2 Finding Y_n

Here are the conditions for the the second solution, Y_n:

1. Needs to be independent from J_v as that is a condition to form the full solution space.
2. Varies smoothly as $v \to n$, for all $n \in \mathbb{Z}$ to fulfill the condition set at the end of the previous section ($Y_n(x) = \lim_{v\to n} Y_v\left(x\right)$) to ensure that the function remains well-defined and provides a continuous extension to integer orders.
3. $W(J_v(x), Y_v(x)) = \frac{2}{\pi x}$. This is chosen merely for convenience and to simplify formulas that come out of the Bessel Functions.

Any solution for the equation has to be a linear combination of J_v and J_{-v} (i.e. $aJ_v + bJ_{-v}$) and the Wronskian has to be $\frac{2}{\pi x}$:

$$W(J_v, aJ_v + bJ_{-v}) = a \cdot W(J_v, J_v) + b \cdot W(J_v, J_{-v}) = b \cdot \left(-\frac{2\sin(v\pi)}{\pi x}\right).$$

> ***Remark.*** The Wronskian of a function with itself is always 0 as they are always linearly dependent. The Wronskian can be split up as shown above because it is linear in its second argument due to it being calculated from the determinant which is linear.

Equating this to $\frac{2}{\pi x}$ gives us

$$b \cdot \left(-\frac{2\sin\left(v\pi\right)}{\pi x}\right) = \frac{2}{\pi x} \implies b = -\frac{1}{\sin\left(v\pi\right)}.$$

At integer $v = n$, $J_{-n} = (-1)^n J_n$, so $aJ_v + bJ_{-v}$ approaches

$$aJ_n + (-1)^n \left(-\frac{1}{\sin(v\pi)}\right) J_n = \left(a - \frac{(-1)^n}{\sin(v\pi)}\right) J_n.$$

$\frac{1}{\sin(v\pi)}$ diverges as $v \to n$. To cancel out this divergence, a also needs to diverge in the same manner. We choose $a = \frac{\cos(v\pi)}{\sin(v\pi)}$ because near $v = n$,

$$\frac{\cos(v\pi)}{\sin(v\pi)} \approx \frac{(-1)^n}{\sin(v\pi)}.$$

This produces a $\frac{0}{0}$ form with a finite L'Hôpital limit rather than diverging. This gives us

$$Y_v(x) = \frac{J_v(x)\cos(v\pi) - J_{-v}(x)}{\sin(v\pi)}.$$

4.2.3 Computing with L'Hôpital

At $v = n$, $Y_v(x)$ is $\frac{0}{0}$. We now use L'Hôpital to differentiate with respect to v.
The denominator. Below is the differentiated denominator:

$$\frac{\partial}{\partial v}\sin(v\pi)\bigg|_{v=n} = \pi\cos(v\pi) = \pi(-1)^n.$$

The numerator. Here, we use N to refer to the numerator. Using the product rule:

$$\frac{\partial N}{\partial v} = \frac{\partial J_v}{\partial v}\cos(v\pi) - \pi J_v \sin(v\pi) - \frac{\partial J_{-v}}{\partial v}.$$

At $v = n$, $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so the middle term vanishes:

> ***Remark.*** The reason that $\cos(n\pi) = (-1)^n$ is because this is only at integers and at multiples of π, it only takes on the values of -1 and 1. For example, $\cos(\pi) = -1$, $\cos(2\pi) = 1$, $\cos(3\pi) = -1$, and so on.

$$\frac{\partial N}{\partial v}\bigg|_{v=n} = (-1)^n \frac{\partial J_v}{\partial v}\bigg|_{v=n} - \frac{\partial J_{-v}}{\partial v}\bigg|_{v=n}.$$

For the second term, let $\mu = -v$, so $\frac{\partial}{\partial v}J_{-v} = -\frac{\partial J_\mu}{\partial \mu}\Big|_{\mu=-v}$. At $v = n$, $\mu = -v$:

$$\frac{\partial J_{-v}}{\partial v}\bigg|_{v=n} = -\frac{\partial J_v}{\partial v}\bigg|_{v=-n}.$$

Substituting it in,

$$\frac{\partial N}{\partial v}\bigg|_{v=n} = (-1)^n \frac{\partial J_v}{\partial v}\bigg|_{v=n} + \frac{\partial J_v}{\partial v}\bigg|_{v=n}.$$

This leaves us with

$$Y_n(x) = \frac{(-1)^n \frac{\partial J_v}{\partial v}\big|_{v=n} + \frac{\partial J_v}{\partial v}\big|_{v=-n}}{\pi(-1)^n} = \frac{1}{\pi}\frac{\partial J_v}{\partial v}\bigg|_{v=n} + \frac{(-1)^n}{\pi}\frac{\partial J_v}{\partial v}\bigg|_{v=-n}. \tag{7}$$

4.2.4 Differentiating J_v with respect to v

In the series of J_v,

$$J_v\left(x\right) = \sum_{m=0}^{\infty} \frac{\left(-1\right)^m}{2^{2m} m! \Gamma(m+v+1)} \left(\frac{x}{2}\right)^{2m+v},$$

there are two factors dependent on v. We differentiate this using the product rule. For the first factor (Note that in this paper we use log to signify the natural log of base e):

$$\frac{\partial}{\partial v}\left(\frac{x}{2}\right)^{2m+v} = \left(\frac{x}{2}\right)^{2m+v} \log\frac{x}{2}.$$

> ***Remark.*** $\psi(x)$ is the Digamma Function. It is defined as the logarithmic derivative of the Gamma function:
>
> $$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For the second factor:

$$\frac{\partial}{\partial v}\frac{1}{\Gamma(m+v+1)} = -\frac{\psi\left(m+v+1\right)}{\Gamma(m+v+1)}.$$

Putting this together:

$$\frac{\partial J_v}{\partial v} = \sum_{m=0}^{\infty} \frac{\left(-1\right)^m}{m! \Gamma(m+v+1)} \left(\frac{x}{2}\right)^{2m+v} \left[\log\frac{x}{2} - \psi\left(m+v+1\right)\right].$$

4.2.5 Evaluating

Evaluating at $v = n$. Substitute and use $\Gamma(m+n+1) = (m+n)!$:

$$\left.\frac{\partial J_v}{\partial v}\right|_{v=n} = \sum_{m=0}^{\infty} \frac{\left(-1\right)^m}{m!\left(m+n\right)!} \left(\frac{x}{2}\right)^{2m+n} \left[\log\frac{x}{2} - \psi\left(m+n+1\right)\right]$$

$$= \log\frac{x}{2} \underbrace{\sum_{m=0}^{\infty} \frac{\left(-1\right)^m}{m!\left(m+n\right)!} \left(\frac{x}{2}\right)^{2m+n}}_{J_n(x)} - \sum_{m=0}^{\infty} \frac{\left(-1\right)^m \psi\left(m+n+1\right)}{m!\left(m+n\right)!} \left(\frac{x}{2}\right)^{2m+n}.$$

The sum of the first term is the same as $J_n(x)$, so:

$$\left.\frac{\partial J_v}{\partial v}\right|_{v=n} = J_n\left(x\right) \log\frac{x}{2} - \sum_{m=0}^{\infty} \frac{\left(-1\right)^m \psi\left(m+n+1\right)}{m!\left(m+n\right)!} \left(\frac{x}{2}\right)^{2m+n}. \tag{8}$$

Evaluating at $v = -n$. Using the same method:

$$\left.\frac{\partial J_v}{\partial v}\right|_{v=-n} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m-n} \left[\log\frac{x}{2} - \psi\left(m-n+1\right)\right].$$

> ***Remark.*** Due to the properties of the Gamma Function,
>
> $$\frac{1}{\Gamma(m-n+1)} = 0$$
>
> whenever $m-n+1$ is 0 or a negative integer.

Due to the Gamma function being 0 for those numbers, the first n terms vanish and have to be treated differently:

$$\left.\frac{\partial J_v}{\partial v}\right|_{v=-n} = \sum_{m=0}^{n-1}(\cdots) + \sum_{m=n}^{\infty}(\cdots).$$

For the first part, the sum starts at $m = n$:

$$= \sum_{m=n}^{\infty} \frac{(-1)^m}{m!\Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m-n} \left[\log\frac{x}{2} - \psi\left(m-n+1\right)\right].$$

This can be reindexed with $k = m - n$ to produce:

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(k+n)!\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+n} \left[\log\frac{x}{2} - \psi\left(k+1\right)\right].$$

Using $\Gamma(k+1) = k!$ and splitting,

$$= (-1)^n \log\frac{x}{2} \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{x}{2}\right)^{2k+n}}_{J_n(x)} - (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k \psi\left(k+1\right)}{k!\,(k+n)!} \left(\frac{x}{2}\right)^{2k+n}.$$

For the other part of the series, let $k = n-m-1$, so $-k = m+v+1$. The logarithmic term vanishes as $\frac{1}{\Gamma(-k)} = 0$ and there is no other factor with n. This leaves only the digamma term:

$$\sum_{m=0}^{n-1} \frac{(-1)^m \psi\left(-k\right)}{m!\Gamma(-k)} \left(\frac{x}{2}\right)^{2m-n}.$$

Remark. In this case, we cannot leave it as $\frac{\psi(z)}{\Gamma(z)}$ because that is undefined at negative integers. Instead we use the original formula to get:

$$\frac{d}{dz}\left(\frac{1}{\Gamma(z)}\right) = -\frac{\Gamma'(z)}{\Gamma^2(z)} = -\frac{1}{\Gamma^2(z)}\left(\psi(z)\,\Gamma(z)\right) = -\frac{\psi(z)}{\Gamma(z)}$$

To find the derivative of $1/\Gamma$, we use the Gamma recurrence:

$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)\cdots(z+k)} \implies \frac{z(z+1)\cdots(z+k)}{\Gamma(z+k+1)}.$$

Let

$$f(z) = z(z+1)\cdots(z+k).$$

$$\frac{d}{dz}\left(\frac{1}{\Gamma(z)}\right) = \frac{f'(z)\,\Gamma(z+k+1) - f(z)\,\Gamma'(z+k+1)}{\Gamma(z+k+1)^2}.$$

Evaluating this at $z=-k$, (this causes $f(-k)=0$):

$$\frac{d}{dz}\left(\frac{1}{\Gamma(z)}\right)\Bigg|_{z=-k} = \frac{f'(-k)}{\Gamma(1)} = f'(-k).$$

To find $f'(z)$, we apply the product rule to $f(z)$:

$$f'(z) = \sum_{j=0}^{k}\prod_{\substack{i=0\\ i\neq j}}^{k}(z+i).$$

The term corresponding to j is

$$\prod_{i\neq j}(z+i).$$

If $j \neq k$, then the product will contain the factor $-k+k$, so they will vanish. The only surviving term is $j=k$:

$$f'(-k) = \prod_{i=0}^{k-1}(-k+i) = (-k)(-k+1)\cdots(-1) = (-1)^k k!.$$

$$\frac{d}{dz}\left(\frac{1}{\Gamma(z)}\right)\Bigg|_{z=-k} = (-1)^k k! = (-1)^{n-m-1}(n-m-1)!.$$

Substituting this back into the sum:

$$\sum_{m=0}^{n-1} \frac{(-1)^m \psi(-k)}{m!\Gamma(-k)} \left(\frac{x}{2}\right)^{2m-n} = \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} \left(\frac{d}{dz} \left(\frac{1}{\Gamma(z)} \right) \Big|_{z=-k} \right) \left(\frac{x}{2}\right)^{2m-n}$$

$$= \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} (-1)^{n-m-1} (n-m-1)! \left(\frac{x}{2}\right)^{2m-n} = \sum_{m=0}^{n-1} \frac{(-1)^{n-1} (n-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-n}$$

$$= (-1)^{n-1} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-n}.$$

The first term is the series of $J_n(x)$, so

$$\begin{aligned} \frac{\partial J_v}{\partial v}\Big|_{v=-n} = (-1)^n J_n(x) \log \frac{x}{2} + (-1)^{n-1} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-n} \\ - (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k \psi(k+1)}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}. \end{aligned} \tag{9}$$

4.2.6 The Result

From Equation (7), we have:

$$Y_n(x) = \frac{1}{\pi} \frac{\partial J_v}{\partial v}\Big|_{v=n} + \frac{(-1)^n}{\pi} \frac{\partial J_v}{\partial v}\Big|_{v=-n}.$$

Substituting in Equations (8) and (9),

$$\begin{aligned} Y_n(x) = \frac{1}{\pi} \left[J_n(x) \log \frac{x}{2} - \sum_{m=0}^{\infty} \frac{(-1)^m \psi(m+n+1)}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n} \right] \\ + \frac{(-1)^n}{\pi} \left[(-1)^n J_n(x) \log \frac{x}{2} + (-1)^{n-1} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-n} \right. \\ \left. - (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k \psi(k+1)}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n} \right]. \end{aligned} \tag{10}$$

For the second part, the $(-1)^n$ outside the brackets gets multiplied with the $(-1)^n$ inside in each term to form $(-1)^{2n}$ which is equal to 1:

$$\begin{aligned} Y_n(x) = \frac{2}{\pi} J_n(x) \log \frac{x}{2} - \frac{1}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-n} \\ - \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n} \left[\psi(m+n+1) + \psi(m+1)\right]. \end{aligned} \tag{11}$$

4.2.7 Rewriting with Harmonic Numbers

> ***Remark.*** Harmonic numbers are partial sums of the Harmonic series and are defined by:
>
> $$H_k = \sum_{j=1}^{k} \frac{1}{j}.$$
>
> The digamma function satisfies:
>
> $$\psi(k+1) = -\gamma + H_k,$$
>
> where γ is the Euler-Mascheroni constant.

So,

$$\psi(m+1) + \psi(m+n+1) = (-\gamma + H_m) + (-\gamma + H_{m+n}) = H_m + H_{m+n} - 2\gamma.$$

Substituting them into Equation (11),

$$Y_n\left(x\right) = \frac{2}{\pi} J_n\left(x\right) \log \frac{x}{2} - \frac{1}{\pi}\left(-1\right)^{n-1} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-n},$$

$$-\frac{1}{\pi} \sum_{m=0}^{\infty} \frac{\left(-1\right)^m \left(H_m + H_{m+n} - 2\gamma\right)}{m!\left(m+n\right)!} \left(\frac{x}{2}\right)^{2m+n}.$$

The last term can be split into:

$$-\frac{1}{\pi} \sum_{m=0}^{\infty} \frac{\left(-1\right)^m \left(H_m + H_{m+n}\right)}{m!\left(m+n\right)!} \left(\frac{x}{2}\right)^{2m+n} + \frac{2\gamma}{\pi} \underbrace{\sum_{m=0}^{\infty} \frac{\left(-1\right)^m}{m!\left(m+n\right)!} \left(\frac{x}{2}\right)^{2m+n}}_{J_n(x)}.$$

The second part of this is a multiple of $J_n(x)$ and will get absorbed into the logarithmic term to form the full equation for Bessel Functions of the Second Kind:

$$\boxed{\begin{aligned} Y_n\left(x\right) = \frac{2}{\pi} J_n\left(x\right)\left(\log \frac{x}{2} + \gamma\right) &- \frac{1}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-n} \\ &- \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{\left(-1\right)^m \left(H_m + H_{m+n}\right)}{m!\left(m+n\right)!} \left(\frac{x}{2}\right)^{2m+n}. \end{aligned}} \qquad (12)$$

5 Properties of Bessel Functions

Below are a few key properties of the Bessel Functions

5.1 Derivatives

This subsection will prove the following identities:

$$\left(x^v J_v\right)' = x^v J_{v-1}(x),$$

$$\left(x^{-v} J_v(x)\right)' = -x^{-v} J_{v+1}(x),$$

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x),$$

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x).$$

To start with, we multiply Equation (6) by x^v and take x^{2v} into the summation sign:

$$x^v J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2v}}{2^{2m+v} m! \Gamma(v+m+1)}.$$

We can now differentiate this:

$$\left(x^v J_v\right)' = \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+v) x^{2m+2v-1}}{2^{2m+v} m! \Gamma(v+m+1)}.$$

The 2 in the numerator can be moved to the denominator, the x^{2v-1} is moved out of the summation and the property of the Gamma Function ($\Gamma(n+1) = n\Gamma(n)$) is used:

$$= x^{2v-1} \sum_{m=0}^{\infty} \frac{(-1)^m (m+v) x^{2m}}{2^{2m+v-1} m! (v+m) \Gamma(v+m)}.$$

The $(v+m)$ cancels and the x^{2v-1} on the outside can be split up:

$$= x^v x^{v-1} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v-1} m! \Gamma(v+m)}.$$

The right side can be rewritten as

$$x^v \left(x^{v-1} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+(v-1)} m! \Gamma((v-1)+m+1)} \right).$$

The inner sum is the formula for $J_{v-1}(x)$. So, this can be rewritten as:

$$\boxed{\left(x^{v} J_{v}\right)' = x^{v} J_{v-1}\left(x\right).} \tag{13}$$

Similarly, we can multiply Equation (6) by x^{-v}, which cancels out with x^{v} to give:

$$\left(x^{-v} J_{v}\right)' = \sum_{m=1}^{\infty} \frac{\left(-1\right)^{m} \left(2m\right) x^{2m-1}}{2^{2m+v} m! \Gamma(v+m+1)}.$$

> ***Remark.*** The new series starts at $m = 1$ because the first term becomes 0 due to differentiation.

The 2 in the numerator gets moved to the denominator and using $m! = m(m-1)!$, the m cancels out:

$$= \sum_{m=1}^{\infty} \frac{\left(-1\right)^{m} x^{2m-1}}{2^{2m+v-1} \left(m-1\right)! \Gamma(v+m+1)}.$$

Now, with $m = s+1$, we get:

$$= \sum_{s=0}^{\infty} \frac{\left(-1\right)^{s+1} x^{2s+1}}{2^{2s+v+1} s! \Gamma(v+s+2)}.$$

This can be rewritten by factoring out the -1, adding x^{-v} outside and x^{v} inside (this can be done as they cancel out):

$$= -x^{-v} \left(x^{v+1} \sum_{s=0}^{\infty} \frac{\left(-1\right)^{s} x^{2s}}{2^{2s+(v+1)} s! \Gamma((v+1)+s+1)} \right).$$

The inner part is the formula for $J_{v+1}(x)$. So this can be written as:

$$\boxed{\left(x^{-v} J_{v}\left(x\right)\right)' = -x^{-v} J_{v+1}\left(x\right).} \tag{14}$$

The left side of Equation (13) can be differentiated to give the equation:

$$v x^{v-1} J_{v}\left(x\right) + x^{v} J_{v}'\left(x\right) = x^{v} J_{v-1}\left(x\right). \tag{15}$$

Doing the same with Equation (14) and multiplying by x^{2v} gives

$$-v x^{v-1} J_{v}\left(x\right) + x^{v} J_{v}'\left(x\right) = -x^{v} J_{v+1}\left(x\right). \tag{16}$$

Subtracting Equation (16) from Equation (15) and rearranging gives

$$\boxed{J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_{v}(x).} \tag{17}$$

Adding Equation (15) to Equation (16) and dividing by x^{v} gives

$$\boxed{J_{v-1}(x) - J_{v+1}(x) = 2 J_{v}'(x).} \tag{18}$$

5.2 Half integer v

This subsection will show that Bessel Functions of the First Kind with half integer v are elementary. We start with $J_{1/2}(x)$:

$$J_{\frac{1}{2}}(x) = \sqrt{x}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+1/2}m!\Gamma(m+\frac{3}{2})} = \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}m!\Gamma(m+\frac{3}{2})}.$$

We can separate the denominator into two parts: A ($2^m m!$) and B ($2^{m+1}\Gamma(m+\frac{3}{2})$):

$$A = 2m(2m-2)(2m-4)\cdots 4\cdot 2,$$

$$B = 2^{m+1}\left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = (2m+1)(2m-1)\cdots 3\cdot 1\cdot\sqrt{\pi}.$$

> ***Remark.*** For B, we use the result $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. This is proved in the section on Gamma Functions.

Multiplying both A and B, we get:

$$AB = (2m+1)(2m)(2m-1)\cdots 3\cdot 2\cdot 1\cdot\sqrt{\pi} = (2m+1)!\sqrt{\pi}.$$

Substituting this into the Bessel Function:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!}.$$

The infinite sum is the Taylor series of the sin function, so this can be written as:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin(x). \tag{19}$$

Differentiating this and using Equation (13)

$$\left(\sqrt{x}J_{\frac{1}{2}}(x)\right)' = \sqrt{\frac{2}{\pi}}\cos(x) = x^{\frac{1}{2}}J_{-\frac{1}{2}}(x). \tag{20}$$

Using Equations (19) and (20) along with Equation (17), any of the half-integer Bessel Functions can be found.

5.3 Linear Dependence

This subsection will explain why $J_{-v}(x)$ and $J_v(x)$ are linearly dependent for integer $v = n$. For $v = n$, we have:

$$J_{-n}(x) = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-n} m! \Gamma(m-n+1)}.$$

Due to the properties of the Gamma Function, it has poles at $0, -1, -2, -3, \ldots$. So $\frac{1}{\Gamma(m-n+1)} = 0$ for the first n terms, leading to them vanishing. This gives:

$$J_{-n}(x) = x^{-n} \sum_{m=m}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-n} m! \Gamma(m-n+1)}.$$

Now let $m = s + n$ and use $\Gamma(s+1) = s!$ to get:

$$J_{-n}(x) = x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^{s+n} x^{2s+2n}}{2^{2s+n} (s+n)! s!} = (-1)^n x^n \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{2^{2s+n} (s+n)! s!}.$$

The infinite sum is the same as the one for $J_n(x)$. So, we get:

$$J_{-n}(x) = (-1)^n J_n(x).$$

Since $J_{-n}(x)$ is a multiple of $J_n(x)$ for integer n, they are linearly dependent.

6 Modified Bessel Functions

For the modified Bessel Functions, we substitute ix instead of x into the equation:

$$\tilde{y}(x) = y(ix).$$

This leads us to:

$$\frac{d}{dx} y(ix) = i \cdot y'(ix), \; \frac{d^2}{dx^2} y(ix) = i^2 \cdot y''(ix) = -y''(ix),$$

$$(ix)^2 y''(ix) + (ix) y'(ix) + \left((ix)^2 - v^2\right) y(ix) = 0,$$
$$-x^2 y''(ix) + ix \cdot y'(ix) + \left(-x^2 - v^2\right) y(ix) = 0,$$
$$-x^2 \left(-\tilde{y}''\right) + ix \cdot \left(-i\tilde{y}'\right) + \left(-x^2 - v^2\right) \tilde{y} = 0,$$
$$x^2 \tilde{y}'' + x\tilde{y}' - \left(x^2 + v^2\right) \tilde{y} = 0.$$

This is the modified Bessel Equation.

6.1 Main Difference

The main difference between the Bessel Functions and the modified Bessel Functions is that for large x in Bessel Functions, we get:

$$y'' + y \approx 0.$$

which has solutions $e^{\pm ix}$ which are oscillatory. Whereas, with modified Bessel Functions at large x, we get:

$$y'' - y \approx 0.$$

which has solutions $e^{\pm x}$ which represent exponential growth and decay.

6.2 Solution 1

This is defined as:

$$I_v\left(x\right) = i^{-v} J_v\left(ix\right).$$

The i^{-v} term at the front ensures the result is not imaginary. Substituting:

$$J_v\left(ix\right) = \sum_{m=0}^{\infty} \frac{\left(-1\right)^m \left(i\right)^{2m+v}}{m!\Gamma(m+v+1)} \left(\frac{x}{2}\right)^{2m+v} = i^v \sum_{m=0}^{\infty} \frac{\left(-1\right)^m \left(-1\right)^m}{m!\Gamma(m+v+1)} \left(\frac{x}{2}\right)^{2m+v}.$$

$$i^v \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+v+1)} \left(\frac{x}{2}\right)^{2m+v}.$$

The i^v and the i^{-v} terms cancel out to give:

$$I_v\left(x\right) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+v+1)} \left(\frac{x}{2}\right)^{2m+v}.$$

6.3 Solution 2

The Modified Bessel Function of the Second Kind is derived very similarly to the normal one. The main difference, however, is that $I_{-v}(x)$ does not need to be multiplied with $\cos(\pi v)$ as is done for the normal ones. This is because, the normal ones require it to eliminate the oscillating $(-1)^n$ term that appears in the Bessel Function of the First Kind. There is no such term in the modified Bessel Function of the First Kind. Here, the factor $\frac{\pi}{2}$ is arbitrary and is chosen for convenience:

$$K_v(x) = \frac{\pi}{2} \frac{I_{-v}(x) - I_v(x)}{\sin(\pi v)}.$$

The same idea is used here with:

$$K_n(x) = \lim_{v \to n} \frac{\pi}{2} \frac{I_{-v}(x) - I_v(x)}{\sin(\pi v)}.$$

7 Gauss Circle Problem

This is the first application of Bessel Functions that will be covered in this paper. This problem is shown in [1]. The main question of the problem is, in a circle of radius R, how many points (m, n) with $m, n \in \mathbb{Z}$ land on or in the circle.
The number of points is denoted as $N(R)$. For example, $N(1) = 5$ with the points being $(0,0), (\pm 1, 0), (0, \pm 1)$. The most obvious guess is that it is πR^2 as each unit of area would have roughly one point inside it. This gives an error term of

$$E(R) = N(R) - \pi R^2.$$

Gauss proved that $|E(R)| \leq C \cdot R$ for some constant C. This is because the error can only come from the boundary strip of width 1 of the circle, which is proportional to R. The problem asks about how much closer we can get.

7.1 Integral Representation of the Bessel Function

The problem involves the usage of the Bessel Function through an integral representation. This subsection will cover the derivation of that representation. Using Equation (6):

$$J_0\left(x\right) = \sum_{m=0}^{\infty} \frac{\left(-1\right)^m}{\left(m!\right)^2} \left(\frac{x}{2}\right)^{2m}.$$

This will use the following trigonometric identity:

$$\frac{(2m)!}{2^{2m}\left(m!\right)^2} = \frac{1}{2\pi} \int_0^{2\pi} \cos^{2m}\left(\theta\right)\ d\theta.$$

Substituting it in:

$$\begin{aligned} J_0\left(x\right) &= \sum_{m=0}^{\infty} \frac{\left(-1\right)^m x^{2m}}{(2m)!} \left(\frac{1}{2\pi} \int_0^{2\pi} \cos^{2m}\left(\theta\right)\ d\theta\right), \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=0}^{\infty} \frac{\left(-1\right)^m \left(x\cos\left(\theta\right)\right)^{2m}}{(2m)!}\ d\theta. \end{aligned}$$

The inner sum is the Taylor Series of cos with $x\cos\theta$ inside rather than x:

$$J_0\left(x\right) = \frac{1}{2\pi} \int_0^{2\pi} \cos\left(x\cos\left(\theta\right)\right)\ d\theta.$$

Using the exponential representations:

$$e^{ix\cos(\theta)} = \cos\left(x\cos\left(\theta\right)\right) + i\sin\left(x\cos\left(\theta\right)\right).$$

The imaginary part integrates to 0 due to symmetry:

$$\int_0^{2\pi} e^{ix\cos(\theta)}\ d\theta = \int_0^{2\pi} \cos\left(x\cos\left(\theta\right)\right)\ d\theta.$$

Substituting into the equation above, we get the integral representation of the Bessel Function:

$$J_0\left(x\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos(\theta)}\ d\theta. \tag{21}$$

7.2 Rephrasing

Here we use

$$r_2\left(n\right) = \#\left\{(a,b)\in\mathbb{Z}^2 : a^2+b^2=n\right\}.$$

Using this, we can define $N(R)$ as

$$N\left(R\right) = \sum_{n=0}^{\lfloor R^2\rfloor} r_2\left(n\right).$$

This formula is summing up each shell of radius $\sqrt{n}$.

7.3 Fourier Transform

The Poisson Summation formula (as shown by [4]) says that

$$\sum_{(m,n)\in\mathbb{Z}^2} f(m,n) = \sum_{(j,k)\in\mathbb{Z}^2} \hat{f}(j,k)$$

where $\hat{f}$ is the Fourier Transform of f. Let f be the indicator function of the disk, i. e. $f(x)=1$ if $x^2+y^2\leq R^2$ and 0 otherwise. This means that the left side of the formula will be $N(R)$ and the right side will be a sum of Fourier Transforms of the disk.
The Fourier Transform of the indicator function of the disk of radius R is:

$$\hat{f}\left(\xi\right) = \int_{\mathbb{R}^2} 1_{x^2+y^2\leq R^2} e^{-2\pi i(x\xi_1+y\xi_2)}\ dx\ dy.$$

We can now switch to polar coordinates to make the function simpler, with $\rho=|\xi|$:

$$\hat{f}\left(\xi\right) = \int_0^R r\left(\int_0^{2\pi} e^{-2\pi i r\rho\cos(\theta)}\ d\theta\right)\ dr.$$

The inner integral is a representation of the Bessel Function:

$$\int_0^{2\pi} e^{-2\pi i r\rho\cos(\theta)}\, d\theta = 2\pi J_0\left(2\pi r\rho\right),$$

$$\hat{f}\left(\xi\right) = 2\pi \int_0^R r J_0\left(2\pi r\rho\right)\, dr.$$

Using the Bessel Recurrence Identity $\frac{d}{dt}\left[tJ_1\left(t\right)\right] = tJ_0\left(t\right)$ and $t = 2\pi r\rho$,

$$\hat{f}\left(\xi\right) = \frac{R}{\rho} J_1\left(2\pi R\rho\right) = \frac{R}{|\xi|} J_1\left(2\pi R|\xi|\right).$$

Putting it together, we get:

$$E\left(R\right) = N\left(R\right) - \pi R^2,$$

$$E\left(R\right) = \sum_{(j,k)\neq(0,0)} \frac{R}{\sqrt{j^2+k^2}} J_1\left(2\pi R\sqrt{j^2+k^2}\right).$$

The $(0,0)$ term omitted from the sum is the πR^2 and the rest of it forms the error term. Simplifying with $n = \sqrt{j^2+k^2}$ and the number of pairs being $r_2(n)$:

$$E\left(R\right) = R\sum_{n=1}^{\infty} \frac{r_2\left(n\right)}{\sqrt{n}} J_1\left(2\pi\sqrt{n}R\right).$$

8 Heat Equation on a Cylinder

This section is another application of the Bessel Functions. The main topic here is modelling the heat over time on a cylinder as covered by [2].
Consider a cylinder of radius a and height L. In this section, we will be working with cylindrical coordinates (r, θ, z). This is an extension of the polar coordinates with z as the height. The bounds for these are: $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq L$. The heat equation is:

$$u_t = \kappa\Delta u.$$

where κ is the thermal diffusivity and Δ is the Laplacian. The Laplacian in cylindrical coordinates is:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

This leaves the PDE as:

$$u_t = \kappa\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right]. \tag{22}$$

The boundary conditions we will be using are:

$$u(a,\theta,z,t) = 0,\ u(r,\theta,0,t) = 0,\ u(r,\theta,L,t) = 0.$$

These conditions specify u as being 0 along the boundaries of the cylinder.

8.1 Separation of Variables

Assume that $u(r,\theta,z,t) = R(r)\,\Theta(\theta)\,Z(z)\,T(t)$. Substitute this into Equation (22) and divide by $R\Theta ZT$:

$$\frac{T'}{T} = \kappa\left[\frac{1}{rR}(rR')' + \frac{\Theta''}{r^2\Theta} + \frac{Z''}{Z}\right].$$

8.1.1 Time

Since the left side only depends on t and the right side only depends on r, θ, and z, they must be equal to a constant. Here, we take the constant $-\kappa\lambda^2$. The negative sign is used so that T decays with time. This is important as T is a factor of u. So, we are left with the following differential equation:

$$\frac{T'}{T} = -\kappa\lambda^2 \implies T' = -\kappa\lambda^2 T \implies T(t) = e^{-\kappa\lambda^2 t}.$$

And the rest of it (the spacial part) satisfies:

$$\frac{1}{rR}(rR')' + \frac{\Theta''}{r^2\Theta} + \frac{Z''}{Z} = -\lambda^2.$$

8.1.2 Axial

We can now rearrange the equation to get:

$$\frac{Z''}{Z} = -\lambda^2 - \frac{1}{rR}(rR')' - \frac{\Theta''}{r^2\Theta}.$$

Once again, both sides depend on different things, so we choose a constant $-\mu^2$. To solve this equation:

$$\frac{Z''}{Z} = -\mu^2 \implies Z'' + \mu^2 Z = 0.$$

try solution of e^{rz}:

$$r^2 e^{rz} + \mu^2 e^{rz} = 0,$$

$$r^2 + \mu^2 = 0 \implies r = \pm i\mu.$$

This gives us the general solution of

$$Z(z) = A\sin(\mu z) + B\cos(\mu z).$$

The boundary conditions are that $Z(0) = Z(L) = 0$. Substituting this in, we get:

$$0 = A\sin(0) + B\cos(0) = B \implies B = 0,$$

$$0 = A\sin(\mu L).$$

For a nontrivial solution $A \neq 0$, so $\sin(\mu L) = 0$. Therefore:

$$\mu L = p\pi, \quad p = 1, 2, 3, \ldots,$$

$$\mu_p = \frac{p\pi}{L} \implies Z_p(z) = \sin\left(\frac{p\pi z}{L}\right).$$

8.1.3 Angular

We can now substitute in $-\mu^2$ and multiply through with r^2 to get:

$$\frac{r}{R}(rR')' + r^2\left(\lambda^2 - \mu_p^2\right) + \frac{\Theta''}{\Theta} = 0.$$

Isolating the θ term and using the same method as before, both sides depend on different things, so we choose a constant $-m^2$:

$$\frac{\Theta''}{\Theta} = -m^2.$$

To solve this, we use the same method as we used for Z as it is the same equation. So, we are left with:

$$\Theta(\theta) = A\cos(m\theta) + B\sin(m\theta).$$

This one does not have a boundary condition, however since it is angular, it has a periodicity condition. i. e. $\Theta(\theta + 2\pi) = \Theta(\theta)$:

$$A\cos(m(\theta + 2\pi)) + B\sin(m(\theta + 2\pi)) = A\cos(m\theta) + B\sin(m\theta).$$

This is only possible if $m \cdot 2\pi$ is a multiple of 2π. Therefore, $m \in \mathbb{Z}$. Negative values of m give no new solutions as $\cos(-m\theta) = \cos(m\theta)$ and $\sin(m\theta) = -\sin(-m\theta)$. So, $m \in \mathbb{Z}_{\geq 0}$. This gives us:

$$\Theta_m(\theta) = A_m\cos(m\theta) + B_m\sin(m\theta), m \in \mathbb{Z}_{\geq 0}.$$

8.2 Bessel Functions

After separating Θ and Z, we get:

$$\frac{1}{r}\left(rR'\right)' + \left[\left(\lambda^2 - \mu_p^2\right) - \frac{m^2}{r^2}\right] R = 0.$$

Let $v^2 = \lambda^2 - \mu_p^2$, multiply through with r^2 and rearrange:

$$r^2 R'' + rR' + \left(v^2 r^2 - m^2\right) R = 0.$$

Use the substitution $\rho = vr$, to get:

$$\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} + \left(\rho^2 - m^2\right) R = 0.$$

This is a Bessel Equation of order m. The two solutions are $J_m(\rho)$ and $Y_m(\rho)$. However, $Y_m(\rho)$ is singular at $\rho = 0$. Since u needs to be finite at $r = 0$, we discard $Y_m(\rho)$ and take

$$R\left(r\right) = J_m\left(vr\right).$$

The boundary condition of $R(a) = 0$ requires that

$$J_m\left(va\right) = 0.$$

Here, we denote the n-th positive zero of the Bessel Function with $j_{m,n}$. Then,

$$v_{m,n} = \frac{j_{m,n}}{a}, \quad n = 1, 2, 3, \ldots.$$

Therefore,

$$R_{m,n}\left(r\right) = J_m\left(\frac{j_{m,n} r}{a}\right).$$

Since $\lambda^2 = v_{m,n}^2 + \mu_p^2$, we get

$$\lambda_{m,n,p}^2 = \left(\frac{j_{m,n}}{a}\right)^2 + \left(\frac{p\pi}{L}\right)^2.$$

And the time factor is:

$$T_{m,n,p}\left(t\right) = \exp\left(-\kappa \lambda_{m,n,p}^2 t\right).$$

8.3 The result

We now have the separated solutions for all 4. Multiplying them together leaves us with

$$u_{m,n,p} = J_m\left(\frac{j_{m,n}r}{a}\right)\left[A_{m,n,p}\cos(m\theta) + B_{m,n,p}\sin(m\theta)\right]\sin\left(\frac{p\pi z}{L}\right)\exp\left(-\kappa\lambda_{m,n,p}^2 t\right).$$

Since the heat equation is linear, any combination of solutions is also allowed. Therefore, we can sum over all valid indices to get

$$\boxed{\begin{aligned} u(r,\theta,z,t) = &\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} J_m\left(\frac{j_{m,n}r}{a}\right)\left[A_{m,n,p}\cos(m\theta) + B_{m,n,p}\sin(m\theta)\right] \\ &\sin\left(\frac{p\pi z}{L}\right)\exp\left(-\kappa\lambda_{m,n,p}^2 t\right). \end{aligned}} \tag{23}$$

Appendix A. The Frobenius Method

The Frobenius method (as shown by [3]) is a technique for obtaining a series solution to a linear differential equation near a regular singular point. It differs from the normal power series method by the addition of an x^r term to the solution. When using this, we assume a solution of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

This is then differentiated term-by-term into:

$$y'(x) = \sum_{n=1}^{\infty}(n+r)\,a_n x^{n+r-1}, \quad y''(x) = \sum_{n=2}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}.$$

and so on. These are then substituted into the differential equation. Notice that the indices are being shifted from $n=0$ to $n=1$ and $n=2$. This is because, as each term is differentiated, some of them become 0, which happens to the first 2 terms after two differentiations. As an example, let us consider the differential equation

$$y'' + y = 0.$$

After substitution, we get

$$\sum_{n=2}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

The next step of the method is to combine everything into a single series. We do this by using index shifts. In this case we would switch from $n = 2$ to $n = 0$ and replace all the n's in the equation with $n + 2$. This gives us:

$$\sum_{n=0}^{\infty} (n + r)(n + r - 1)\, a_{n+2} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0,$$

$$\sum_{n=0}^{\infty} [(n + r)(n + r - 1)\, a_{n+2} + a_n] x^{n+r} = 0.$$

If a power series needs to be 0 for all x, then for each term or each n, the sum also has to be 0. This gives us

$$(n + 2)(n + 1)a_{n+2} + a_n = 0,$$

$$a_{n+2} = -\frac{a_n}{(n + 2)(n + 1)}, n = 0, 1, 2, 3, \ldots.$$

These are the coefficients and a_0 and a_1 will be arbitrary.

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