

# Theta Functions

Zixin Zhou

## Abstract

In this paper we introduce theta functions associated with quadratic forms, explain their generating functions, and describe how similar spectral theta functions appear in geometry. We then discuss Kac's famous question "Can one hear the shape of a drum?".

## 1 Introduction & Examples:

### Definition 1.1 (Quadratic Form)

A *quadratic form* in  $d$  variables over a field  $k$  is a homogeneous polynomial of degree two in variables  $(x_1, \dots, x_d)$ . In general it can be written in the form

$$q(x_1, \dots, x_d) = \sum_{i,j} a_{ij} x_i x_j$$

where the coefficients  $(a_{ij} \in k)$  form a symmetric matrix.

A simple example is the quadratic form in two variables

$$q(x, y) = x^2 + y^2$$

Now it is natural for us to consider how many integer solutions satisfy the equation

$$q(x_1, \dots, x_d) = n$$

To study this problem, we define

$$r_q(n) = \#\{(x_1, \dots, x_d) \in \mathbb{Z}^d : q(x_1, \dots, x_d) = n\}$$

which counts the number of integer vectors whose quadratic form equals  $(n)$ .

For example, consider the quadratic form  $(q(x, y) = x^2 + y^2)$ . The first few values of  $(r_q(n))$  are obtained by listing the integer solutions to  $(x^2 + y^2 = n)$  as listde in the chart below:

$n$	solutions	$r_q(n)$
0	(0, 0)	1
1	$(\pm 1, 0), (0, \pm 1)$	4
2	$(\pm 1, \pm 1)$	4

Geometrically, these solutions correspond to lattice points lying on circles centered at the origin.

For instance, the solutions to  $(x^2 + y^2 = 1)$  are the four integer points on the unit circle.

Rather than studying each number  $(r_q(n))$  individually, we simply can just collect them into a single generating function.

**Definition 1.2 (Theta Function)**

The *theta function* associated with a quadratic form  $(q)$  is the infinite series

$$\theta_q(x) = \sum_{n=0}^{\infty} r_q(n)x^n$$

**Theorem 1.3**

Let  $q$  be a quadratic form and let

$$r_q(n) = \#\{x \in \mathbb{Z}^d : q(x) = n\}$$

Then the coefficient of  $x^n$  in the theta function

$$\theta_q(x) = \sum_{n=0}^{\infty} r_q(n)x^n$$

is exactly the number of integer solutions of the equation  $q(x) = n$ .

**Proof:**

By definition,

$$\theta_q(x) = \sum_{n=0}^{\infty} r_q(n)x^n$$

Each quantity  $r_q(n)$  counts the number of integer vectors  $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$  satisfying  $q(x) = n$ .

Since  $r_q(n)$  appears as the coefficient of  $x^n$  in the series, the coefficient of  $x^n$  will records precisely the number of representations of  $n$  by the quadratic form  $q$  where we could encode all in a single infinite series.  $\square$

For the example  $(q(x, y) = x^2 + y^2)$ , the theta function begins

$$\theta_q(x) = 1 + 4x + 4x^2 + \dots$$

Thus the theta function encodes an infinite amount of arithmetic information that make we able to analyze properties of the entire series.

Theta functions are particularly important because they often have cool symmetry properties.

In many cases they are examples of modular forms which are special functions that remain invariant under certain transformations of the complex plane. We will not explore these properties in detail here.

## 2 Spectral Theta Functions:

We could observe similar use of infinite series appears in spectral theory. This connection is crucial to answering Kac's Question in **Section 3** Recall that the behavior of many physical system is governed by a sequence of frequencies:

$$\lambda_1, \lambda_2, \lambda_3, \dots$$

These values often arise as eigenvalues of the Laplace operator which are called the Laplacian.

### Definition 2.1 (Laplacian)

In Euclidean space  $\mathbb{R}^d$ , the Laplacian of a function  $u(x_1, \dots, x_d)$  is defined by

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_d^2}$$

The eigenvalue problem for the Laplacian is equivalent to finding functions  $u$  and numbers  $\lambda$  satisfying:

$$\Delta u = -\lambda u$$

The numbers

$$\lambda_1, \lambda_2, \lambda_3, \dots$$

obtained in this way form the spectrum of the system where these eigenvalues describe the natural vibration frequencies of the domain or the rate at which heat diffuses.

Apply similar logic from **Section 1**, instead of studying each eigenvalue individually, we will encode the entire sequence into a single function.

### Definition 2.2 (Spectral Theta Function)

$$\Theta(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}$$

This infinite series is called a spectral theta function or the heat trace.

Just as the theta function of a quadratic form encodes the number of lattice points satisfying a quadratic equation, the spectral theta function encodes information about the entire spectrum of the Laplacian. Each term in the series records the contribution of one eigenvalue.

### Proposition 2.3

If  $t > 0$ , then the spectral theta function

$$\Theta(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}$$

converges.

**Proof:**

The eigenvalues of the Laplacian on a bounded domain satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

and  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

For  $t > 0$ , the terms  $e^{-\lambda_n t}$  decrease rapidly to zero because of exponential decay.

Since exponential functions decrease faster than any polynomial growth, the series

$$\sum_{n=1}^{\infty} e^{-\lambda_n t}$$

converges comparatively.

Therefore  $\Theta(t)$  is well defined for every  $t > 0$ .  $\square$

It's interesting to notice that the spectral theta function actually arises quite naturally from the heat equation:

$$\frac{\partial u}{\partial t} = \Delta u$$

which models the diffusion of heat across a surface or region.

**Proposition 2.4**

Let  $u_n$  be an eigenfunction of the Laplacian with eigenvalue  $\lambda_n$ .

Then the function

$$u(x, t) = e^{-\lambda_n t} u_n(x)$$

is a solution of the heat equation.

**Proof:**

Compute the time derivative:

$$\frac{\partial u}{\partial t} = -\lambda_n e^{-\lambda_n t} u_n(x)$$

compute the Laplacian:

$$\Delta u = e^{-\lambda_n t} \Delta u_n$$

Since  $u_n$  satisfies the eigenvalue equation  $\Delta u_n = -\lambda_n u_n$ , we obtain

$$\Delta u = -\lambda_n e^{-\lambda_n t} u_n(x)$$

Thus

$$\frac{\partial u}{\partial t} = \Delta u$$

so  $u(x, t)$  satisfies the heat equation.  $\square$

Notice that the full solution to the heat equation can be written as a sum of such eigenfunction solutions.

**Theorem 2.5**

If  $\{u_n\}$  is a complete set of Laplacian eigenfunctions with eigenvalues  $\lambda_n$ , then the solution to the heat

equation with initial condition  $f(x)$  can be written as

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} u_n(x)$$

where the constants  $a_n$  depend on the initial function  $f$ .

**Proof:**

The initial condition  $f(x)$  can be expanded as

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x)$$

Each term evolves according to Proposition 2.4 proved above.

Therefore our solution is obtained by multiplying each term by  $e^{-\lambda_n t}$ ,

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} u_n(x)$$

This function satisfies the heat equation and the initial condition  $u(x, 0) = f(x)$ .  $\square$

Now if we take the trace of the heat operator, then we actually get back to the spectral theta function introduced earlier.

**Corollary 2.6**

The spectral theta function summarizes the contribution of all Laplacian eigenvalues to heat diffusion on the domain.

**Proof:**

From Theorem 2.5, the solution of the heat equation is built from terms of the form  $e^{-\lambda_n t}$ .

When the heat operator is viewed spectrally, each eigenvalue contributes exactly one exponential term.

Taking the trace amounts to summing these contributions, producing the series

$$\Theta(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}$$

Thus the spectral theta function records the influence of the entire spectrum on the evolution of heat.  $\square$

### 3 Kac's Question:

In 1966 the mathematician Mark Kac posed a famous question,

*Can one hear the shape of a drum?*

Mathematically, the vibrations of a drumhead are governed by the eigenvalue problem for the Laplacian

$$\Delta u = -\lambda u$$

with boundary conditions along the edge of the drum.

The numbers

$$\lambda_1, \lambda_2, \lambda_3, \dots$$

represent the natural frequencies of vibration where these eigenvalues form the *spectrum* of the drum.

Kac's question can therefore be reformulated as follows:

Does the spectrum of the Laplacian determine the geometric shape of the domain?

As shown before in the **Section 2**, the spectrum can be encoded in the spectral theta function

$$\Theta(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}$$

which summarizes the entire collection of vibrational frequencies in a single infinite series.

### **Proposition 3.1**

If two domains have identical Laplacian eigenvalues, then they have the same spectral theta function.

#### **Proof:**

Suppose two domains have the same spectrum

$$\lambda_1, \lambda_2, \lambda_3, \dots$$

The spectral theta function is defined by

$$\Theta(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}$$

Since both domains have exactly the same eigenvalues then the terms in the series are identical.

Thus, the resulting functions are equal for all  $t > 0$  and therefore identical spectra produce identical spectral theta functions.  $\square$

### **Theorem 3.2 (Weyl's Law)**

Let  $\lambda_n$  denote the eigenvalues of the Laplacian on a bounded planar domain with area  $A$ .

Then for large  $n$ ,

$$\lambda_n \sim \frac{4\pi n}{A}$$

Equivalently, the number  $N(\lambda)$  of eigenvalues less than or equal to  $\lambda$  satisfying

$$N(\lambda) \sim \frac{A}{4\pi} \lambda$$

This proof is too complicated and thus we will take such Theorem for granted, assuming it is true.

**Corollary 3.3**

The spectrum of the Laplacian determines the area of the drum.

**Proof:**

From Weyl's law we have

$$N(\lambda) \sim \frac{A}{4\pi} \lambda.$$

Since the function  $N(\lambda)$  depends only on the eigenvalues of the Laplacian, the constant  $A$  can be recovered from the spectrum.

Therefore the area of the domain is determined by its eigenvalues.  $\square$

We look at the geometric information can also be extracted from the spectral theta function.

**Proposition 3.4**

The small- $t$  expansion of the spectral theta function has the form

$$\Theta(t) \sim \frac{A}{4\pi t} + \frac{L}{8\sqrt{\pi t}} + \dots$$

where  $A$  is the area of the domain and  $L$  is the length of its boundary.

Again, proof to this proposition is overly complicated, thus we will take it for granted.

**Corollary 3.5:**

The spectrum of the Laplacian determines both the area and the boundary length of the drum.

From above, we are able to show that the spectrum contains a lot of geometric information about the domain.

Now we circle back to Kac's questions,

**Theorem 3.6**

There exist distinct planar domains with identical Laplacian spectra.

**Proof:** The whole proof will not be discussed in this paper, however, the initial step for the proof could trace back to Gordon, Webb, and Wolpert's paper in 1992, which construct isospectral domains.

Thus, the answer to Kac's question is that although the spectrum determines many geometric properties such as area and boundary length, it does not uniquely determine the shape of the drum.

In simple words, Two drums may sound identical but their shapes differ.

## References

- [1] Massachusetts Institute of Technology *18.782 Introduction to Arithmetic Geometry*. Cambridge, Massachusetts, USA, 2013
- [2] Carolyn Gordon, David L. Webb, Scott Wolpert “One cannot hear the shape of a drum” *Bull. Amer. Math. Soc. (N.S.)* 27 (1992) 134-138, Ithaca, NY 14850
- [3] Walter A. Strauss *Partial Differential Equations: An Introduction, 2nd Edition*. Brown University
- [4] S. Minakshisundaram, A. Pleijel *Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds*, *Can.J.Math.* 1 (1949) 242-256
- [5] Steven Rosenberg *The Laplacian on a Riemannian Manifold, An Introduction to Analysis on Manifolds*. Boston University