RAMANUJAN'S FORMULA FOR π

ERCHAMION

Abstract. The goal of this paper is to summarize the proof of Ramanujan's pi formula as found in [BB98], the intention being to urge the reader to then go and read [BB98] and [Ram14].

1. Introduction

The calculation of π has been a major driving force of progress in mathematics for many ages. These efforts have resulted in the discovery of a great multitude and variety of formulas for π, many of which are very beautiful in themselves and some of which are useful in the actual computation of the digits of π. One very famous result, for example, is the Madhava-Gregory-Leibniz series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

This is an extremely important theoretical result. However, it is in itself not quite useful to obtain many digits of π, on account of its very slow convergence. On the other hand, the series

$$\begin{aligned}\frac{1}{2\pi\sqrt{2}} &= \frac{1103}{99^2} + \frac{27493}{99^6}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{53883}{99^{10}}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \dots \\ &= \sum_{n=0}^{\infty}\frac{(26390n+1103)}{99^{4n+2}}\frac{1\cdot 3\cdot \ldots (2n-1)}{2\cdot 4\ldots 2n}\frac{1\cdot 3\cdot\ldots(4n-1)}{4^2\cdot 8^2\cdot\ldots(4n)^2},\end{aligned}$$

found by Ramanujan [Ram14], is not only very elegant achievement of theory, but is also extremely convenient for calculating π because of its very fast convergence; it gives about 8 digits of π per term. For example, with only slate and chalk, one can obtain 20 correct decimal places of $1/(2\pi\sqrt{2})$ as follows:

$$\begin{aligned}\frac{1103}{99^2} &= 0.112539536781961024385 \\ \frac{27493}{99^6}\frac{3}{2\cdot 4^2} &= 0.000000002737677211399 \\ \frac{53883}{99^{10}}\frac{3\cdot 3\cdot 5\cdot 7}{1\cdot 4\cdot 4^2\cdot 8^2} &= 0.000000000000000022909 \\ \sum &= 0.112539539519638258693,\end{aligned}$$

in a very reasonable amount of time.

Now this formula is an achievement of elliptic function theory, and Ramanujan obtained it (along with 13 other similar formulas) in his very original version of this theory based on hypergeometric series. Unfortunately, Ramanujan did not live long enough to publish his

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Date: March 26, 2026.

proof in full details. People after him have tried to construct proofs as Ramanujan indicated. The goal of this short paper is only to urge the reader to read [BB98] by making the path of the proof there more visible hopefully. The reader is also encouraged to read [Ram14].

2. DEFINITIONS

We will give most of the basic notions and definitions in this section.

Definition 2.1 (Elliptic Integrals of the First Kind)**.** This is

$$\int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

where $0 \le k < 1$ is called the *modulus* and $0 \le \phi \le \pi/2$ is the *amplitude.* The *Complete Elliptic Integral of the first kind* is

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

and is denoted by $K(k)$.

Definition 2.2 (Elliptic Integrals of the Second Kind)**.** This is

$$\int_0^{\phi} \sqrt{1-k^2\sin^2\theta}d\theta.$$

$$E(k) := \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}d\theta$$

is the *Complete Elliptic Integral of the Second Kind.*

Definition 2.3 (Complementary Modulus)**.**

$$k' = \sqrt{1-k^2}$$

is the *complementary modulus.*

Definition 2.4 (Complementary Elliptic Integrals)**.**

$$K'(k) := K(k')$$
$$E'(k) := E(k').$$

Remark 2.5. To denote derivatives we will use dots. Thus $\dot{K}$ is the first derivative of K with respect to k.

Definition 2.6 (Singular Moduli)**.** Let $r > 0$ be any real number. The modulus k_r which makes

$$\frac{K'}{K} = r\frac{K}{K'},$$

i.e.

$$\frac{K'}{K} = \sqrt{r},$$

is called a *singular modulus.*

Definition 2.7 (Singular Value Function)**.** For $r > 0$, letting k_r be the singular modulus such that

$$\frac{K'(k_r)}{K(k_r)} = \sqrt{r},$$

we define

$$\alpha(r) := \frac{E'(k_r)}{K(k_r)} - \frac{\pi}{4K(k_r)^2}.$$

Definition 2.8 (Rising Factorial)**.** For any $a \in \mathbb{C}$,

$$(a)_0 := 1$$

and

$$(a)_n = a(a+1)\dots(a+n-1)$$

for all positive integers n.

Definition 2.9 (Hypergeometric Series)**.**

$$ {}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$$

$$ {}_3F_2(a,b,c;d,e;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{x^n}{n!}.$$

Definition 2.10 (Generalized Complete Elliptic Integrals)**.** For $|s| < 1/2$ and $0 \le k < 1$,

$$K_s(k) := \frac{\pi}{2}{}_2F_1\left(\frac{1}{2}-s, \frac{1}{2}+s; 1; k^2\right)$$

$$E_s(k) := \frac{\pi}{2}{}_2F_1\left(-\frac{1}{2}-s, \frac{1}{2}+s; 1; k^2\right).$$

The complementary functions are still defined by $K_s'(k) := K_s(k')$ and $E_s'(k) := E_s(k')$. We will motivate this definition in a while.

Definition 2.11 (Ramanujan's invariants)**.**

$$G := (2kk')^{-\frac{1}{12}}$$

$$g := \left(\frac{2k}{k'^2}\right)^{-\frac{1}{12}}$$

Definition 2.12. Let k_r be the singular modulus for some $r > 0$. We denote

$$g_r = \left(\frac{2k_r}{k_r'^2}\right)^{-\frac{1}{12}}.$$

3. BASIC FORMULAE FOR ELLIPTIC INTEGRALS

In this section, we will give the most important formulae concerning elliptic integrals which will be of use later. First we will express the complete elliptic integrals $K(k)$ and $E(k)$ in terms of the hypergeometric series ${}_2F_1$.

Theorem 3.1.

$$K(k) = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

Proof. Let $0 \leq k < 1$ and $0 \leq \theta \leq \pi/2$. Then, by the binomial theorem,

$$\begin{aligned}
\frac{1}{\sqrt{1-k^2\sin^2\theta}} &= 1 + \frac{1}{2}k^2\sin^2\theta + \frac{1\cdot 3}{2\cdot 4}k^4\sin^4\theta + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^6\sin^2\theta + \ldots \\
\therefore \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} &= \int_0^{\frac{\pi}{2}} 1 d\theta + \frac{1}{2}k^2 \int_0^{\frac{\pi}{2}} \sin^2\theta d\theta + \frac{1\cdot 3}{2\cdot 4}k^4 \int_0^{\frac{\pi}{2}} \sin^4\theta d\theta + \ldots \\
&= \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \ldots\right).
\end{aligned}$$

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Theorem 3.2.

$$E(k) = \frac{\pi}{2} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

Proof. The proof is similar to the previous one except that we now use the function

$$(1-k^2\sin^2\theta)^{\frac{1}{2}}.$$

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Now we would like to motivate the definitions of the generalized elliptic integrals given in Definition 2.10. The polynomial formula for $\cos nx$ when n is odd is actually a special case of a general formula (given in the next lemma) which holds for all real values of n. When we appropriately modify the parameters in this formula and integrate, we naturally arrive at the generalized elliptic integrals.

Lemma 3.3. *For all real n and all real x, we have*

$$\cos(nx) = \cos x \left\{1 + \frac{1^2 - n^2}{2!}\sin^2 x + \frac{(1^2-n^2)(3^2-n^2)}{4!}\sin^4 x + \ldots\right\}.$$

Proof. We have

$$
\begin{aligned}
&t^n \frac{d}{dt}\left\{ t^{1-2n} \frac{d}{dt}\left(\frac{t^{1+n}\cos(n\sin^{-1}t)}{\sqrt{1-t^2}} \right) \right\} \\
&= \frac{-n^2t^2\cos(n\sin^{-1}t)}{(1-t^2)^{\frac{3}{2}}} - \frac{n(2-n)t\sin(n\sin^{-1}t)}{1-t^2} - \frac{2nt^3\sin(n\sin^{-1}t)}{(1-t^2)^2} \\
&- \frac{n(1+n)t\sin(n\sin^{-1}t)}{(1-t^2)} + \frac{(1+n^2)\cos(n\sin^{-1}t)}{\sqrt{1-t^2}} + \frac{(1+n)t^2\cos(n\sin^{-1}t)}{(1-t^2)^{\frac{3}{2}}} \\
&- \frac{nt^3\sin(n\sin^{-1}t)}{(1-t^2)^2} + \frac{(3-n)t^2\cos(n\sin^{-1}t)}{(1-t^2)^{\frac{3}{2}}} + \frac{3t^4\cos(n\sin^{-1}t)}{(1-t^2)^{\frac{5}{2}}} \\
&= \left\{ \frac{-n^2t^2\cos(n\sin^{-1}t)}{(1-t^2)^{\frac{3}{2}}} + \frac{(1-n^2)\cos(n\sin^{-1)}t}{\sqrt{1-t^2}} + \frac{t^2\cos(n\sin^{-1})t}{(1-t^2)^{\frac{3}{2}}} \right\} \\
&+ \left\{ \frac{nt^2\cos(n\sin^{-1}t)}{(1-t^2)^{\frac{3}{2}}} + \frac{(3-n)t^2\cos(n\sin^{-1}t)}{(1-t^2)^{\frac{3}{2}}} + \frac{3t^4\cos(n\sin^{-1})t}{(1-t^2)^{\frac{5}{2}}} \right\} \\
&- \left\{ \frac{n(2-n)t\sin(n\sin^{-1}t)}{1-t^2} + \frac{2nt^3\sin(n\sin^{-1}t)}{(1-t^2)^2} + \frac{n(1+n)t\sin^{-1}(n\sin^{-1}t)}{1-t^2} + \frac{nt^3\sin(n\sin^{-1}t)}{(1-t^2)^2} \right\} \\
&= \frac{(1-n^2)\cos(n\sin^{-1}t)}{(1-t^2)^{\frac{3}{2}}} + \frac{3t^2\cos(n\sin^{-1}t)}{(1-t^2)^{\frac{5}{2}}} - \frac{3nt\sin(n\sin^{-1}t)}{(1-t^2)^2}.
\end{aligned}
$$

Letting

$$\frac{\cos(n\sin^{-1}t)}{\sqrt{1-t^2}} = a_0 + a_2t^2 + a_4t^4 + \dots,$$

we therefore have

$$a_0 = 1, a_2 = \frac{1^2-n^2}{1\cdot 2}a_0, a_4 = \frac{3^2-n^2}{3\cdot 4}a_2, a_6 = \frac{5^2-n^2}{5\cdot 6}a_4, \dots$$

and the result follows. ■

Now to motivate the definition of K_s in Definition 2.10, we note that from the previous lemma we have

$$\frac{\cos(n\sin^{-1}x)}{\sqrt{1-x^2}} = F\left(\frac{1}{2}n+\frac{1}{2}, -\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}, x^2\right).$$

Now we can put $x = \sin\phi$, let $n = 2s$ and integrate termwise and we would get $K_s(k)$. We would get $E_s(k)$ similarly from a consideration of the formula

$$\cos nx = F\left(\frac{1}{2}n, -\frac{1}{2}n, \frac{1}{2}, \sin^2 x\right).$$

We can express the derivative of K in terms of E, K and k as follows:

Theorem 3.4.

$$\frac{dK}{dk} = \frac{E - k'^2K}{k(k')^2}.$$

Proof. This can be proved straightforwardly using the definitions of K and E. ■

A relation also exists among $E(k), E'(k), K(k)$ and $K'(k)$ as follows:

Theorem 3.5 (Legendre's Relation)**.** *For* $0 < k < 1$*,*

$$E(k)K'(k) + E'(k)K(k) - K(k)K'(k) = \frac{\pi}{2}.$$

Proof. See [BB98]. ■

Using Legendre's relation and Definition 2.7

Theorem 3.6.

$$\alpha(r) = \frac{1}{\pi}\left(\frac{\pi}{2K}\right)^2 - \sqrt{r}\left(kk'^2\frac{\dot{K}}{K} - k^2\right).$$

Proof. See [BB98]. ■

4. RAMANUJAN'S PI FORMULA

Now we can give a proof of Ramanujan's Pi Formula

$$\frac{1}{2\pi\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(26390n + 1103)}{99^{4n+2}} \frac{1 \cdot 3 \cdot \ldots (2n-1)}{2 \cdot 4 \cdot \ldots 2n} \frac{1 \cdot 3 \cdot \ldots (4n-1)}{4^2 \cdot 8^2 \cdot \ldots (4n)^2}.$$

Theorem 4.1. *For* $0 \leq k \leq \sqrt{2} - 1$*,*

$$\left[\frac{2K(k)}{\pi}\right]^2 = (1+k^2)^{-1} {}_3F_2\left(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}; 1, 1; \left(\frac{g^{12} + g^{-12}}{2}\right)^{-2}\right).$$

Proof. See [BB98]. ■

Let now $N \geq 2$ be a positive integer. We write Theorem 3.6 as

$$\frac{1}{\pi} = \sqrt{N}k_N k_N'^2 \frac{4K\dot{K}}{\pi^2} + [\alpha(N) - \sqrt{N}k_N^2]\frac{4K^2}{\pi^2}.$$

By differentiating both sides of Theorem 4.1, and then substituting in the previous formula, we deduce

Theorem 4.2. *For* $N > 2$*, if* $x_N = \left(\frac{g_N^{12}+g_N^{-12}}{2}\right)^{-1} = \frac{4k_N k_N'^2}{(1+k_N^2)^2}$*, then*

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} d_n(N) x_N^{2n+1}$$

where

$$d_n(N) := \left(\frac{\alpha(N)x_N^{-1}}{1+k_N^2} - \frac{\sqrt{N}}{4}g_N^{-12}\right) + n\sqrt{N}\left(\frac{g_N^{12} - g_N^{-12}}{2}\right).$$

Proof. See [BB98]. ■

Theorem 4.3.

$$g_{58}^2 = \frac{\sqrt{29} + 5}{2}.$$

Proof. See [BB98]. ■

After calculating $\alpha(58)$ (cf. [BB98]), we can now specialize Theorem 4.2 for $N = 58$ to obtain

$$\frac{1}{\pi} = 2\sqrt{2}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3}(1103 + 26390n)\left(\frac{1}{99^2}\right)^{2n+1}.$$

Proof.

$$\begin{aligned}
g_{58} &= \sqrt{\frac{\sqrt{29}+5}{2}} \\
g_{58}^{-1} &= \sqrt{\frac{\sqrt{29}-5}{2}} \\
g_{58}^{12} &= 9801 + 1820\sqrt{29} \\
g_{58}^{-12} &= 9801 - 1820\sqrt{29} \\
\therefore x_{58} &= \frac{g_{58}^{12} + g_{58}^{-12}}{2} = 9801 = 99^2 \\
\frac{g_{58}^{12} - g_{58}^{-12}}{2} &= 1820\sqrt{29} \\
\therefore \sqrt{58}\left(\frac{g_{58}^{12} - g_{58}^{-12}}{2}\right) &= 2\sqrt{2}(26390)
\end{aligned}$$

As in [BB98], we can calculate

$$\frac{\alpha(58)x_{58}^{-1}}{1+k_{58}^2} - \frac{\sqrt{58}}{4}g_{58}^{-12} = 2\sqrt{2}\cdot 1103.$$

The theorem now follows using Theorem 4.2. ■

References

[BB98] Jonathan M. Borwein and Peter B. Borwein. *Pi and the AGM. A study in analytic number theory and computational complexity.* Can. Math. Soc. Ser. Monogr. Adv. Texts. New York, NY: Wiley, 1998.

[Ram14] S. Ramanujan. Modular equations and approximations to π. *Quart. J.*, 45:350–372, 1914.