Simple Continued Fractions and Optimal Rational Approximation

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Abstract

This paper presents a self-contained treatment of simple continued fractions, covering their algebraic structure, convergent recurrences, and role in optimal rational approximation.

Contents

1 Definition and Basic Structure

Definition 1.1. A **simple continued fraction** is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

which we denote compactly by $[a_0; a_1, a_2, a_3, \ldots]$, where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{Z}_{>0}$ for all $n \geq 1$. The integers a_n are called the partial quotients of the continued fraction.

The continued fraction expansion of a real number x is produced by the **continued fraction algorithm**: set $x_0 = x$, and for $n \geq 0$ define

$$a_n = \lfloor x_n \rfloor, \qquad x_{n+1} = \frac{1}{x_n - a_n},$$

provided $x_n \notin \mathbb{Z}$. If x is rational the algorithm terminates in finitely many steps; if x is irrational it produces an infinite sequence of partial quotients.

Theorem 1.2. *Every irrational number $x \in \mathbb{R}$ admits a unique infinite simple continued fraction expansion $x = [a_0; a_1, a_2, \ldots]$.*

Proof. Existence follows from the continued fraction algorithm: since x is irrational, $x_n - a_n \neq 0$ at every stage, so the algorithm never terminates and produces an infinite sequence $(a_n)_{n \geq 0}$.

For uniqueness, suppose $x = [a_0; a_1, a_2, \ldots] = [b_0; b_1, b_2, \ldots]$. Then $a_0 = \lfloor x \rfloor = b_0$. Setting $x' = 1/(x - a_0)$, we have $x' = [a_1; a_2, \ldots] = [b_1; b_2, \ldots]$, and by induction $a_n = b_n$ for all n. ☐

1.1 Convergents and their recurrences

Definition 1.3. The n**th convergent** of the continued fraction $[a_0; a_1, a_2, \ldots]$ is the rational number

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n].$$

The numerators and denominators of the convergents satisfy an important recurrence.

Theorem 1.4 (Convergent Recurrence)**.** *Define sequences (p_n) and (q_n) by the initial conditions*

$$p_{-1} = 1, \quad p_0 = a_0, \qquad q_{-1} = 0, \quad q_0 = 1,$$

and the recurrences

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2}, \qquad n \geq 1.$$

Then $[a_0; a_1, \ldots, a_n] = p_n/q_n$ for all $n \geq 0$.

Proof. We proceed by induction on n. For $n = 0$, $p_0/q_0 = a_0/1 = a_0 = [a_0]$. For $n = 1$, $p_1/q_1 = (a_1a_0 + 1)/a_1 = a_0 + 1/a_1 = [a_0; a_1]$.

For the inductive step, note that

$$[a_0; a_1, \ldots, a_{n-1}, a_n] = \left[a_0; a_1, \ldots, a_{n-1} + \frac{1}{a_n}\right].$$

By the inductive hypothesis applied with the last partial quotient replaced by $a_{n-1} + 1/a_n$, we get

$$[a_0; a_1, \ldots, a_n] = \frac{\left(a_{n-1} + \frac{1}{a_n}\right) p_{n-2} + p_{n-3}}{\left(a_{n-1} + \frac{1}{a_n}\right) q_{n-2} + q_{n-3}}.$$

Multiplying numerator and denominator by a_n and using the recurrence relations for p_{n-1} and q_{n-1} yields p_n/q_n. □

Theorem 1.5 (Determinant Identity)**.** *For all* $n \geq 0$*,*

$$p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}.$$

Proof. For $n = 0$: $p_0q_{-1} - p_{-1}q_0 = a_0 \cdot 0 - 1 \cdot 1 = -1 = (-1)^{-1}$. For $n \geq 1$, use the recurrence:

$$\begin{aligned} p_nq_{n-1} - p_{n-1}q_n &= (a_np_{n-1} + p_{n-2})q_{n-1} - p_{n-1}(a_nq_{n-1} + q_{n-2}) \\ &= p_{n-2}q_{n-1} - p_{n-1}q_{n-2} \\ &= -(p_{n-1}q_{n-2} - p_{n-2}q_{n-1}). \end{aligned}$$

By induction, the result follows. □

An important structural consequence is that the even convergents $p_0/q_0, p_2/q_2, p_4/q_4, \ldots$ form a strictly increasing sequence, the odd convergents $p_1/q_1, p_3/q_3, p_5/q_5, \ldots$ form a strictly decreasing sequence, and every even convergent is less than every odd convergent. The value x is sandwiched between consecutive convergents:

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < x < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

2 Classical Examples

2.1 Quadratic irrationals and periodicity

Theorem 2.1 (Lagrange, 1770)**.** *A real number* α *has an eventually periodic continued fraction expansion if and only if* α *is a quadratic irrational, i.e.,* α *is irrational and satisfies a quadratic equation* $a\alpha^2 + b\alpha + c = 0$ *with* $a, b, c \in \mathbb{Z}$*,* $a \neq 0$*.*

We write $[\overline{a_0; a_1, \ldots, a_k}]$ for a purely periodic continued fraction and $[a_0; a_1, \ldots, a_m, \overline{b_1, \ldots, b_k}]$ for an eventually periodic one.

Example 2.2 ($\sqrt{2}$)**.** We have $\sqrt{2} = [1;\overline{2}] = [1;2,2,2,\ldots]$. To verify this, set $x = [1;\overline{2}]$. Then

$$x = 1 + \frac{1}{1+x} \quad\Longrightarrow\quad x(1+x) = (1+x) + 1 \quad\Longrightarrow\quad x^2 = 2.$$

Since $x > 0$, we conclude $x = \sqrt{2}$. The convergents

$$\frac{1}{1}, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}, \quad \frac{99}{70}, \quad \ldots$$

give increasingly accurate rational approximations to $\sqrt{2}$.

Example 2.3 (The golden ratio φ)**.** The golden ratio $\varphi = (1+\sqrt{5})/2$ has the simplest possible continued fraction:

$$\varphi = [1;\overline{1}] = [1;1,1,1,\ldots].$$

Setting $x = 1 + 1/x$ gives $x^2 - x - 1 = 0$, whose positive root is φ. Because all partial quotients equal 1—the smallest possible value—the convergents of φ approach their limit more slowly than for any other irrational number. The convergents are ratios of consecutive Fibonacci numbers:

$$\frac{p_n}{q_n} = \frac{F_{n+2}}{F_{n+1}},$$

where $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$.

2.2 Euler's continued fraction formula

Beyond quadratic irrationals, continued fractions provide remarkable representations of transcendental constants. Euler developed a powerful tool for converting infinite series into continued fractions, that we prove in our class pset.

Theorem 2.4 (Euler's Continued Fraction Formula)**.** *If $a_0, a_1, a_2, \ldots$ is a sequence of complex numbers with $a_n \neq 0$ for $n \geq 1$, then*

$$a_0 + a_0a_1 + a_0a_1a_2 + a_0a_1a_2a_3 + \cdots = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{a_2}{1 + a_2 - \cfrac{a_3}{1 + a_3 - \cdots}}}},$$

This is a powerful identity that transforms series into continued fractions and vice versa. As a classical application, Euler used this method to establish continued fraction expansions for the number e.

Example 2.5 (Continued fraction for e)**.**

$$e = [2;1,2,1,1,4,1,1,6,1,1,8,\ldots] = [2;\overline{1,2k,1}]_{k=1}^{\infty}.$$

The partial quotients follow the pattern $2,1,2,1,1,4,1,1,6,1,1,8,\ldots$ where every third quotient increases by 2 and the rest are 1. This expansion can be derived from the series $e = \sum_{n=0}^{\infty} 1/n!$ via Euler's continued fraction formula.

The first several convergents are

$$2, \quad 3, \quad \frac{8}{3}, \quad \frac{11}{4}, \quad \frac{19}{7}, \quad \frac{87}{32}, \quad \frac{106}{39}, \quad \frac{193}{71}, \quad \frac{1264}{465}, \quad \dots$$

The convergent $193/71 \approx 2.71831$ already approximates e to four decimal places.

Remark 2.6. The continued fraction for e was instrumental in the first proof that e is irrational, given by Euler in 1737. The argument runs as follows: if e were rational, its continued fraction would terminate, but the pattern $1, 2k, 1$ continues indefinitely, so e is irrational.

2.3 Integration and continued fractions

Continued fractions also arise naturally in the evaluation of certain integrals. We illustrate this connection with Brouncker's classical result.

Example 2.7 (Brouncker's continued fraction for $4/\pi$)**.** Lord Brouncker (1656) discovered the remarkable identity

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}.$$

This is a *generalized* continued fraction (the numerators are not all 1), and it arises from manipulating Wallis's integral formula

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdots$$

which itself is derived from the integral

$$\int_0^{\pi/2} \sin^n(x)\, dx.$$

Example 2.8 (The arctangent continued fraction)**.** The integral representation

$$\arctan(x) = \int_0^x \frac{dt}{1 + t^2}$$

leads, via Euler's techniques on the taylor expansion, to the continued fraction

$$\arctan(x) = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{4x^2}{5 + \cfrac{9x^2}{7 + \cdots}}}} = \cfrac{x}{1 + \cfrac{(1x)^2}{3 + \cfrac{(2x)^2}{5 + \cfrac{(3x)^2}{7 + \cdots}}}}.$$

Setting $x = 1$ gives a continued fraction for $\pi/4$:

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cdots}}}}.$$

3 Rational Approximation

We now develop the approximation theory of continued fractions. The central theme is that convergents provide the best possible rational approximations in a precise sense.

3.1 Basic approximation bounds

Theorem 3.1. *Let $x = [a_0; a_1, a_2, \ldots]$ be irrational with convergents p_n/q_n. Then for all $n \geq 0$,*

$$\frac{1}{q_n(q_{n+1} + q_n)} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

In particular, since $q_{n+1} \geq q_n + 1 > q_n$,

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

Proof. Since x lies between p_n/q_n and p_{n+1}/q_{n+1}, we have

$$\left| x - \frac{p_n}{q_n} \right| < \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n q_{n+1}},$$

using the determinant identity (Theorem 1.5).

For the lower bound, write $x = (x_{n+1} p_n + p_{n-1})/(x_{n+1} q_n + q_{n-1})$, where $x_{n+1} = [a_{n+1}; a_{n+2}, \ldots]$. Then

$$x - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(x_{n+1} q_n + q_{n-1})}.$$

Since $a_{n+1} < x_{n+1} < a_{n+1} + 1$, we get $q_{n+1} < x_{n+1} q_n + q_{n-1} < q_{n+1} + q_n$, which gives the stated lower bound. ☐

3.2 Best approximation property

Definition 3.2. A rational number p/q (in lowest terms, $q > 0$) is a **best rational approximation** to x if for every rational a/b with $0 < b \leq q$ and $a/b \neq p/q$,

$$\left| x - \frac{p}{q} \right| < \left| x - \frac{a}{b} \right|.$$

Theorem 3.3 (Best Approximation Theorem)**.** *Every convergent p_n/q_n of an irrational number x is a best rational approximation to x. Conversely, every best rational approximation to x is a convergent of x.*

Proof. Let p/q be a rational with $0 < q \leq q_n$ and $p/q \neq p_n/q_n$. Consider the system of linear equations

$$p = p_n u + p_{n-1} v, \qquad q = q_n u + q_{n-1} v.$$

By the determinant identity, this system has the unique integer solution

$$u = (-1)^{n-1}(q_{n-1}p - p_{n-1}q), \qquad v = (-1)^n (q_n p - p_n q).$$

Since $p/q \neq p_n/q_n$, we have $v \neq 0$. From $q = q_n u + q_{n-1} v \leq q_n$ with $q > 0$, one can show that u and v must have opposite signs (or $u = 0$).

Now

$$qx - p = u(q_n x - p_n) + v(q_{n-1}x - p_{n-1}).$$

Since $q_n x - p_n$ and $q_{n-1}x - p_{n-1}$ have opposite signs (the convergents alternate around x), and u, v have opposite signs, the two terms on the right have the *same* sign. Therefore

$$|qx - p| = |u| \cdot |q_n x - p_n| + |v| \cdot |q_{n-1}x - p_{n-1}| \geq |q_n x - p_n|.$$

Dividing by $q \leq q_n$ gives

$$\left| x - \frac{p}{q} \right| \geq \frac{|q_n x - p_n|}{q} \geq \frac{|q_n x - p_n|}{q_n} = \left| x - \frac{p_n}{q_n} \right|.$$

Moreover, the inequality is strict since $|v| \geq 1$ contributes a positive term. □

3.3 The notable example π

Example 3.4. The continued fraction expansion of π begins

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \ldots].$$

The first several convergents are:

$$3, \quad \frac{22}{7}, \quad \frac{333}{106}, \quad \frac{355}{113}, \quad \frac{103993}{33102}, \quad \ldots$$

The convergent 22/7 is Archimedes' famous approximation. The convergent 355/113, known to the Chinese mathematician Zu Chongzhi (5th century), is spectacularly accurate:

$$\left| \pi - \frac{355}{113} \right| \approx 2.67 \times 10^{-7} < \frac{1}{113^2} \approx 7.83 \times 10^{-5}.$$

3.4 The Hurwitz Approximation Theorem

Lemma 3.5. *Let x be irrational with convergents p_n/q_n. For any three consecutive convergents, at least one satisfies*

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2}.$$

Proof. Suppose to the contrary that $|x - p_n/q_n| \geq 1/(2q_n^2)$ and $|x - p_{n+1}/q_{n+1}| \geq 1/(2q_{n+1}^2)$. Since consecutive convergents lie on opposite sides of x,

$$\frac{1}{q_n q_{n+1}} = \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| = \left|x - \frac{p_n}{q_n}\right| + \left|x - \frac{p_{n+1}}{q_{n+1}}\right| \geq \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}.$$

Multiplying through by $2q_n q_{n+1}$:

$$2 \geq \frac{q_{n+1}}{q_n} + \frac{q_n}{q_{n+1}}.$$

But by the AM-GM inequality, $t + 1/t \geq 2$ for all $t > 0$, with equality only when $t = 1$, i.e., $q_n = q_{n+1}$. Since $q_{n+1} = a_{n+1}q_n + q_{n-1} \geq q_n + 1 > q_n$ for $n \geq 1$, this is a contradiction. $\square$

Theorem 3.6 (Hurwitz, 1891)**.** *For every irrational number α, there exist infinitely many rationals p/q such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\,q^2}.$$

Moreover, the constant $\sqrt{5}$ is best possible.

Proof. **Existence.** We strengthen Lemma 3.5 by considering three consecutive convergents p_{n-1}/q_{n-1}, p_n/q_n, p_{n+1}/q_{n+1}. Suppose none of them satisfies the Hurwitz bound. Then

$$\left|\alpha - \frac{p_k}{q_k}\right| \geq \frac{1}{\sqrt{5}\,q_k^2} \quad \text{for } k = n-1, n, n+1.$$

Using the identity $1/(q_n q_{n+1}) = |\alpha - p_n/q_n| + |\alpha - p_{n+1}/q_{n+1}|$ (since consecutive convergents bracket α), we obtain

$$\frac{1}{q_n q_{n+1}} \geq \frac{1}{\sqrt{5}\,q_n^2} + \frac{1}{\sqrt{5}\,q_{n+1}^2}.$$

Setting $r = q_{n+1}/q_n$, this gives $\sqrt{5} \geq r + 1/r$, hence $r^2 - \sqrt{5}\,r + 1 \leq 0$, so

$$r \leq \frac{\sqrt{5}+1}{2} = \varphi.$$

Similarly, from the pair $(n-1, n)$:

$$\frac{1}{q_{n-1}q_n} \geq \frac{1}{\sqrt{5}\,q_{n-1}^2} + \frac{1}{\sqrt{5}\,q_n^2},$$

which gives $q_n/q_{n-1} \leq \varphi$. But $q_{n+1} = a_{n+1}q_n + q_{n-1} \geq q_n + q_{n-1}$, so

$$r = \frac{q_{n+1}}{q_n} \geq 1 + \frac{q_{n-1}}{q_n} \geq 1 + \frac{1}{\varphi} = 1 + \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2} = \varphi.$$

Combined with $r \leq \varphi$, we get $r = \varphi$ exactly, which forces $a_{n+1} = 1$ and $q_n/q_{n-1} = \varphi$ as well. Since q_n/q_{n-1} is rational and φ is irrational, this is a contradiction.

Since this argument applies for every n, infinitely many convergents satisfy the Hurwitz bound.

Sharpness. Let $\alpha = \varphi = [1; \overline{1}]$. All partial quotients are 1, so $q_n = F_{n+1}$ (Fibonacci numbers). A classical identity gives

$$q_n^2 \left| \varphi - \frac{p_n}{q_n} \right| = \frac{q_n^2}{q_n(x_{n+1}q_n + q_{n-1})} \to \frac{1}{\varphi + 1/\varphi} = \frac{1}{\sqrt{5}}$$

as $n \to \infty$, since $x_{n+1} \to \varphi$ and $q_{n-1}/q_n \to 1/\varphi$. Thus $|\varphi - p_n/q_n| \sim 1/(\sqrt{5}\, q_n^2)$, and no constant $c > \sqrt{5}$ allows $|\varphi - p/q| < 1/(c\, q^2)$ for infinitely many p/q. $\square$

4 Concluding Remarks

The theory of continued fractions is incredibly relevant to both algebra and number theory. They are incredible approximation tools with the best approximation theorem guaranteeing optimality among all rational approximations, and the Hurwitz theorem gives a sharp universal bound on how well irrationals can be approximated. This paper covers a few of the introductory techniques dealing with simple continued fractions, but there is still more to be said about evaluating general continued fractions and continued fractions with random variables as mathematicians continue exploring these structures.

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