BROWNIAN MOTION: RANDOM FOURIER SERIES AND ITÔ CALCULUS

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1. Introduction

Brownian motion is a fundamental example of a stochastic process and arises naturally as the scaling limit of random walks. Despite its simple definition, it exhibits highly irregular behavior and is nowhere differentiable, making classical analytical tools insufficient for its study. One way to understand Brownian motion is through its covariance structure. This perspective leads to the Karhunen–Loève expansion, which represents Brownian motion as an infinite series with random coefficients. In this way, Brownian motion can be viewed as a random Fourier series, connecting probabilistic ideas with classical harmonic analysis. The irregularity of Brownian motion also motivates the development of stochastic calculus. In particular, the quadratic variation of Brownian motion leads to a modified chain rule known as Itô's lemma. This allows one to analyze functions of stochastic processes and to solve stochastic differential equations. As an application, we derive an explicit formula for geometric Brownian motion. The goal of this paper is to develop these ideas and illustrate how analytic and probabilistic methods combine to describe the structure of Brownian motion.

2. Preliminaries

In this section we review several analytic and probabilistic concepts that will be used throughout the paper. We begin with orthogonal expansions in $L^2[0,1]$, then discuss Gaussian random variables and random series, and finally recall the definition and covariance structure of Brownian motion. Let $L^2[0,1]$ denote the space of square-integrable functions on the interval $[0,1]$, that is

$$L^2[0,1] = \left\{ f : [0,1] \to \mathbb{R},\ \int_0^1 |f(t)|^2 dt < \infty \right\}.$$

This space is a Hilbert space with inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

A sequence $\{e_n\}_{n=1}^{\infty}$ in $L^2[0,1]$ is called orthonormal if

$$\langle e_m, e_n \rangle = \begin{cases} 1 & m = n, \\ 0 & m \neq n. \end{cases}$$

Date: March 25, 2026.

If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2[0,1]$, then every function $f \in L^2[0,1]$ admits an expansion $f(t) = \sum_{n=1}^{\infty} a_n e_n(t)$, where $a_n = \langle f, e_n \rangle$. Parseval's identity states that

$$\int_0^1 |f(t)|^2 dt = \sum_{n=1}^{\infty} a_n^2.$$

Orthogonal expansions of this type play a central role in Fourier analysis and will later allow us to represent stochastic processes as infinite series.

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable is a measurable function $X : \Omega \to \mathbb{R}$. The expectation of X is defined by

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega).$$

The variance of X is $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$.

Definition 2.2. Let X and Y be random variables with finite expectations. The covariance of X and Y is defined by

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

Definition 2.3. A random variable X is called Gaussian, or normally distributed, with mean μ and variance $\sigma^2 > 0$ if it has density

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)},\ x \in \mathbb{R}.$$

In this case we write $X \sim N(\mu, \sigma^2)$. The special case $\mu = 0$ and $\sigma^2 = 1$ is called the standard Gaussian distribution. A random variable Z with this distribution is called a standard Gaussian random variable, and we write $Z \sim N(0, 1)$.

Because of this, if $Z_1, Z_2, \ldots$ are independent standard Gaussian random variables, then $\mathbb{E}[Z_n] = 0$, $\mathbb{E}[Z_n^2] = 1$ and for $m \neq n$, $\mathbb{E}[Z_m Z_n] = 0$.

Definition 2.4. A collection of random variables $\{X_n\}_{n=1}^{\infty}$ is said to be independent if for every positive integer k and every choice of real numbers $a_1, \ldots, a_k$,

$$\mathbb{P}(X_1 \leq a_1, \ldots, X_k \leq a_k) = \prod_{n=1}^{k} \mathbb{P}(X_n \leq a_n).$$

If X and Y are independent and have finite expectations, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

In this paper we will consider random series of the form

$$X(t) = \sum_{n=1}^{\infty} a_n Z_n e_n(t),$$

where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal family in $L^2[0,1]$, the coefficients a_n are real numbers, and the random variables Z_n are independent standard Gaussian variables.

Definition 2.5. A sequence of random variables X_N converges to X in mean square if

$$\mathbb{E}[(X_N - X)^2] \to 0$$

as $N \to \infty$.

Mean-square convergence will be the natural notion of convergence for the random series considered in this paper as each of the series are just the sum of many random variables so are a random variable themselves.

Definition 2.6. A stochastic process is a collection of random variables $\{X_t\}_{t\in T}$ defined on the same probability space, indexed by a set T. For each $t \in T$, the random variable X_t represents the value of the process at time t.

We now introduce Brownian motion, which will be the central stochastic process in this paper.

Definition 2.7. [Wie23] A stochastic process $\{B_t\}_{t\geq 0}$ is called standard Brownian motion if it satisfies the following properties:

(1) $B_0 = 0$,
(2) for $0 \leq s < t$, the increment $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$,
(3) increments over disjoint intervals are independent,
(4) the sample paths $t \mapsto B_t$ are continuous.

Brownian motion may be viewed as a continuous limit of a simple random walk. Let $X_1, X_2, \dots$ be independent random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$. Define the partial sums $S_n = X_1 + \cdots + X_n$. The scaled process

$$B_n(t) = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$$

approximates Brownian motion as $n \to \infty$. In this sense Brownian motion may be interpreted as the continuous limit of a random walk.

An important quantity associated with a stochastic process is its covariance function $K(s,t) = \mathbb{E}[X_s X_t]$. Notice that this is the case because $\mathbb{E}[X_s] = \mathbb{E}[X_t] = 0$. For Brownian motion the covariance can be computed explicitly. If $s < t$, we write $B_t = B_s + (B_t - B_s)$. Using independence of increments, we obtain $\mathbb{E}[B_s(B_t - B_s)] = 0$, and therefore

$$\mathbb{E}[B_s B_t] = \mathbb{E}[B_s^2] = s$$

since the variance of $B_s - B_0 = B_s$ is $s - 0 = s$. Thus the covariance function of Brownian motion is

$$\mathbb{E}[B_s B_t] = \min(s,t).$$

This covariance function will play a central role in the remainder of the paper. In particular, it will allow us to construct an orthogonal expansion of Brownian motion through the eigenfunctions of the associated covariance operator.

3. The Covariance Operator

An important object associated with a stochastic process is its covariance function. If $\{X_t\}_{t\geq 0}$ is a stochastic process with finite second moments, its covariance function is defined by

$$K(s,t) = \mathbb{E}[X_s X_t].$$

As shown in the previous section, the covariance function of Brownian motion is $K(s,t) = \min(s,t)$. The covariance function allows us to construct a linear operator acting on functions in $L^2[0,1]$.

Definition 3.1. Let $K(s,t)$ be the covariance function of a stochastic process defined on $[0,1]$. The covariance operator T is the linear operator defined by

$$(Tf)(t) = \int_0^1 K(s,t)f(s)\,ds.$$

For Brownian motion the kernel is $K(s,t) = \min(s,t)$, and therefore the covariance operator takes the form

$$(Tf)(t) = \int_0^1 \min(s,t)f(s)\,ds.$$

The significance of this operator arises from the following observation. Suppose a stochastic process admits a representation of the form

$$X(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} Z_n e_n(t),$$

where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal family in $L^2[0,1]$, the constants λ_n are nonnegative, and $\{Z_n\}_{n=1}^{\infty}$ are independent standard Gaussian random variables. Then the covariance of the process can be computed as

$$\mathbb{E}[X_s X_t] = \mathbb{E}\left[\sum_{m=1}^{\infty} \sqrt{\lambda_m} Z_m e_m(s) \sum_{n=1}^{\infty} \sqrt{\lambda_n} Z_n e_n(t)\right].$$

Expanding the product gives

$$\mathbb{E}[X_s X_t] = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sqrt{\lambda_m \lambda_n}\, \mathbb{E}[Z_m Z_n]\, e_m(s) e_n(t).$$

Since the random variables Z_n are independent standard Gaussians, we have $\mathbb{E}[Z_m Z_n] = \delta_{mn}$, where δ_{mn} denotes the Kronecker delta. Therefore

$$\mathbb{E}[X_s X_t] = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sqrt{\lambda_m \lambda_n}\, \delta_{mn}\, e_m(s) e_n(t) = \sum_{n=1}^{\infty} \lambda_n e_n(s) e_n(t).$$

Thus any process with such a representation has covariance function

$$K(s,t) = \sum_{n=1}^{\infty} \lambda_n e_n(s) e_n(t).$$

In particular, if we wish to represent Brownian motion in this way, the functions e_n and constants λ_n must satisfy $\min(s,t) = \sum_{n=1}^{\infty} \lambda_n e_n(s) e_n(t)$. Applying the covariance operator to a function e gives

$$(Te)(t) = \int_0^1 \min(s,t) e(s)\,ds.$$

If the kernel admits the expansion above, then the functions e_n must satisfy $Te_n = \lambda_n e_n$. In other words, the functions appearing in the expansion must be eigenfunctions of the covariance operator. We are therefore led to the following eigenvalue problem

$$\int_0^1 \min(s,t)e(s)\,ds = \lambda e(t).$$

Solving this equation will determine the functions that form the orthogonal basis in the expansion of Brownian motion.

4. Eigenfunctions and Eigenvalues of the Covariance Operator

In the previous section we introduced the covariance operator $(Tf)(t) = \int_0^1 \min(s,t)f(s)\,ds$. To obtain an orthogonal expansion for Brownian motion we must solve the eigenvalue problem $Te = \lambda e$. In integral form this becomes

$$\int_0^1 \min(s,t)e(s)\,ds = \lambda e(t).$$

To analyze this equation we use the fact that $\min(s,t) = \begin{cases} s & s < t \\ t & s \geq t \end{cases}$. Splitting the integral at t gives

$$\int_0^t s\,e(s)\,ds + t\int_t^1 e(s)\,ds = \lambda e(t).$$

We now differentiate this expression with respect to t. Differentiating the first term using the fundamental theorem of calculus gives

$$\frac{d}{dt}\left(\int_0^t s\,e(s)\,ds\right) = t\,e(t).$$

For the second term we obtain

$$\frac{d}{dt}\left(t\int_t^1 e(s)\,ds\right) = \int_t^1 e(s)\,ds - t\,e(t).$$

Adding these derivatives yields

$$\int_t^1 e(s)\,ds = \lambda e'(t).$$

Differentiating once more gives $-e(t) = \lambda e''(t)$, and therefore $e''(t) = -\frac{1}{\lambda}e(t)$. Thus the eigenfunctions satisfy the second-order differential equation $e''(t) + \omega^2 e(t) = 0$, where $\omega^2 = \frac{1}{\lambda}$. We now determine the boundary conditions. Setting $t = 0$ in the integral equation gives $\int_0^1 \min(s,0)e(s)\,ds = 0$, which implies $\lambda e(0) = 0$. Since $\lambda \neq 0$, we obtain $e(0) = 0$. From the first derivative equation $\int_t^1 e(s)\,ds = \lambda e'(t)$, we set $t = 1$ and obtain $0 = \lambda e'(1)$, which implies $e'(1) = 0$. The general solution of the differential equation is

$$e(t) = A\sin(\omega t) + B\cos(\omega t).$$

Using the boundary condition $e(0) = 0$ gives $B = 0$, so $e(t) = A\sin(\omega t)$. Differentiating yields $e'(t) = A\omega\cos(\omega t)$. Applying the condition $e'(1) = 0$ gives $\cos(\omega) = 0$. Hence

$$\omega = (n - \tfrac{1}{2})\pi, \qquad n = 1, 2, 3, \ldots$$

Since $\omega^2 = \frac{1}{\lambda}$, we obtain the eigenvalues

$$\lambda_n = \frac{1}{(n - \frac{1}{2})^2 \pi^2}.$$

The corresponding eigenfunctions are

$$e_n(t) = A_n \sin((n - \tfrac{1}{2})\pi t).$$

Choosing the constant A_n so that the functions are normalized in $L^2[0,1]$ gives

$$e_n(t) = \sqrt{2} \sin((n - \tfrac{1}{2})\pi t).$$

We summarize the result in the following theorem.

Theorem 4.1. *The eigenvalues and eigenfunctions of the covariance operator*

$$(Tf)(t) = \int_0^1 \min(s,t) f(s)\, ds$$

are given by

$$\lambda_n = \frac{1}{(n - \frac{1}{2})^2 \pi^2}, \qquad e_n(t) = \sqrt{2} \sin((n - \tfrac{1}{2})\pi t),$$

for $n = 1, 2, 3, \ldots$.

These functions form an orthonormal system in $L^2[0,1]$. In the next section we use these eigenfunctions and eigenvalues to obtain the Karhunen–Loève expansion of Brownian motion.

5. The Karhunen–Loève Expansion

The Karhunen–Loève theorem provides a general method for representing stochastic processes as orthogonal series with random coefficients. The idea is that the eigenfunctions of the covariance operator provide a natural orthogonal basis. In this way the deterministic part of the process is in the eigenfunctions while the randomness appears through independent Gaussian coefficients.

We first state the theorem in general form and then apply it to Brownian motion.

Theorem 5.1 (Karhunen–Loève)**.** [Kar47][Loé45] *Let* $\{X_t\}_{t \in [0,1]}$ *be a mean-zero stochastic process with covariance function*

$$K(s,t) = \mathbb{E}[X_s X_t].$$

Let T be the covariance operator defined by

$$(Tf)(t) = \int_0^1 K(s,t) f(s)\, ds.$$

Suppose T has eigenvalues λ_n and orthonormal eigenfunctions e_n. Then the process admits a representation

$$X(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, Z_n e_n(t),$$

where $\{Z_n\}_{n=1}^{\infty}$ are independent standard Gaussian random variables.

We will not prove the theorem in full generality. Instead we verify it explicitly for Brownian motion.

Recall from the previous section that the covariance operator for Brownian motion is

$$(Tf)(t) = \int_0^1 \min(s,t) f(s)\, ds,$$

and that its eigenvalues and eigenfunctions are

$$\lambda_n = \frac{1}{(n-\frac{1}{2})^2\pi^2}, \qquad e_n(t) = \sqrt{2}\sin((n-\tfrac{1}{2})\pi t), \qquad n = 1, 2, 3, \ldots.$$

Define a stochastic process $X = \{X(t)\}_{t\in[0,1]}$ by

$$X(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, Z_n e_n(t),$$

where $\{Z_n\}_{n=1}^{\infty}$ are independent standard Gaussian random variables. Explicitly,

$$X(t) = \sum_{n=1}^{\infty} Z_n \frac{\sqrt{2}}{(n-\frac{1}{2})\pi} \sin\big((n-\tfrac{1}{2})\pi t\big).$$

We now compute the covariance of this process. Using independence of the Gaussian variables and the identity $\mathbb{E}[Z_m Z_n] = \delta_{mn}$, we obtain

$$\mathbb{E}[X_s X_t] = \sum_{n=1}^{\infty} \lambda_n e_n(s) e_n(t).$$

Because the eigenfunctions and eigenvalues were obtained from the covariance operator with kernel $\min(s,t)$, the series above equals $\min(s,t)$. Therefore

$$\mathbb{E}[X_s X_t] = \min(s,t),$$

which is precisely the covariance function of Brownian motion.

We summarize this in the following theorem.

Theorem 5.2. *Standard Brownian motion on* $[0,1]$ *admits the expansion*

$$B_t = \sum_{n=1}^{\infty} Z_n \frac{\sqrt{2}}{(n-\frac{1}{2})\pi} \sin\big((n-\tfrac{1}{2})\pi t\big),$$

where $\{Z_n\}_{n=1}^{\infty}$ *are independent standard Gaussian random variables.*

Now we need to see that this expansion converges. For each $t \in [0,1]$, define the partial sums

$$X_N(t) = \sum_{n=1}^{N} \sqrt{\lambda_n}\, Z_n e_n(t).$$

We will show that $\{X_N(t)\}$ converges in mean square.

Proposition 5.1. *For each fixed* $t \in [0,1]$*, the sequence* $X_N(t)$ *converges in mean square.*

Proof. Let $M > N$. Then

$$X_M(t) - X_N(t) = \sum_{n=N+1}^{M} \sqrt{\lambda_n}\, Z_n e_n(t).$$

Taking expectations and using independence of the Gaussian variables, we obtain

$$\mathbb{E}\big[(X_M(t) - X_N(t))^2\big] = \sum_{n=N+1}^{M} \lambda_n e_n(t)^2.$$

Using the explicit form of $e_n(t)$, we have $e_n(t)^2 = 2\sin^2((n-\frac{1}{2})\pi t) \le 2$. Hence

$$\mathbb{E}\big[(X_M(t) - X_N(t))^2\big] \le 2 \sum_{n=N+1}^{M} \lambda_n.$$

Since $\lambda_n = \frac{1}{(n-\frac{1}{2})^2\pi^2}$, the series $\sum_{n=1}^{\infty} \lambda_n$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Therefore the tail $\sum_{n=N+1}^{\infty} \lambda_n \to 0$ as $N \to \infty$, and it follows that

$$\mathbb{E}\big[(X_M(t) - X_N(t))^2\big] \to 0.$$

Thus $\{X_N(t)\}$ is a Cauchy sequence in mean square and hence converges in L^2. □

We denote the limit by $X(t) = \lim_{N\to\infty} X_N(t)$. By construction, $X(t)$ is a Gaussian random variable with mean zero, since each partial sum is a finite linear combination of independent Gaussian variables. We can also compute the variance of $X(t)$ directly. Using independence and the identity $\mathbb{E}[Z_n^2] = 1$, we obtain

$$\mathbb{E}[X(t)^2] = \sum_{n=1}^{\infty} \lambda_n e_n(t)^2.$$

6. Itô Calculus

A fundamental difficulty is that Brownian motion is nowhere differentiable. Its increments satisfy $B_{t+h} - B_t \sim N(0,h)$, so their typical size is of order $\sqrt{h}$ rather than h. As a result, the difference quotient

$$\frac{B_{t+h} - B_t}{h}$$

diverges as $h \to 0$, and classical notions of differentiation and integration do not apply. This motivates the development of a new theory of integration with respect to Brownian motion. The key observation is that while first-order approximations don't work, second-order approximations behave in a regular way. In particular, sums of squared increments converge to t, a phenomenon known as the quadratic variation of Brownian motion. This property underlies the construction of the Itô integral and leads to a modified chain rule, known as Itô's lemma.

Theorem 6.1 (Quadratic variation)**.** *For every $\tau > 0$ and standard Brownian motion $B(t)$,*

$$\sum_{i=0}^{n-1} \left(B\left(\frac{i+1}{n}\tau\right) - B\left(\frac{i}{n}\tau\right)\right)^2 \to \tau$$

in mean square as $n \to \infty$. Equivalently, one writes that $(dB)^2 = d\tau$.

Proof. Set $X_i = B\left(\frac{i+1}{n}\tau\right) - B\left(\frac{i}{n}\tau\right)$. The random variables X_i are independent and each satisfies $X_i \sim N\left(0, \frac{\tau}{n}\right)$. Let

$$Q_n = \sum_{i=0}^{n-1} X_i^2 = \sum_{i=0}^{n-1}\left(B\left(\frac{i+1}{n}\tau\right) - B\left(\frac{i}{n}\tau\right)\right)^2.$$

We show that $Q_n \to \tau$ in mean square. First, since $\mathbb{E}[X_i^2] = \frac{\tau}{n}$, we have

$$\mathbb{E}[Q_n] = \sum_{i=0}^{n-1}\mathbb{E}[X_i^2] = \sum_{i=0}^{n-1}\frac{\tau}{n} = \tau.$$

Next, because the X_i are independent, the random variables X_i^2 are also independent, so $\mathrm{Var}(Q_n) = \sum_{i=0}^{n-1}\mathrm{Var}(X_i^2)$. If $X \sim N(0,\sigma^2)$, then we can use the well known result $\mathbb{E}[X^4] = 3\sigma^4$, and therefore

$$\mathrm{Var}(X^2) = \mathbb{E}[X^4] - \mathbb{E}[X^2]^2 = 3\sigma^4 - \sigma^4 = 2\sigma^4.$$

Applying this with $\sigma^2 = \frac{\tau}{n}$ gives

$$\mathrm{Var}(X_i^2) = 2\left(\frac{\tau}{n}\right)^2.$$

Hence

$$\mathrm{Var}(Q_n) = \sum_{i=0}^{n-1} 2\left(\frac{\tau}{n}\right)^2 = 2n\left(\frac{\tau}{n}\right)^2 = \frac{2\tau^2}{n}.$$

Observe that

$$Q_n - \tau = \big(Q_n - \mathbb{E}[Q_n]\big) + \big(\mathbb{E}[Q_n] - \tau\big).$$

Squaring and taking expectations gives

$$\mathbb{E}\big[(Q_n-\tau)^2\big] = \mathbb{E}\big[(Q_n - \mathbb{E}[Q_n])^2\big] + 2(\mathbb{E}[Q_n]-\tau)\mathbb{E}\big[Q_n - \mathbb{E}[Q_n]\big] + (\mathbb{E}[Q_n]-\tau)^2.$$

Since $\mathbb{E}[Q_n - \mathbb{E}[Q_n]] = 0$, the middle term vanishes, and therefore

$$\mathbb{E}\big[(Q_n-\tau)^2\big] = \mathrm{Var}(Q_n) + (\mathbb{E}[Q_n]-\tau)^2.$$

But $\mathbb{E}[Q_n] = \tau$, so this simplifies to

$$\mathbb{E}\big[(Q_n-\tau)^2\big] = \mathrm{Var}(Q_n) = \frac{2\tau^2}{n} \to 0$$

as $n \to \infty$. Thus $Q_n \to \tau$ in mean square, as claimed. □

We now define integration with respect to Brownian motion. Let $f : [0,T] \to \mathbb{R}$ be a continuous function. The Itô integral is defined as the limit of sums

$$\int_0^T f(t)\,dB_t = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(t_i)\big(B_{t_{i+1}} - B_{t_i}\big),$$

where $0 = t_0 < \cdots < t_n = T$ is a partition of $[0,T]$, the function is evaluated at the left endpoints, and the limit is taken in mean square.

Theorem 6.2 (Itô isometry)**.** *Let $f : [0,T] \to \mathbb{R}$ be continuous. Then*
$$\mathbb{E}\left[\left(\int_0^T f(t)\,dB_t\right)^2\right] = \int_0^T f(t)^2\,dt.$$

Proof. Let
$$S_n = \sum_{i=0}^{n-1} f(t_i)\big(B_{t_{i+1}} - B_{t_i}\big).$$
Then
$$\mathbb{E}[S_n^2] = \mathbb{E}\left[\sum_{i,j} f(t_i)f(t_j)(B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j})\right].$$
By independence of increments, the cross terms vanish when $i \neq j$, so
$$\mathbb{E}[S_n^2] = \sum_{i=0}^{n-1} f(t_i)^2\,\mathbb{E}\big[(B_{t_{i+1}} - B_{t_i})^2\big].$$
Since $\mathbb{E}[(B_{t_{i+1}} - B_{t_i})^2] = t_{i+1} - t_i$, we obtain
$$\mathbb{E}[S_n^2] = \sum_{i=0}^{n-1} f(t_i)^2(t_{i+1} - t_i).$$
Taking the limit gives
$$\mathbb{E}\left[\left(\int_0^T f(t)\,dB_t\right)^2\right] = \int_0^T f(t)^2\,dt.$$
□

Before stating Itô's lemma, we explain why its form differs from the ordinary chain rule. If X_t satisfies an equation of the form
$$dX_t = \mu_t\,dt + \sigma_t\,dB_t,$$
then the random increment ΔX_t contains a Brownian term and is therefore typically of size $\sqrt{\Delta t}$. Consequently $(\Delta X_t)^2$ is of size Δt and contributes in the limit, while terms such as $\Delta t\,\Delta X_t$ and $(\Delta t)^2$ are of smaller order and vanish. For this reason, in the Taylor expansion underlying Itô's lemma one keeps terms up to second order in the space variable but only first order in the time variable.

Theorem 6.3 (Itô's lemma)**.** [Itô44] *Let X_t be a stochastic process and let it satisfy*
$$dX_t = \mu_t\,dt + \sigma_t\,dB_t,$$
and let $f(t,x)$ be a function with continuous derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, and $\frac{\partial^2 f}{\partial x^2}$. Then
$$df = \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t\frac{\partial f}{\partial x}\,dB_t.$$

Proof. Let $0 = t_0 < \cdots < t_n = t$ be a partition, and write

$$f(t, X_t) - f(0, X_0) = \sum_{k=0}^{n-1} \big(f(t_{k+1}, X_{t_{k+1}}) - f(t_k, X_{t_k}) \big).$$

Let $\Delta t_k = t_{k+1} - t_k$ and $\Delta X_k = X_{t_{k+1}} - X_{t_k}$. By Taylor expansion,

$$f(t_{k+1}, X_{t_{k+1}}) = f(t_k, X_{t_k}) + \frac{\partial f}{\partial t}(t_k, X_{t_k})\Delta t_k + \frac{\partial f}{\partial x}(t_k, X_{t_k})\Delta X_k + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t_k, X_{t_k})(\Delta X_k)^2 + R_k,$$

where R_k collects higher-order terms. Summing over k, we obtain

$$f(t, X_t) - f(0, X_0) = \sum \frac{\partial f}{\partial t}\Delta t_k + \sum \frac{\partial f}{\partial x}\Delta X_k + \frac{1}{2}\sum \frac{\partial^2 f}{\partial x^2}(\Delta X_k)^2 + \sum R_k.$$

Now write

$$\Delta X_k = \mu_{t_k}\Delta t_k + \sigma_{t_k}\Delta B_k.$$

Expanding, we obtain

$$(\Delta X_k)^2 = \mu_{t_k}^2(\Delta t_k)^2 + 2\mu_{t_k}\sigma_{t_k}\Delta t_k \Delta B_k + \sigma_{t_k}^2(\Delta B_k)^2.$$

The first two terms are of smaller order and vanish in the limit, while the third term contributes, since $(\Delta B_k)^2$ is of order Δt_k. Thus

$$(\Delta X_k)^2 = \sigma_{t_k}^2(\Delta B_k)^2 + \text{terms that vanish}.$$

Passing to the limit yields

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, dB_s + \int_0^t \left(\frac{\partial f}{\partial t}(s, X_s) + \mu_s \frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2}\sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right) ds.$$

Taking differentials gives the form in the lemma. □

We now apply Itô's lemma to solve a stochastic differential equation.

Definition 6.4. A stochastic differential equation is an equation of the form

$$dX_t = \mu_t\, dt + \sigma_t\, dB_t,$$

where B_t is Brownian motion and the processes μ_t and σ_t are called the drift and diffusion coefficients, respectively.

We now consider an important example of a stochastic differential equation. A stochastic process X_t is called a geometric Brownian motion if it satisfies

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants. The term $\mu X_t\, dt$ represents deterministic growth proportional to the current value of the process, while the term $\sigma X_t\, dB_t$ represents random fluctuation, also proportional to the current value. Thus geometric Brownian motion evolves multiplicatively rather than additively. This makes it a natural model for quantities whose relative change is random. Because the coefficients are proportional to X_t, it is natural to apply Itô's lemma to the function $f(x) = \log x$, which converts multiplicative behavior into additive behavior.

Theorem 6.5 (Geometric Brownian motion)**.** *Let X_t satisfy*

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants. Then

$$X_t = X_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right).$$

Proof. Take $f(t, x) = \log x$. Since f has no explicit time dependence, we have $\frac{\partial f}{\partial t} = 0$, and

$$\frac{\partial f}{\partial x}(x) = \frac{1}{x}, \qquad \frac{\partial^2 f}{\partial x^2}(x) = -\frac{1}{x^2}.$$

In the notation of Itô's lemma, the process X_t has drift coefficient $\mu_t = \mu X_t$ and diffusion coefficient $\sigma_t = \sigma X_t$. Therefore

$$d(\log X_t) = \left(\mu_t \frac{\partial f}{\partial x}(X_t) + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2}(X_t)\right) dt + \sigma_t \frac{\partial f}{\partial x}(X_t) \, dB_t.$$

Substituting the expressions above gives

$$d(\log X_t) = \left(\mu X_t \cdot \frac{1}{X_t} + \frac{1}{2}(\sigma X_t)^2 \left(-\frac{1}{X_t^2}\right)\right) dt + \sigma X_t \cdot \frac{1}{X_t} \, dB_t.$$

Simplifying, we obtain

$$d(\log X_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma \, dB_t.$$

Integrating from 0 to t yields

$$\log X_t - \log X_0 = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t.$$

Exponentiating both sides, we conclude that

$$X_t = X_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right).$$

□

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