The Poisson Summation Formula

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Abstract

We define Fourier series and prove the Poisson summation formula. We apply this formula to sum several infinite series.

1 Introduction

You have probably seen Fourier series before.

Definition 1.1. A Fourier series is a sum of form

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}. \tag{1.1}$$

It is known that any piecewise continuous bounded 1-periodic function has a Fourier series, and that the coefficients can be found with the following formula.

Theorem 1.2. *Let f be a bounded piecewise continuous 1 periodic function. Then*

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}, \tag{1.2}$$

where $a_n = \frac{1}{2\pi} \int_0^1 f(x) e^{-2\pi i n x}\, dx$.

We will prove the Poisson summation formula, and use it to evaluate several infinite series. We first must define the Fourier transform.

Definition 1.3. $f : \mathbb{R} \to \mathbb{C}$ is a Schwartz function is $\lim_{|x|\to\infty} f(x)x^N = 0$ for all positive integers N.

Example. The function e^{-x^2} is Schwartz.

Example. The function x^{-n} is not Schwartz, since $\lim_{x\to\infty} x^{-n}x^{n+1} = \infty$.

Definition 1.4. The Fourier transform $\hat{f}$ of a function f is defined as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi t} f(t)\, dt. \tag{1.3}$$

2 The Poisson Summation Formula

The Poisson summation formula is the following.

Theorem 2.1 ([1]). *Let f be a Schwartz function. Then*

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) = \sum_{n=-\infty}^{\infty} f(n) \tag{2.1}$$

Proof. Define

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n). \tag{2.2}$$

Notice that $F(x)$ is 1-periodic. So F has a Fourier series. We calculate the coefficients.

$$\int_0^1 e^{-2\pi ikx} \sum_{n\in\mathbb{Z}} f(x+n)\,dx = \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi ikx}\,dx = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)e^{-2\pi ikx}\,dx \tag{2.3}$$

We can swap the sum and integral to get

$$\sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x)e^{-2\pi ikx}\,dx = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx = \hat{f}(k).$$

So the Fourier series of F is

$$F(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi ikx}. \tag{2.4}$$

Setting $x = 0$, we get that

$$F(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k) \tag{2.5}$$

as desired. ∎

We can apply this to some infinite sums.

3 Infinite Sums

The Poisson summation formula helps us evaluate many infinite sums.

For example, consider this weird looking sum.

Theorem 3.1.

$$\theta(a) = \sum_{k\in\mathbb{Z}} e^{-\pi k^2 a} \tag{3.1}$$

be the Jacobi theta function*. Then* $\theta(a) = \frac{1}{\sqrt{a}}\theta(1/a)$.

This sum shows that $\theta(a) = \frac{1}{\sqrt{a}}\theta(1/a)$. This actually is a connection to modular forms. Practically: for small a, we can apply this identity to turn a slowly converging series into a quickly converging one.

We need a well known lemma on $e^{-\pi x^2}$.

Lemma 3.2. *Let $f(x) = e^{-\pi x^2}$. Then $f(x) = \hat{f}(x)$, or $\hat{f}(\xi) = e^{-\pi\xi^2}$.*

Proof. By definition,

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi t} e^{-\pi t^2}\, dt = \int_{\mathbb{R}} e^{-2\pi i\xi t - \pi t^2}\, dt.$$

Notice that $2i\xi t + \pi t^2 = (t + i\xi)^2 - \xi^2$. So

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{(t+i\xi)^2} e^{-\pi t^2}\, dt = e^{-\pi\xi^2} \int_{\mathbb{R}} e^{-\pi(t+i\xi)^2}\, dt,$$

since we can factor out the $e^{-\pi\xi^2}$ term. So it remains to show that

$$\int_{\mathbb{R}} e^{-\pi(t+i\xi)^2}\, dt = 1.$$

The integral $\int_{\mathbb{R}} e^{-\pi(t+i\xi)^2}\, dt$ closely resembles the well known gaussian integral $e^{-\pi t^2}$, which can be evaluated as 1 through a change in variables. We will show that

$$\int_{\mathbb{R}} e^{-\pi(t+i\xi)^2}\, dt = \int_{\mathbb{R}} e^{-\pi u^2}\, du$$

by differentiating under the integral sign. Let

$$F(\xi) = \int_{\mathbb{R}} e^{-\pi(t+i\xi)^2}\, dt.$$

We will prove that F is a constant by computing that $F(\xi)' = 0$. We calculate that

$$\frac{d}{d\xi}\int_{\mathbb{R}} e^{-\pi(t+i\xi)^2}\, dt = \int_{\mathbb{R}} \frac{d}{d\xi} e^{-\pi(t+i\xi)^2}\, dt = -\int_{\mathbb{R}} 2\pi i(t+i\xi)e^{-\pi(t+i\xi)^2}\, dt.$$

But $\frac{d}{dt}e^{-\pi(t+i\xi)^2} = -2\pi(t+i\xi)e^{-\pi(t+i\xi^2)}$, so we get that

$$-\int_{\mathbb{R}} 2\pi i(t+i\xi)e^{-\pi(t+i\xi)^2}\, dt = i\int_{\mathbb{R}} \frac{d}{dt} e^{-(t+i\xi)^2}\, dt$$

and since $e^{-(t+i\xi)^2} \to 0$ as $t \to \pm\infty$, so $F'(\xi) = 0$, and thus F is a constant. Finally, we have shown that

$$\int_{\mathbb{R}} e^{-\pi(t+i\xi)^2}\, dt = \int_{\mathbb{R}} e^{-\pi u^2}\, du = 1.$$

Thus, $\hat{f}(\xi) = e^{-\pi\xi^2}$ as claimed. □

The proof of the theorem is now relatively simple.

Proof of Theorem 3.1. Let

$$f(x) = e^{-\pi a x^2}. \tag{3.2}$$

If we can show that $\widehat{f}(x) = \frac{1}{\sqrt{a}} e^{-\pi x^2/a}$, then the Poisson summation formula proves the theorem. By Lemma 3.2, we have that

$$\hat{f}(\xi) = \int_{\mathbb{R}}$$

So the Poisson summation identity gives us that

$$\sum_{n\in\mathbb{Z}} e^{-\pi a x^2} = \frac{1}{\sqrt{a}} \sum_{k\in\mathbb{Z}} e^{-\pi k^2/a} \tag{3.3}$$

. □

Our next sum, which resembles the sum of $\zeta(2)$, might be more interesting.

Theorem 3.3.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} coth(\pi a). \tag{3.4}$$

Proof. We want to apply the Poisson summation formula with $f(n) = \frac{1}{n^2+a^2}$. Although f isn't Schwartz, we will take it for granted that the Poisson summation formula holds in this case as well. Then we want to calculate

$$\hat{f}(\xi) = \int_{\mathbb{R}} \frac{e^{-2\pi i \xi t}}{t^2+a^2}\, dt = \int_{\mathbb{R}} \frac{\cos(2\pi\xi t)}{t^2+a^2}\, dt$$

This integral is even, so we just need to find

$$2\int_0^\infty \frac{\cos(2\pi\xi t)}{t^2+a^2}\, dt = \frac{2}{|a|}\int_0^\infty \frac{\cos(2\pi\xi t a)}{t^2+1}\, dt$$

which we will do by differentiating under the integral sign. Let

$$I(b) = \int_0^\infty \frac{\cos(bt)}{t^2+1}\, dt.$$

Since the integral converges uniformly, we get that

$$\begin{aligned} \frac{d}{db} I(b) = \int_0^\infty \frac{d}{db}\frac{\cos bt}{t^2+1}\, dt = -\int_0^\infty \frac{b\sin bt}{t^2+1}\, dt \\ = \int_0^\infty -\frac{\sin bt}{t} + \frac{\sin bt}{t(t^2+1)}\, dt. \end{aligned} \tag{3.5}$$

Now we must evaluate

$$\int_0^\infty \frac{\sin bt}{t}\, dt$$

for $b \neq 0$. We differentiate under the integral sign again. Let

$$J_\varepsilon(b) = \int_0^\infty e^{-\varepsilon t} \frac{\sin bt}{t}\, dt.$$

For $\varepsilon > 0$. We will eventually take $\varepsilon \to 0$. Then

$$\begin{aligned}\frac{d}{db} J_\varepsilon(b) &= \int_0^\infty e^{-\varepsilon t} \cos(bt)\, dt = \operatorname{Re}\left(\int_0^\infty e^{-\varepsilon t} e^{bit}\, dt\right) \\ &= \operatorname{Re}\left(\frac{e^{bit-\varepsilon t}}{bi - \varepsilon}|_{t=0}^{t=\infty}\right) = \operatorname{Re}\left(\frac{1}{bi-\varepsilon}\right) = \operatorname{Re}\left(\frac{bi+\varepsilon}{b^2+\varepsilon^2}\right) = \frac{\varepsilon}{b^2+\varepsilon^2}\end{aligned}$$

Thus

$$J_\varepsilon(a) = \int \frac{\varepsilon}{b^2+\varepsilon^2} = \tan^{-1}\left(\frac{b}{\varepsilon}\right) + C$$

At $J_\varepsilon(0) = 0$, so $C = 0$ and $J_\varepsilon(b) = \tan^{-1}\left(\frac{b}{\varepsilon}\right)$. Letting $\varepsilon \to 0$ from above, we see that

$$\int_0^\infty \frac{\sin bt}{t}\, dt = \frac{\pi}{2}\operatorname{sgn}(b)$$

where $\operatorname{sgn}(b)$ is the sign of b. It follows that

$$\frac{d}{db} I(b) = -\frac{\pi}{2}\operatorname{sgn}(b) + \int_0^\infty \frac{\sin bt}{t(t^2+1)}\, dt$$

Notice that $\lim_{b\to 0^+} \frac{d}{db} I(b) = -\frac{\pi}{2}$ and $\lim_{b\to 0^-} \frac{d}{db} I(b) = +\frac{\pi}{2}$. Differentiating again, we see that

$$\frac{d^2}{db^2} I(b) = \int_0^\infty \frac{\cos bt}{x^2+1} = I(b).$$

For all $b \neq 0$. Thus: I is the solution to a linear differential equation for $b \neq 0$, so $I(b) = C_0 e^b + C_1 e^{-b}$ for some constants C_0, C_1 on the intervals $(-\infty, 0)$ and $(0, \infty)$. We know that $I(0) = \frac{\pi}{2}$ and $\lim_{b\to 0^+} \frac{d}{db} I(b) = -\frac{\pi}{2}$ and $\lim_{b\to 0^-} \frac{d}{db} I(b) = \frac{\pi}{2}$. So for $b > 0$, $I(b) = \frac{\pi e^{-b}}{2}$ and for $b < 0$, $I(b) = \frac{\pi e^b}{2}$. In general, $I(b) = \frac{\pi e^{-|b|}}{2}$. Letting $b = 2\pi\xi a$ and multiplying by $2/|a|$, we see that

$$\hat{f}(\xi) = \frac{\pi e^{-|2\pi\xi a|}}{|a|}.$$

Thus the Poisson summation fprmula shows that

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2+a^2} = \sum_{n\in\mathbb{Z}} \frac{\pi e^{-|2\pi n a|}}{|a|}.$$

We simplify

$$\sum_{n\in\mathbb{Z}} e^{-|2\pi n a|} = 1 + 2\sum_{n=1}^\infty e^{-2\pi n a} = 1 + 2\frac{e^{-2\pi a}}{1 - e^{-2\pi a}} = \frac{e^{\pi a} + e^{-\pi a}}{e^{\pi a} - e^{-\pi a}} = \coth(\pi a)$$

by the definition of coth, giving us our Theorem. □

In general, even ζ values can be evaluated through similar methods. An alternative proof of the previous identity can be found through the Weierstrauss product for sin.

References

[1] Elias M. Stein and Rami Shakarchi. *Fourier analysis. An Introduction.* English. Vol. 1. Princeton Lect. Anal. Princeton, NJ: Princeton University Press, 2003. ISBN: 0-691-11384-X.