Polylogarithms

Sri Srinivasan

March 2026

1 Introduction

Polylogarithms are a family of special functions defined by the power series $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n$, which generalize the classical logarithm ($n = 1$) to higher orders $n \geq 2$. They are connected to the Riemann zeta function via the special values $\mathrm{Li}_n(1) = \zeta(n)$, and to iterated integrals via the representation $\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{Li}_{n-1}(t)}{t}\, dt$. The shuffle algebra of iterated integrals and the notion of transcendental weight will be discussed in later sections.

Historically, special cases such as the dilogarithm $\mathrm{Li}_2(z)$ were studied by Euler and later by Spence, Abel, and Kummer, who discovered functional equations such as

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z)$$

and the five-term identity. The dilogarithm also computes volumes of ideal tetrahedra in hyperbolic three-space and appears in the Bloch regulator in algebraic K-theory [4]. We will also discuss Goncharov polylogarithms [3], the recent generalization that allows poles of the integrand at arbitrary complex points rather than just 0 and 1.

In this paper, we begin with the power series definition and its convergence, following the treatment in [1]. We derive the differential recursion $z\, d\,\mathrm{Li}_n / dz = \mathrm{Li}_{n-1}$, which leads to the iterated integral representation. We then define Goncharov polylogarithms and state the theorem, due to Goncharov [3], that every iterated integral of rational one-forms on a rational variety can be expressed in terms of these functions. We then introduce the shuffle product, which describes how products of iterated integrals decompose into sums of iterated integrals of the same total length, and derive related formulas including consequences for multiple zeta values. Finally, we close with a discussion of transcendental weight and the symbol map, a tensor-valued invariant that reduces identities between polylogarithms to linear algebra over $\mathbb{Q}$.

2 Power Series Definition

Definition 1. *For an integer $n \geq 1$ and $|z| < 1$, the* polylogarithm *of order n is*

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

Proposition 1. *The series converges absolutely for $|z| < 1$.*

Proof. Since $k^n \geq 1$ for all $k \geq 1$ and $n \geq 1$, we have $|z^k/k^n| \leq |z|^k$. The geometric series $\sum_{k=1}^{\infty} |z|^k = |z|/(1-|z|)$ converges for $|z| < 1$, so the comparison test gives absolute convergence. □

Although the series converges only for $|z| < 1$, the function Li_n extends to a multivalued holomorphic function on $\mathbb{C} \setminus [1, \infty)$ by analytic continuation. Since $\mathrm{Li}_1(z) = -\log(1-z)$ is holomorphic on $\mathbb{C} \setminus [1, \infty)$ with a branch cut along $[1, \infty)$, and each Li_n satisfies $\frac{d}{dz}\mathrm{Li}_n(z) = \mathrm{Li}_{n-1}(z)/z$, one constructs Li_n for $n \geq 2$ inductively by integrating $\mathrm{Li}_{n-1}(t)/t$ along any path in $\mathbb{C} \setminus [1, \infty)$ from 0 to

z. Homotopy invariance of the resulting integral (on the simply connected domain $\mathbb{C} \setminus [1, \infty)$, which requires checking that $d(\mathrm{Li}_{n-1}(t)/t\, dt) = 0$, which holds since the integrand is holomorphic there) shows the value depends only on z, not on the path. The branch cut thus propagates to all orders $n \geq 1$.

2.1 The Case $n = 1$

Theorem 1. *For* $|z| < 1,\ \mathrm{Li}_1(z) = -\log(1 - z)$.

Proof. The geometric series $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ converges uniformly on compact subsets of $|z| < 1$, so integrating term by term from 0 to z is valid and gives $-\log(1 - z) = \sum_{k=1}^{\infty} z^k/k = \mathrm{Li}_1(z)$. ☐

As a direct consequence, $\mathrm{Li}_1(1/2) = \log 2$.

2.2 Low-Order Examples and Special Values

The first three polylogarithms are:

$$\mathrm{Li}_1(z) = -\log(1 - z), \qquad \mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \qquad \mathrm{Li}_3(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^3}.$$

The function $\mathrm{Li}_2(z)$ is called the **dilogarithm**. Evaluating the series at $z = 1$ gives $\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} k^{-n} = \zeta(n)$ for $n \geq 2$. In particular, $\mathrm{Li}_2(1) = \pi^2/6$.

We now prove the functional equation

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1 - z) = \frac{\pi^2}{6} - \log(z) \log(1 - z), \tag{1}$$

which also gives $\mathrm{Li}_2(1) = \pi^2/6$ as a special case.

Proof of (1). Let $F(z) = \mathrm{Li}_2(z) + \mathrm{Li}_2(1 - z) + \log(z) \log(1 - z)$. We show F is constant on $(0, 1)$. Differentiating, and using $\frac{d}{dz} \mathrm{Li}_2(z) = -\log(1 - z)/z$,

$$F'(z) = -\frac{\log(1 - z)}{z} + \frac{\log z}{1 - z} + \frac{\log(1 - z)}{z} - \frac{\log z}{1 - z} = 0.$$

Hence F is constant. To find the constant, we take the limit $z \to 0^+$. We have $\mathrm{Li}_2(z) \to 0$, $\log(1 - z) \to 0$, and we need $\lim_{z \to 0^+} \mathrm{Li}_2(1 - z)$. Substituting $w = 1 - z$ and using the power series, $\mathrm{Li}_2(1 - z) \to \mathrm{Li}_2(1) = \zeta(2)$. Also $\log(z) \log(1 - z) \to 0$ as $z \to 0^+$ (since $z \log z \to 0$). Therefore $F(z) \equiv \zeta(2) = \pi^2/6$. ☐

Setting $z = 1$ in (1) (more precisely by continuity) recovers $\mathrm{Li}_2(1) = \pi^2/6$, consistently.

The connection to the Riemann zeta function extends to multiple polylogarithms, defined for indices $n_1, \ldots, n_k \geq 1$ and $|x_i| < 1$ by

$$\mathrm{Li}_{n_1, \ldots, n_k}(x_1, \ldots, x_k) = \sum_{1 \leq p_1 < \cdots < p_k} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}.$$

Setting all $x_i = 1$ gives the *multiple zeta value* $\zeta(n_1, \ldots, n_k)$, provided $n_k \geq 2$ for convergence. The simplest non-trivial example is Euler's identity $\zeta(1, 2) = \zeta(3)$, which states that $\sum_{1 \leq p < q} \frac{1}{p\, q^2} = \sum_{q=1}^{\infty} \frac{1}{q^3}$. Identities of this type (expressing a multiple zeta value as a linear combination of products of single zeta values) are the subject of the double shuffle relations, which arise from applying the shuffle product (Section 6) and a second "stuffle" product to the power series representation simultaneously.

3 Differential Recursion

Theorem 2. *For* $n \geq 2$, $z\dfrac{d}{dz}\operatorname{Li}_n(z) = \operatorname{Li}_{n-1}(z)$.

Proof. Differentiating $\operatorname{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n$ term by term gives $\frac{d}{dz}\operatorname{Li}_n(z) = \sum_{k=1}^{\infty} z^{k-1}/k^{n-1}$, and multiplying by z yields $\sum_{k=1}^{\infty} z^k/k^{n-1} = \operatorname{Li}_{n-1}(z)$. □

Rewriting as $\frac{d}{dz}\operatorname{Li}_n(z) = \operatorname{Li}_{n-1}(z)/z$ and integrating from 0 to z gives

$$\operatorname{Li}_n(z) = \int_0^z \frac{\operatorname{Li}_{n-1}(t)}{t}\,dt,$$

since $\operatorname{Li}_n(0) = 0$ by the power series.

Example 1. *For* $n = 2$*:* $\frac{d}{dz}\operatorname{Li}_2(z) = -\log(1-z)/z$*, so* $\operatorname{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt$*. The integrand* $-\log(1-t)/t$ *has no antiderivative in terms of elementary functions, which is why* Li_2 *is not reducible to logarithms and rational functions.*

Example 2. *For* $n = 3$*:* $\frac{d}{dz}\operatorname{Li}_3(z) = \operatorname{Li}_2(z)/z$*, so* $\operatorname{Li}_3(z) = \int_0^z \frac{\operatorname{Li}_2(t)}{t}\,dt$*. In the same way,* Li_3 *is not expressible as a* $\mathbb{Q}$*-linear combination of products of* Li_2 *and logarithms: the weight-three functions* Li_3 *and* $\log(z)\operatorname{Li}_2(z)$ *are linearly independent over* $\mathbb{Q}$ *(a consequence of the weight grading in Section 7), so* Li_3 *extends the family.*

4 Iterated Integral Representation

Applying $\operatorname{Li}_n(z) = \int_0^z \operatorname{Li}_{n-1}(t)/t\,dt$ repeatedly, starting from $\operatorname{Li}_1(t) = -\log(1-t)$, expresses each Li_n as a nested integral. To write this uniformly, we have the two differential one-forms

$$\omega_0 = \frac{dt}{t}, \qquad \omega_1 = \frac{dt}{1-t},$$

and define the iterated integral of a sequence of one-forms $\alpha_1, \ldots, \alpha_n$ along a path from 0 to z by

$$\int_0^z \alpha_1 \circ \cdots \circ \alpha_n := \int_0^z \alpha_n(t_n) \int_0^{t_n} \alpha_{n-1}(t_{n-1}) \cdots \int_0^{t_2} \alpha_1(t_1).$$

With this notation,

$$\operatorname{Li}_n(z) = \int_0^z \omega_1 \circ \underbrace{\omega_0 \circ \cdots \circ \omega_0}_{n-1}.$$

This follows by induction. We have that $\operatorname{Li}_1(z) = \int_0^z \omega_1$ since $\omega_1 = dt/(1-t)$ integrates to $-\log(1-z)$, and if the formula holds for $n-1$ then

$$\int_0^z \frac{\operatorname{Li}_{n-1}(t)}{t}\,dt = \int_0^z \frac{1}{t}\left(\int_0^t \omega_1 \circ \omega_0^{n-2}\right) dt = \int_0^z \omega_1 \circ \omega_0^{n-1},$$

where ω_0^{n-1} abbreviates $\omega_0 \circ \cdots \circ \omega_0$ ($n-1$ times). The first two cases written out explicitly are as follows

$$\operatorname{Li}_2(z) = \int_0^z \omega_1 \circ \omega_0 = -\int_0^z \frac{\log(1-t)}{t}\,dt,$$

$$\operatorname{Li}_3(z) = \int_0^z \omega_1 \circ \omega_0 \circ \omega_0 = \int_0^z \frac{\operatorname{Li}_2(t)}{t}\,dt.$$

The form ω_1 has a simple pole at $t = 1$ and ω_0 has a simple pole at $t = 0$. The order in which these poles appear in the iterated integral, with ω_1 first (outermost), then ω_0 repeated, encodes which function we are computing. Swapping the order or using different poles gives different functions, which is the starting point for the Goncharov generalization.

5 Goncharov Polylogarithms

Definition 2. *For $a_1, \dots, a_n, x \in \mathbb{C}$, the Goncharov polylogarithm is defined by*

$$G(a_1, \dots, a_n;\, x) = \int_0^x \frac{dt}{t - a_1}\, G(a_2, \dots, a_n;\, t), \qquad G(\,;\, x) = 1.$$

The integrand $dt/(t - a_1)$ has a simple pole at $t = a_1$, so $G(a_1, \dots, a_n; x)$ is an iterated integral of n rational one-forms $dt/(t - a_i)$, each with a pole at one of the parameters a_i. The weight is n. Setting all parameters to 0 or 1 gives the classical case. Since $dt/(t-0) = \omega_0$ and $dt/(t-1) = -\omega_1$, one computes

$$\mathrm{Li}_n(z) = (-1)^{n-1}\, G(\underbrace{0, \dots, 0}_{n-1}, 1;\, z).$$

For weight two this gives $G(0, 1; z) = -\mathrm{Li}_2(z)$, which can be shown as follows. $G(0, 1; z) = \int_0^z \frac{dt}{t-0}\, G(1; t) = \int_0^z \frac{1}{t} \int_0^t \frac{ds}{s-1}\, ds\, dt = -\int_0^z \frac{\log(1-t)}{t}\, dt = -\mathrm{Li}_2(z)$.

An equivalent notation used is

$$I(a_0;\, a_1, \dots, a_n;\, a_{n+1}) = \int_{a_0}^{a_{n+1}} \frac{dt}{t - a_n}\, I(a_0;\, a_1, \dots, a_{n-1};\, t),$$

related to G by $G(a_1, \dots, a_n;\, x) = I(0;\, a_n, \dots, a_1;\, x)$ (the interior parameters are listed in reverse order). The significance of the Goncharov polylogarithms is captured by the following theorem, which shows they are universal among iterated integrals of rational one-forms.

Theorem 3 (Goncharov [3])**.** *Every iterated integral of rational one-forms $dt/(t - a_i)$ on a rational variety (that is, a variety defined by polynomial equations with rational coefficients) can be expressed in terms of hyperlogarithms $G(a_1, \dots, a_n; x)$ with a_i rational.*

Concretely, Theorem 3 says the following. Any function built by integrating sequences of forms of the shape $dt/(t - r(z))$, where $r(z)$ is a rational function with rational coefficients, can be written as a finite $\mathbb{Q}$-linear combination of Goncharov polylogarithms with rational parameters.

6 Shuffle Algebra of Iterated Integrals

The product of two iterated integrals along the same path can always be written as a sum of iterated integrals of the same total length. This is the shuffle product.

Definition 3. *An (r, s)*-shuffle *is a permutation σ of $\{1, \dots, r+s\}$ such that $\sigma^{-1}(1) < \cdots < \sigma^{-1}(r)$ and $\sigma^{-1}(r+1) < \cdots \sigma^{-1}(r+s)$.*

Theorem 4 (Shuffle product)**.** *Let $I(\alpha_1, \dots, \alpha_r)$ and $I(\beta_1, \dots, \beta_s)$ be iterated integrals along the same path from 0 to z. Then*

$$I(\alpha_1, \dots, \alpha_r) \cdot I(\beta_1, \dots, \beta_s) = \sum_{\sigma \,\in\, \mathrm{Sh}(r,s)} I(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(r+s)}),$$

where $(\gamma_1, \dots, \gamma_{r+s}) = (\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s)$.

Proof. We prove the theorem by induction on $r + s$. The base case $r = s = 1$ follows from decomposing the square $[0, z]^2$ into the two open triangles $\{0 \le t_1 < t_2 \le z\}$ and $\{0 \le t_2 < t_1 \le z\}$ (the diagonal has measure zero):

$$\begin{aligned}\left(\int_0^z f_1\, dt\right)\left(\int_0^z f_2\, dt\right) &= \int_{0 \le t_1 \le t_2 \le z} f_1(t_1) f_2(t_2)\, dt_1\, dt_2 + \int_{0 \le t_2 \le t_1 \le z} f_1(t_1) f_2(t_2)\, dt_1\, dt_2 \\ &= I(\alpha_1, \alpha_2) + I(\alpha_2, \alpha_1),\end{aligned}$$

which is the sum over the two $(1,1)$-shuffles, as required.

For the inductive step, suppose the theorem holds for all pairs (r', s') with $r'+s' < r+s$. We use the recursive structure of iterated integrals. Write $I(\alpha_1, \ldots, \alpha_r) = \int_0^z \alpha_r(t)\, I(\alpha_1, \ldots, \alpha_{r-1})|_{[0,t]}\, dt$ and similarly for the β sequence. By the product rule for the outermost integral and the inductive hypothesis applied to the inner product, and we have the recursive formula

$$(\alpha_1 \circ \cdots \circ \alpha_r)(\beta_1 \circ \cdots \circ \beta_s) = \alpha_1 \circ \big((\alpha_2 \circ \cdots \circ \alpha_r)(\beta_1 \circ \cdots \circ \beta_s)\big) + \beta_1 \circ \big((\alpha_1 \circ \cdots \circ \alpha_r)(\beta_2 \circ \cdots \circ \beta_s)\big).$$

This recursion, together with the base case, uniquely characterizes the sum over all (r,s)-shuffles, completing the induction. ☐

Two consequences demonstrate the power of the shuffle product. First, since $\mathrm{Li}_1(z) = I(\omega_1)$ and the two $(1,1)$-shuffles of (ω_1) and (ω_1) both give $I(\omega_1, \omega_1)$, as

$$\mathrm{Li}_1(z)^2 = I(\omega_1)^2 = 2\, I(\omega_1, \omega_1) = 2\ \mathrm{Li}_{1,1}(z),$$

where $\mathrm{Li}_{1,1}(z) = \sum_{1 \le p < q} z^p z^q/(pq)$ is the depth-two multiple polylogarithm. We can see this identity directly from the power series, since $\left(\sum_{p\ge1} z^p/p\right)^2 = 2\sum_{p<q} z^{p+q}/(pq)$. Second, since $\log z = I(\omega_0)$ and the n one-forms are all identical, all $n!$ shuffles of n copies of ω_0 produce the same word, so

$$\log^n(z) = I(\omega_0)^n = n!\, I(\underbrace{\omega_0, \ldots, \omega_0}_{n}),$$

recovering the standard Taylor coefficient $\log^n(z)/n!$ as an iterated integral of depth n.

6.1 Consequences for Multiple Zeta Values

These shuffle identities, together with the *stuffle* (or quasi-shuffle) product that arises from multiplying the power series representations of multiple polylogarithms, impose two independent sets of $\mathbb{Q}$-linear relations among multiple zeta values. For instance, the shuffle relation $\mathrm{Li}_1(z)^2 = 2\,\mathrm{Li}_{1,1}(z)$, combined with the stuffle relation

$$\mathrm{Li}_1(z)^2 = 2\,\mathrm{Li}_{1,1}(z) + \mathrm{Li}_2(z),$$

which follows from the power series identity $\left(\sum_p z^p/p\right)^2 = 2\sum_{p<q} z^{p+q}/(pq) + \sum_p z^{2p}/p^2$, gives $\mathrm{Li}_2(z) = 0$ upon subtracting. This is a vacuous identity. However, at depth two with $n_1 + n_2 \ge 3$ the shuffle and stuffle relations together give non-trivial identities. A concrete example is Euler's identity $\zeta(1,2) = \zeta(3)$. To derive it from the double shuffle relations, one uses the stuffle product formula

$$\zeta(1)\zeta(2) = \zeta(1,2) + \zeta(2,1) + \zeta(3)$$

and the shuffle product formula

$$\zeta(1)\zeta(2) = 2\zeta(1,2) + \zeta(2,1)$$

(both require regularization since $\zeta(1)$ diverges). Subtracting gives $0 = \zeta(1,2) - \zeta(3)$, i.e. $\zeta(1,2) = \zeta(3)$.

7 Weight and Transcendentality

Definition 4. *The* weight *of* $\mathrm{Li}_n(z)$ *is* n*. The weight of an iterated integral* $I(\alpha_1, \ldots, \alpha_n)$ *is* n*.*

The weight of a product of two functions is the sum of their weights, consistent with the shuffle product. Shuffling a word of length r with one of length s produces words of length $r+s$. For example,

$$\begin{aligned} \log(z) &\quad \text{weight } 1, \\ \mathrm{Li}_2(z) &\quad \text{weight } 2, \\ \log(z)\,\mathrm{Li}_2(z) &\quad \text{weight } 3, \\ \mathrm{Li}_3(z) &\quad \text{weight } 3. \end{aligned}$$

A theorem states that there are no non-trivial $\mathbb{Q}$-linear relations among transcendental functions of *different* weights [1]. Stated formally, if $\sum_i c_i F_i = 0$ where the F_i are iterated integrals of $d\log$ forms with rational arguments and the $c_i \in \mathbb{Q}$, then the terms of each weight must individually sum to zero. This implies, for instance, that $\mathrm{Li}_3(z)$ cannot be written as any $\mathbb{Q}$-linear combination of $\log(z)\,\mathrm{Li}_2(z)$, $\log^2(z)\,\mathrm{Li}_1(z)$, and $\log^3(z)$. All three of those expressions have weight three, but each arises as a product of lower-weight functions and therefore has symbol equal to a shuffle of shorter tensors.

The *symbol map* makes weight grading formal as a tensor invariant. If F is a weight-n function expressible as an iterated integral $F = \int_0^z d\log f_1 \circ \cdots \circ d\log f_n$, its symbol is

$$\mathcal{S}(F) = f_1 \otimes f_2 \otimes \cdots \otimes f_n \ \in \ (\mathbb{Q}^\times)^{\otimes n}.$$

For the classical polylogarithm, reading off the iterated integral $\mathrm{Li}_n(z) = \int_0^z \omega_1 \circ \omega_0^{n-1}$ directly gives

$$\mathcal{S}(\mathrm{Li}_n(z)) = -(1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{n-1}.$$

The minus sign arises because $\omega_1 = dt/(1-t) = -d\log(1-t)$, so the first entry is $-(1-z)$. The symbol satisfies $\mathcal{S}(FG) = \mathcal{S}(F)\mathcal{S}(G)$, consistent with Theorem 4. Constants, multiples of π, and zeta values all have symbol zero (since $d\log c = 0$ for any constant c), so the symbol discards these but retains all functional dependence on z.

We can now make the earlier claim about Li_3 precise. The symbols of the weight-three products of lower-weight functions are

$$\begin{aligned} \mathcal{S}(\log(z)\,\mathrm{Li}_2(z)) &= \mathcal{S}(\log z)\mathcal{S}(\mathrm{Li}_2(z)) = z(-(1-z)\otimes z) \\ &= -z \otimes (1-z) \otimes z - (1-z) \otimes z \otimes z, \\ \mathcal{S}(\log^2(z)\,\mathrm{Li}_1(z)) &= \mathcal{S}(\log^2 z)\mathcal{S}(\mathrm{Li}_1(z)) = 2(z\otimes z)(-(1-z)) \\ &= -2\big(z\otimes z\otimes(1-z) + z\otimes(1-z)\otimes z + (1-z)\otimes z\otimes z\big), \\ \mathcal{S}(\log^3(z)) &= 6\, z\otimes z\otimes z. \end{aligned}$$

By contrast, $\mathcal{S}(\mathrm{Li}_3(z)) = -(1-z)\otimes z\otimes z$. We can see that $-(1-z)\otimes z\otimes z$ is not a $\mathbb{Q}$-linear combination of the three tensors above. For instance, the coefficient of $z\otimes z\otimes z$ in $\mathcal{S}(\mathrm{Li}_3(z))$ is 0, while any linear combination involving $\mathcal{S}(\log^3 z)$ would contribute a nonzero $z\otimes z\otimes z$ term unless the coefficient of $\log^3 z$ is zero. Setting that coefficient to zero and examining the remaining two symbols shows $-(1-z)\otimes z\otimes z$ cannot be achieved. Therefore $\mathrm{Li}_3(z)$ is not a $\mathbb{Q}$-linear combination of weight-three products of Li_2 and logarithms.

To check an identity $F = G$ between two weight-n functions, we first check that $\mathcal{S}(F) = \mathcal{S}(G)$, which is just linear algebra on tensors with rational entries. If the symbols agree, the difference $F - G$ has symbol zero, meaning it is a $\mathbb{Q}$-linear combination of products of π^{2k} and lower-weight functions. We can then determine these remaining terms by evaluating $F - G$ at a specific point, such as $z = 0$ or $z = 1$.

References

[1] D. C. Wood, *The Computation of Polylogarithms*, University of Kent Technical Report, 1992.

[2] C. Vergu, *Polylogarithms and Physical Applications*, Lecture notes, Summer School “Polylogarithms as a Bridge between Number Theory and Particle Physics,” Durham, UK, 2013.

[3] A. B. Goncharov, *Polylogarithms in arithmetic and geometry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2, 1994, 374–387.

[4] S. J. Bloch, *Higher Regulators, Algebraic K-Theory, and Zeta Functions of Elliptic Curves*, CRM Monograph Series, Vol. 11, American Mathematical Society, 1977.