HYPERGEOMETRIC FUNCTIONS AND ORTHOGONAL POLYNOMIALS

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1. Preliminaries: The Beta Function

Definition 1.1 (Beta function)**.** For complex numbers z_1 and z_2 such that $\Re(z_1) > 0$ and $\Re(z_1) > 0$, define the Beta function, denoted by B, by

$$B(z_1, z_2) = \int_0^1 t^{z_1 - 1}(1-t)^{z_2 - 1}\, dt$$

The Beta function is closely related to the Gamma function; proposition 1.2 below gives the precise relationship.

Proposition 1.2. *For all admissible z_1 and z_2, we have*

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}.$$

Proof. Recall the definition of the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dz.$$

Combining integrals, we have

$$\begin{aligned}\Gamma(z_1)\Gamma(z_2) &= \int_0^\infty e^{-t} t^{z_1 - 1}\, dt \int_0^\infty e^{-s} s^{z_2 - 1}\, ds \\ &= \int_0^\infty \int_0^\infty e^{-t} e^{-s} t^{z_1 - 1} s^{z_2 - 1}\, dt\, ds.\end{aligned}$$

Now, we make the substitutions $t(u, v) = uv$ and $s(u, v) = u(1-v)$. Note that $u = s + t$ and $v = \frac{t}{s+t}$, so u ranges from 0 to ∞ and v ranges from 0 to 1. Moreover, the Jacobian is

$$\left|\frac{\partial(s,t)}{\partial(u,v)}\right| = \det\begin{bmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v}\end{bmatrix} = \det\begin{bmatrix} 1-v & -u \\ v & u \end{bmatrix} = (1-v)u + uv = u.$$

Putting all this together, we have

$$\begin{aligned}\Gamma(z_1)\Gamma(z_2) &= \int_0^\infty \int_0^1 e^{-u}(uv)^{z_1 - 1}(u(1-v))^{z_2 - 1} u\, dv\, du \\ &= \int_0^\infty e^{-u} u^{z_1 + z_2 - 1}\, du \int_0^1 v^{z_2 - 1}(1-v)^{z_2 - 1}\, dv \\ &= \Gamma(z_1 + z_2) B(z_1, z_2),\end{aligned}$$

which completes the proof. ■

2. Hypergeometric series: Definition and First Properties

Definition 2.1 (Unilateral and bilateral hypergeometric series)**.** Let p and q be non-negative integers. Furthermore, let $a_1, \cdots, a_p$ and $b_1, \cdots, b_q$ be complex parameters such that none of the b_j are non-positive integers. The *unilateral hypergeometric series* ${}_pF_q$ is defined by

$$ {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. $$

Similarly, we define the *bilateral hypergeometric series* ${}_pH_q$ by

$$ {}_pH_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} z^n, $$

where

$$ (a)_{-n} := \frac{(-1)^n}{(1-a)_n}, $$

for all $n > 0$.

Example. For instance, we have

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. $$

Differentiating both sides, we get

$$ \frac{d}{dz} {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{1}{n!} \frac{d}{dz}(z^n) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^{n-1}}{(n-1)!}. $$

Let $m = n - 1$. Then,

$$ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{(a)_{m+1} (b)_{m+1}}{(c)_{m+1}} \frac{z^m}{m!} = \sum_{m=0}^{\infty} \frac{a(a+1)_m b(b+1)_m}{c(c+1)_m} \frac{z^m}{m!} = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z). $$

By induction on k, one can easily establish that

$$ \frac{d^k}{dz^k} {}_2F_1(a, b; c; z) = \frac{(a)_n (b)_n}{(c)_n} {}_2F_1(a+n, b+n; c+n; z). $$

A more involved example of a transformation formula is as follows.

Proposition 2.2 (Euler's integral for the Gauss hypergeometric function)**.** *We have*

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \, dt. $$

Proof. The key ingredient in the proof is the power series expansion for $(1 - zt)^{-a}$. For $|zt| < 1$, we have

$$ (1 - zt)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (zt)^n. $$

Substituting this into the integral in the proposition, and recognizing the Beta function, we get

$$\begin{aligned}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}(zt)^n \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n\int_0^1 t^{n+b-1}(1-t)^{c-b-1}\,dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n B(n+b,c-b) \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)}z^n \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\frac{\Gamma(b)(b)_n\Gamma(c-b)}{\Gamma(c)(c)_n}z^n \\ &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \\ &= {}_2F_1(a,b;c;z).\end{aligned}$$

■

An upshot of this integral representation is that we can understand the symmetries of ${}_2F_1$.

Corollary 2.3. *We have the following transformation formula*

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z).$$

Proof. We have

$$\begin{aligned}{}_2F_1(c-a,c-b;c;z) &= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}(1-zt)^{a-c}\,dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_1^0 (-1)u^{b-1}(1-u)^{c-b-1}(1-z(1-u))^{a-c}\,du \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-z+zu)\end{aligned}$$

■

Proposition 2.4. *The hypergeometric function ${}_2F_1(a,b;c;z)$ satisfies the following second order differential equation*

$$z(1-z)\frac{d^2w}{dz^2} + (c-(a+b+1)z)\frac{dw}{dz} - abw = 0.$$

To prove this, we will use the differentiation formulae we derived earlier. But first, we show that hypergeometric function satisfies a recurrence relation.

Proof. ■

3. Orthogonal Polynomials

Let V be the inner product space of continuous real-valued functions defined on the interval $[-1,1]$, with inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle f,g\rangle = \int_{-1}^1 f(x)g(x)\,dx.$$

Furthermore, let $\mathcal{P} \subseteq V$ be subspace of V consisting of polynomials. Then, the Legendre polynomials, $\{P_n(x)\}_{n\geq 0}$ form an orthonormal basis for V. We frame this as

Definition 3.1 (Legendre polynomials)**.** The Legendre polynomials, $P_n(x)$, are defined to be such that

- $P_n(x)$ is a polynomial of degree n,
- $P_n(1) = 0$;
- for all $n \geq m$, we have

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = 0.$$

Proposition 3.2. *For all $n \geq 0$, we have that*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x-1)^n\right).$$

Proof. ■

Definition 3.3 (Sequence of orthogonal polynomials)**.** A sequence of polynomials $(P_n(x))_{n=0}^{\infty}$ is said to be a sequence of orthogonal polynomials on the interval $[a,b]$ with respect to the weight function $w(x)$ if $\deg P_n(x) = n$ and

$$\int_a^b P_n(x)P_m(x)w(x)\, dx = K_n\delta_{n,m},$$

where K_n is a constant depending on n.

Proposition 3.4 (Rodrigues' formula)**.** *Let $(P_n(x))_{n=0}^{\infty}$ be a sequence of orthogonal polynomials on $[a,b]$ with respect to $w(x)$. Write $w(x) = W(x)/B(x)$ such that $\frac{W'(x)}{W(x)} = \frac{A(x)}{B(x)}$, where $A(x)$ is a polynomial of degree at most 1 and $B(x)$ is a polynomial of degree at most 2. Furthermore, assume that for all $k \geq 0$ we have*

$$\lim_{x\to a} x^k W(x) = 0, \quad \lim_{x\to b} x^k W(x) = 0 \quad \textit{and} \quad \frac{d^k}{dx^k}\Big(B(x)^k w(x)\Big) \neq 0.$$

Then

$$P_n(x) = \frac{c_n}{w(x)}\frac{d^n}{dx^n}(B(x)^n),$$

for some $c_n \in \mathbb{R}$.

In order to prove Rodrigues' formula, we'll need two lemmas.

Lemma 3.5. *For each $k \geq 0$, define the function $F_k : [a,b] \to \mathbb{R}$ by*

$$F_k(x) := \frac{1}{w(x)}\frac{d^k}{dx^k}\left(B^n(x)w(x)\right),$$

for $x \in [a,b]$. We have that $F_k(x) = B^{n-k}(x)p_k(x)$ where $p_k(x)$ is a polynomial such that $\deg p_k(x) \leq k$.

Proof. We give a proof by induction on k. Note that

$$F_0(x) = \frac{1}{w(x)}\left(B(x)^n w(x)\right) = B(x)^{n-0}p_0(x),$$

where $p_0(x) = 1$; this establishes the base case. Next, assume that $F_{k'}(x) = B^{n-k'}(x)p_{k'}(x)$ where $\deg p_{k'}(x) \leq k'$. Differentiating both sides of

$$F_{k'}(x) = \frac{1}{w(x)}\frac{d^{k'}}{dx^{k'}}\left(B^n(x)w(x)\right),$$

we get

$$F'_{k'}(x) = -\frac{w'(x)}{w(x)^2}\frac{d^{k'}}{dx^{k'}}\left(B^n(x)w(x)\right) + \frac{1}{w(x)}\frac{d^{k'+1}}{dx^{k'+1}}\left(B^n(x)w(x)\right) = -\frac{w'(x)}{w(x)}F_{k'}(x) + F_{k'+1}(x).$$

Therefore,

$$\begin{aligned} F_{k'+1}(x) &= \frac{w'(x)}{w(x)}F_{k'}(x) + F'_{k'}(x) = \frac{A(x)}{B(x)}B^{n-k'}(x)p_{k'}(x) + \frac{d}{dx}\left(B^{n-k'}(x)p_{k'}(x)\right) \\ &= B^{n-(k'+1)}(x)A(x)p_{k'}(x) + p'_{k'}(x)B^{n-k'}(x) + (n-k')B^{n-(k'+1)}(x)B'(x)p_{k'}(x) \\ &= B^{n-(k'+1)}(x)\left(A(x)p_{k'}(x) + p'_{k'}(x)B(x) + (n-k')B'(x)p_{k'}(x)\right). \end{aligned}$$

Thus, let

$$p_{k'+1}(x) = A(x)p_{k'}(x) + p'_{k'}(x)B(x) + (n-k')B'(x)p_{k'}(x).$$

From $\deg A(x) \leq 1$, $\deg B(x) \leq 2$, $\deg p'_{k'}(x) \leq k'-1$ and $\deg p_{k'}(x) \leq k'$, it follows that $\deg p_{k'+1}(x) \leq k'+1$, which completes the inductive step. By the principle of mathematical induction, it follows that the statement holds for all k'. ■

Lemma 3.6. *For $0 \leq p \leq n$, we have that*

$$\left[\frac{d^{n-(p+1)}}{dx^{n-(p+1)}}(B^n(x)w(x))Q_m^{(p)}(x)\right]_a^b = 0.$$

Proof. Let

$$f_p(x) = \frac{d^{n-(p+1)}}{dx^{n-(p+1)}}(B^n(x)w(x))Q_m^{(p)}(x).$$

It suffices to show that $\lim_{x\to a} f_p(x) = 0$ and $\lim_{x\to b} f_p(x) = 0$. We have $f_p(x) = w(x)F_{n-(p+1)}(x)Q_m^{(p)}(x)$, and so by Lemma 3.5,

$$f_p(x) = w(x)B^{p+1}(x)p_{n-(p+1)}(x)Q_m^{(p)}(x) = W(x)B^p(x)p_{n-(p+1)}(x)Q_m^{(p)}(x).$$

Furthermore, $Q_m^{(p)}(x)$, $p_{n-(p+1)}(x)$ and $B^p(x)$ are polynomials, so $f_p(x) = W(x)P_p(x)$, for some polynomial $P_p(x) = \sum_{i=0}^{d} T_i x^i$. Therefore,

$$\lim_{x\to c} f_p(x) = \lim_{x\to c}\sum_{i=0}^{d} T_i x^i W(x) = \sum_{i=0}^{d} T_i \lim_{x\to c} x^i W(x) = 0,$$

where $c \in \{a, b\}$. ■

Now we're ready for the proof of Proposition 3.4.

Proof. For $n \geq 0$, define the polynomial $Q_n(x) : [a,b] \to \mathbb{R}$ by

$$Q_n(x) := F_n(x) = \frac{1}{w(x)}\frac{d^n}{dx^n}\left(B^n(x)w(x)\right),$$

for all $x \in [a, b]$. By Lemma 3.5, $\deg Q_n(x) \le n$. For each $0 \le p < n$, we have that

$$\begin{aligned}
I_p &= \int_a^b \frac{d^{n-p}}{dx^{n-p}} \left(B^n(x)w(x)\right) Q_m^{(p)}(x)\, dx = \int_a^b \frac{d}{dx} \left(\frac{d^{n-(p+1)}}{dx^{n-(p+1)}} (B^n(x)w(x)) \right) Q_m^{(p)}(x)\, dx \\
&= \left[\frac{d^{n-(p+1)}}{dx^{n-(p+1)}} (B^n(x)w(x)) Q_m^{(p)}(x) \right]_a^b - \int_a^b \frac{d^{n-(p+1)}}{dx^{n-(p+1)}} \left(B^n(x)w(x)\right) Q_m^{(p+1)}(x)\, dx \\
&= - \int_a^b \frac{d^{n-(p+1)}}{dx^{n-(p+1)}} \left(B^n(x)w(x)\right) Q_m^{(p+1)}(x)\, dx \\
&= -I_{p+1},
\end{aligned}$$

where the last equality follows from Lemma 3.6. Putting these n recurrences together, we have $I_0 = (-1)^n I_n$, which, after substituting the integral representations for I_0 and I_n, yields

$$\int_a^b Q_n(x) Q_m(x) w(x)\, dx = (-1)^n \int_a^b w(x) B^n(x) Q_m^{(n)}(x)\, dx.$$

Notice that $Q_m(x)$ has degree at most m, and so if $n > m$, we have $Q_m^{(n)}(x) = 0$. Therefore

$$\int_a^b Q_n(x) Q_m(x) w(x)\, dx = 0 \quad \text{for } n > m.$$

This completes the proof that that $(Q_m)_{m=0}^{\infty}$ is a sequence of orthogonal polynomials. ■

Proposition 3.7. *Using the notation of Proposition 3.4, we have the second-order differential equation*

$$B(x) \frac{d^2}{dx^2} P_n(x) + A(x) \frac{d}{dx} P_n(x) + \lambda_n(x) P_n(x) = 0, \tag{3.1}$$

where

$$\lambda_n(x) = -\frac{1}{2} n(n-1) B''(x) - nA'(x).$$

Before we state the proof of this proposition, we need a lemma involving a differentiation identity.

Lemma 3.8. *Let $y = f(x)$ be a function, with $A(x)$ and $B(x)$ polynomials of degree at most 1 and 2 respectively. Then,*

$$A(x) \frac{d^n}{dx^n} y(x) = \frac{d^n}{dx^n} (A(x)y(x)) - n \frac{d^{n-1}}{dx^{n-1}} (A'(x)y(x)), \tag{3.2}$$

and

$$B(x) \frac{d^n}{dx^n} y(x) = \frac{d^n}{dx^n} (B(x)y(x)) - n \frac{d^{n-1}}{dx^{n-1}} (B'(x)y(x)) + \frac{n(n-1)}{2} \frac{d^{n-2}}{dx^{n-1}} (B''(x)y(x)). \tag{3.3}$$

Proof. It suffices to show that the identity for $B(x)$ holds; Equation 3.2 is a special case. For $0 \le k \le n-1$, define

$$P_k(x) = \frac{k(k-1)}{2} B''(x) y^{(k-2)}(x) + kB'(x) y^{(k-1)}(x) + B(x) y^{(k)}(x).$$

We have

$$\begin{aligned} P_k^{(n-k)}(x) &= \frac{d^{n-(k+1)}}{dx^{n-(k+1)}} \left(\frac{d}{dx} \left(\frac{k(k-1)}{2} B''(x) y^{(k-2)}(x) + kB'(x) y^{(k-1)}(x) + B(x) y^{(k)}(x) \right) \right) \\ &= \frac{d^{n-(k+1)}}{dx^{n-(k+1)}} \left(\frac{k(k-1)}{2} B''(x) y^{(k-1)}(x) + kB''(x) y^{(k-1)}(x) \right. \\ &\qquad \left. + kB'(x) y^{(k)}(x) + B'(x) y^{(k)}(x) + B(x) y^{(k+1)}(x) \right) \\ &= \frac{d^{n-(k+1)}}{dx^{n-(k+1)}} \left(\frac{k(k+1)}{2} B''(x) y^{(k-1)}(x) + (k+1) B'(x) y^{(k)}(x) + B(x) y^{(k+1)}(x) \right) \\ &= P_{k+1}^{(n-(k+1))}(x). \end{aligned}$$

Putting these n equations together, we get $P_0^{(n)}(x) = P_n^{(0)}(x)$. By the definition of $P_k(x)$, this translates as

$$\frac{d^n}{dx^n}(B(x)y(x)) = \frac{n(n-1)}{2} B''(x) \frac{d^{n-2}}{dx^{n-2}} y(x) + nB'(x) \frac{d^{n-1}}{dx^{n-1}} y(x) + B(x) \frac{d^n}{dx^n} y(x),$$

which can be rearranged to get Equation 3.3. ∎

We're ready for the proof of the differential equation now!

Proof. When $n = 0$, 3.1 is trivial. When $n = 1$, Equation 3.1 simplifies to

$$E = B(x)P_1''(x) + A(x)P_1'(x) + \lambda_1(x)P_1(x) = A(x)P_1'(x) - A'(x)P_1(x).$$

By Proposition 3.4,

$$P_1(x) = \frac{c_1}{w(x)} \frac{d}{dx} (B(x)w(x)) = \frac{c_1}{w(x)} W'(x) = \frac{c_1}{w(x)} \frac{A(x)W(x)}{B(x)} = c_1 A(x).$$

Thus,

$$E = A(x)c_1A'(x) - A'(x)c_1A(x) = 0.$$

Hence, we may assume that $n \geq 2$. Define

$$I_n(x) = \frac{d^n}{dx^n} (B^n(x)w(x)) = w(x)Q_n(x) = \frac{1}{c_n} w(x)P_n(x).$$

Solving for $P_n(x)$ in terms of $I_n(x)$ and substituting it into the LHS of 3.1 yields

$$\frac{d^2}{dx^2} (B_n(x)I_n(x)) - \frac{d}{dx} (A(x)I_n(x)) + \lambda_n(x)I_n(x).$$

Following the notation of Lemma 3.8, set $y(x) = B^n(x)w(x)$, and define

$$J(x) = \frac{d^2}{dx^2}(B(x)y(x)) - n\frac{d}{dx}(B'(x)y(x)) + \frac{n(n-1)}{2} B''(x)y(x),$$

$$K(x) = -\frac{d}{dx}(A(x)y(x)) + nA'(x)y(x), \quad \text{and} \quad L(x) = \lambda_n(x)y(x).$$

Our expression is equal to

$$\frac{d^n}{dx^n}(J(x) + K(x) + L(x)).$$

Call the three terms of $J(x)$ $J_1(x)$, $J_2(x)$ and $J_3(x)$ respectively, and the two terms of $K(x)$ $K_1(x)$ and $K_2(x)$. Then, we have

$$J_3(x) + K_2(x) + L(x) = \left(nA'(x) + \frac{n(n-1)}{2} B''(x) + \lambda_n(x) \right) y(x) = 0.$$

Next, we show that $J_1(x) + J_2(x) + K(x) = 0$. Note that

$$J_1(x) = \frac{d^2}{dx^2}\left(B^n(x)W(x)\right) = \frac{d^2}{dx^2}\left(B^n(x)\int e^{A(x)/B(x)}\,dx\right).$$

Taking the innermost derivative, we get

$$\begin{aligned} J_1(x) &= \frac{d}{dx}\left(\left(nB'(x)B^{n-1}(x) + A(x)B^{n-1}(x)\right)\int e^{A(x)/B(x)}\,dx\right) \\ &= \frac{d}{dx}\left(nB'(x)B^n w(x) + A(x)B^n(x)w(x)\right). \end{aligned}$$

The first term is $-J_2(x)$ and the second term is $-K_1(x)$, which completes the proof. ■

Now its time to try out these formulae!

Example (Legendre polynomials). Let's start simply, by letting $w(x) = 1$, with $a = -1$ and $b = 1$. It might be tempting to set $W(x) = 1$ and $B(x) = 1$, but note that $\lim_{x\to\pm1} x^k \neq 0$ for all k. Thus, keeping the requirement $\lim_{x\to\pm1} x^k W(x) = 0$ in mind, we set $W(x)$ be such that it has roots ±1; let $W(x) = (x+1)(x-1) = x^2 - 1$. Then, $B(x) = W(c) = x^2 - 1$, and $A(x) = 2x$. Then,

$$P_n(x) = c_n \frac{d^n}{dx^n}((x^2-1)^n),$$

for some constants c_n, and the differential equation becomes

$$(x^2-1)\frac{d^2}{dx^2}P_n(x) + 2x\frac{d}{dx}P_n(x) - (n^2+n)P_n(x) = 0.$$

Example (Jacobi Polynomials). Let $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, with $a = -1$ and $b = 1$ as usual. Note that if $\alpha = 0 = \beta$, then $w^{(\alpha,\beta)}(x) = 1$, which is the trivial weight function used for the Legendre polynomials. Thus, this weight is a generalization of the previous one. Set $W(x) = (1-x)^{a+1}(1+x)^{b+1}$ and $B(x) = (1-x)(1+x) = x^2 - 1$. Then, $\frac{W'(x)}{W(x)} = \frac{(b-a)-(a+b+2)x}{x^2-1}$. Hence, $A(x) = (b-a) - (a+b+2)x$. Therefore,

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{(1-x)^{a+1}(1+x)^{b+1}}\frac{d^n}{dx^n}((x^2-1)^n),$$

and the differential equation becomes

$$(x^2-1)\frac{d^2}{dx^2}P_n(x) + ((\beta-\alpha) - (\alpha+\beta+2)x)\frac{d}{dx}P_n(x) + (-n(n-1) + n(\alpha+\beta+2))P_n(x) = 0$$