AN ELEMENTARY PROOF OF PARSEVAL'S IDENTITY IN L^2

ROY EDUARDO YARANGA ALMEIDA

1. Introduction

One of the most important ideas within the study of infinite series is to be able to approximate a function using elementary objects. In the case of periodic functions, trigonometric functions fulfill this role. Roughly speaking, a Fourier series seeks to express a periodic function as an infinite sum of sines and cosines.

The result we will focus on is Parseval's identity. The usual formulation tells us that, under certain conditions, if a function f can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big),$$

then it must hold that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

But we can also express this result using complex exponential functions instead of sines and cosines using Euler's identity $e^{inx} = \cos(nx) + i\sin(nx)$. This will allow us to handle a single family of elementary functions.

The objective of this work is to present a proof of Parseval's identity stated in a more general form. To do this, we will work with periodic functions on the interval $[-\pi, \pi]$.

In this work we will consider 2π-periodic functions $f : \mathbb{R} \to \mathbb{C}$ restricted to the interval $[-\pi, \pi]$. Since a periodic function is completely determined by its values on any interval of length 2π, this choice does not cause us to lose generality and is even convenient due to the orthogonality properties of complex exponential functions on this interval.

Date: 25-03-2025.

The analytical framework we use throughout this paper is standard in real and functional analysis; see [1] for a classical reference.

2. PRELIMINARIES:

To measure the size of a function and define convergence, we introduce the spaces L^1 and L^2, together with an inner product and norm.

We begin by introducing a notion of integrability, which allows us to measure the total size of a function.

Definition 2.1. A function $f : [-\pi, \pi] \to \mathbb{C}$ belongs to $L^1([-\pi, \pi])$ if

$$\int_{-\pi}^{\pi} |f(x)|\, dx < \infty.$$

We now introduce a stronger notion which will be central to our analysis, namely the space L^2 of square-integrable functions.

Definition 2.2. A function $f : [-\pi, \pi] \to \mathbb{C}$ belongs to $L^2([-\pi, \pi])$ if

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx < \infty.$$

In order to study geometric properties such as orthogonality, we equip L^2 with an inner product structure.

Definition 2.3. Let $f, g \in L^2([-\pi, \pi])$. We define

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

from which we define the norm

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx\right)^{1/2}.$$

These notions are fundamental in the study of Hilbert spaces; see [2] for a detailed treatment.

We now define the Fourier coefficients, which can be interpreted as projections of the function onto the orthogonal family of complex exponentials.

Definition 2.4. Let $f \in L^1([-\pi, \pi])$. For each integer $n \in \mathbb{Z}$ define

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx.$$

These numbers are called the Fourier coefficients of the function f. Note that $\widehat{f}(n) = \langle f, e^{inx} \rangle$ for all $n \in \mathbb{Z}$.

We now construct finite approximations of f using these coefficients.

Definition 2.5. For each integer $N \geq 0$ define the function $S_N f : [-\pi, \pi] \to \mathbb{C}$ by

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{inx}.$$

The function $S_N f$ is called the Fourier partial sum of order N, which is a finite approximation of f constructed from its Fourier coefficients.

The series

$$\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}$$

is called the Fourier series of f.

Finally, we introduce the notion of convergence that will be relevant for our purposes.

Definition 2.6. Let $f \in L^2([-\pi, \pi])$. We say that the Fourier series of f converges to f in L^2 if

$$\lim_{N \to \infty} \|f - S_N f\|_2 = 0.$$

3. PARSEVAL'S IDENTITY:

We conclude by proving Parseval's identity. The argument relies on the orthogonality of the complex exponential functions and proceeds by analyzing the behavior of the partial sums $S_N f$, relating their L^2 norms to the Fourier coefficients, and then passing to the limit as $N \to \infty$.

Theorem 3.1 (Parseval's Identity)**.** Let $f \in L^2([-\pi, \pi])$ such that $S_N f$ converges to f in L^2. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2.$$

Remark 3.2. Parseval's identity can be interpreted as a functional analogue of the Pythagorean theorem, where the squared norm of a function is equal to the sum of the squares of its components with respect to the orthonormal system $\{e^{inx}\}_{n \in \mathbb{Z}}$.

Proof. The next lemma shows that the L^2 norm of the partial sum is exactly captured by a finite number of Fourier coefficients.

Lemma 3.3. *Let $N \geq 0$. Then*

$$\|S_N f\|_2^2 = \sum_{n=-N}^{N} |\widehat{f}(n)|^2.$$

Proof. By definition,

$$\|S_N f\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{n=-N}^{N} \widehat{f}(n) e^{inx} \right|^2 dx.$$

Expanding the square we obtain

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-N}^{N} \widehat{f}(n) e^{inx} \right) \overline{\left(\sum_{m=-N}^{N} \widehat{f}(m) e^{imx} \right)} dx.$$

Then

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-N}^{N} \sum_{m=-N}^{N} \widehat{f}(n) \overline{\widehat{f}(m)} e^{i(n-m)x} dx.$$

Therefore

$$= \sum_{n=-N}^{N} \sum_{m=-N}^{N} \widehat{f}(n) \overline{\widehat{f}(m)} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} dx \right).$$

And by the orthogonality relations, this integral is 0 unless $m = n$, so we finally obtain

$$\|S_N f\|_2^2 = \sum_{n=-N}^{N} |\widehat{f}(n)|^2.$$

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The following lemma relates the approximation error to the Fourier coefficients.

Lemma 3.4. *Let $f \in L^2([-\pi, \pi])$ and $N \geq 0$. Then*

$$\|f - S_N f\|_2^2 = \|f\|_2^2 - \sum_{n=-N}^{N} |\widehat{f}(n)|^2.$$

Proof. We have that

$$\|f - S_N f\|_2^2 = \langle f - S_N f, f - S_N f \rangle.$$

$$= \langle f, f \rangle - \langle f, S_N f \rangle - \langle S_N f, f \rangle + \langle S_N f, S_N f \rangle,$$

using the linearity of the inner product.

Now,

$$\langle f, S_N f\rangle = \left\langle f, \sum_{n=-N}^{N} \widehat{f}(n)e^{inx} \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\sum_{n=-N}^{N} \overline{\widehat{f}(n)e^{inx}}\right) dx.$$

Which is equal to

$$\sum_{n=-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{\widehat{f}(n)e^{inx}} = \sum_{n=-N}^{N} \overline{\widehat{f}(n)}\langle f, e^{inx}\rangle = \sum_{n=-N}^{N} |\widehat{f}(n)|^2.$$

And therefore

$$\langle f, S_N f\rangle = \sum_{n=-N}^{N} |\widehat{f}(n)|^2.$$

Similarly, we have that

$$\langle S_N f, f\rangle = \sum_{n=-N}^{N} |\widehat{f}(n)|^2.$$

Moreover, by the previous lemma,

$$\langle S_N f, S_N f\rangle = \sum_{n=-N}^{N} |\widehat{f}(n)|^2.$$

Therefore,

$$\|f - S_N f\|_2^2 = \|f\|_2^2 - \sum_{n=-N}^{N} |\widehat{f}(n)|^2.$$

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Now, taking the limit as $N \to \infty$ we obtain that,

$$0 = \|f\|_2^2 - \lim_{N\to\infty} \sum_{n=-N}^{N} |\widehat{f}(n)|^2.$$

Therefore

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2.$$

■

Fourier analysis has its origins in the study of the heat equation, where such expansions were first introduced to describe the evolution of temperature in a medium. This historical motivation led to the development of Fourier analysis and its connections with several branches of mathematics. For a comprehensive introduction to Fourier analysis, see [3], while a perspective emphasizing applications can be found in [4].

References

[1] Walter Rudin, *Principles of Mathematical Analysis*, McGraw–Hill.
[2] N. Young, *An Introduction to Hilbert Space*, Cambridge University Press.
[3] Elias M. Stein and Rami Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press.
[4] Mark Holmes, *Introduction to the Foundations of Applied Mathematics*, Springer.

Euler Circle, Mountain View, CA 94040
Email address: `r.yaranga.almeida@gmail.com`