# Hecke Operators, Quasimodular Forms, and the Partition Algebra

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**Abstract**

This paper explains the background for an open problem that arises from the interaction between quasimodular forms and partition theory. The Bloch–Okounkov theorem shows that the $q$-bracket of a shifted symmetric function on partitions always produces a quasimodular form, establishing a deep structural bridge between combinatorics and number theory.

The central question considered here is whether the Hecke action on modular and quasimodular forms can be lifted to the partition algebra itself. To make this question precise, we review modular forms, quasimodular forms, Hecke operators, shifted symmetric functions, and the $q$-bracket. We then explain why the lifting problem is natural, what makes it difficult, and what evidence suggests that such a structure might exist. The goal is expository: not to solve the problem, but to clarify its meaning and significance.

## 1 Introduction

This paper grew out of a research question rather than a standard textbook topic. The question is simple to state but surprisingly deep: if the $q$-bracket sends functions on partitions to quasimodular forms, and if quasimodular forms carry a Hecke action, does a corresponding Hecke structure already exist on the partition side?

To understand why this is a reasonable question, it helps to step back and examine the broader mathematical landscape. Modular forms are among the most structured objects in number theory. They satisfy rigid transformation laws, live in finite-dimensional spaces, and carry Hecke operators, which encode deep arithmetic symmetries. These symmetries are not merely formal; they are closely connected to important number-theoretic phenomena such as multiplicative properties of coefficients and connections to $L$-functions.

At the same time, many naturally occurring generating functions do not quite fit into the strict framework of modular forms. Instead, they belong to the slightly larger world of quasimodular forms. This enlargement is not arbitrary; it arises naturally when one considers operations such as differentiation.

Partitions provide one of the most compelling examples of this interaction. If $p(n)$ denotes the number of partitions of $n$, then the generating series

$$\sum_{n\geq 0} p(n)q^n = \prod_{m\geq 1}(1-q^m)^{-1}$$

already suggests a connection to modular objects. Writing $q = e^{2\pi i\tau}$, we can rewrite this as

$$\sum_{\lambda\in\mathcal{P}} q^{|\lambda|} = q^{1/24}\eta(\tau)^{-1}.$$

Thus, even the most basic partition generating function is closely tied to the Dedekind eta function, a central object in modular form theory.

However, the connection goes much deeper than this single example. Once one considers general functions on partitions and applies the $q$-bracket construction, an entire algebraic bridge emerges. The Bloch–Okounkov theorem shows that this bridge is systematic rather than accidental.

The main objects of the paper may be summarized schematically as

$$\Lambda^* \xrightarrow{\langle\cdot\rangle_q} QM_*, \qquad M_* \subset QM_*, \qquad T_m : QM_k \to QM_k.$$

The central question is whether the right-hand Hecke action can be reflected on the left-hand side.

This naturally leads to the main question of the paper.

# 2 Modular Forms and Hecke Operators

We begin with the classical setting. Let

$$\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$$

be the upper half-plane.

**Definition 1.** A modular form of weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and which is holomorphic at infinity.

Equivalently, if one writes the slash operator

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $f$ is modular of weight $k$ precisely when

$$f|_k\gamma = f \qquad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

The requirement of being holomorphic at the cusp means that $f$ has a Fourier expansion

$$f(\tau) = \sum_{n\geq 0} a_n q^n, \qquad q = e^{2\pi i\tau},$$

with no negative powers of $q$.

The most familiar examples are the Eisenstein series:

$$E_4(\tau) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n, \qquad E_6(\tau) = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n.$$

A remarkable fact is that all modular forms for $\mathrm{SL}_2(\mathbb{Z})$ can be expressed in terms of these two:

$$M_* = \bigoplus_{k \geq 0} M_k = \mathbb{C}[E_4, E_6].$$

Thus each homogeneous piece $M_k$ is finite-dimensional, while the entire graded ring is generated by just two explicit series.

One of the most important additional structures on modular forms is the action of Hecke operators. If

$$f(\tau) = \sum_{n \geq 0} a_n q^n$$

is a modular form of weight $k$, then the Hecke operator $T_m$ produces another modular form:

$$(T_m f)(\tau) = \sum_{n \geq 0} \left( \sum_{d \mid (m,n)} d^{k-1} a_{mn/d^2} \right) q^n.$$

There is also a geometric averaging formula,

$$(T_m f)(\tau) = m^{k-1} \sum_{\substack{ad=m \\ d>0}} \sum_{b \,(\mathrm{mod}\, d)} d^{-k} f\left( \frac{a\tau + b}{d} \right),$$

which makes clearer that $T_m$ is not an ad hoc manipulation of Fourier coefficients but an averaging process over index-$m$ data.

For example, if $p$ is prime, then

$$(T_p f)(\tau) = \sum_{n \geq 0} (a_{pn} + p^{k-1} a_{n/p}) q^n,$$

where $a_{n/p} = 0$ if $p \nmid n$. In particular, the coefficient of $q^1$ in $T_p f$ is $a_p$.

Although these formulas appear technical, they reflect deep arithmetic structure. Hecke operators encode symmetries that relate coefficients of modular forms in a highly nontrivial way. In many cases, one can find eigenforms $f$ satisfying

$$T_m f = \lambda_m f \qquad (m \geq 1),$$

and then the eigenvalues $\lambda_m$ inherit multiplicative relations such as

$$\lambda_m \lambda_n = \sum_{d \mid (m,n)} d^{k-1} \lambda_{mn/d^2}.$$

This is one of the central reasons Hecke theory is so important: it lets us read arithmetic structure directly from the analytic object.

## 3 From Modular to Quasimodular Forms

Despite their elegance, modular forms are not closed under differentiation. If $f$ is modular, then

$$Df := q\frac{d}{dq}f = \frac{1}{2\pi i}\frac{df}{d\tau}$$

typically fails to be modular.

This observation motivates the introduction of quasimodular forms. The first example is the Eisenstein series

$$E_2(\tau) = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n.$$

Unlike $E_4$ and $E_6$, this function does not satisfy the usual modular transformation law. Instead,

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2(\tau) - \frac{12}{2\pi i}c(c\tau+d).$$

The extra linear term in $c/(c\tau+d)$ is exactly what leads to the definition of quasimodularity.

**Definition 2.** A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called *quasimodular of weight $k$ and depth at most $p$* if for every

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

one has an identity of the form

$$(c\tau+d)^{-k}f\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{r=0}^{p} f_r(\tau)\left(\frac{c}{c\tau+d}\right)^r,$$

where the $f_r(\tau)$ are holomorphic.

Thus modular forms are exactly the depth-0 quasimodular forms. For the full modular group,

$$QM_* = \mathbb{C}[E_2, E_4, E_6].$$

This enlargement is natural because differentiation closes within it. The Ramanujan identities show

$$DE_2 = \frac{E_2^2 - E_4}{12}, \qquad DE_4 = \frac{E_2E_4 - E_6}{3}, \qquad DE_6 = \frac{E_2E_6 - E_4^2}{2}.$$

In particular,

$$D(QM_k) \subseteq QM_{k+2}.$$

So quasimodular forms form the smallest extension of modular forms that is stable under differentiation.

There is also a useful almost holomorphic perspective. The corrected function

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi\,\mathrm{Im}(\tau)}$$

is modular, though no longer holomorphic. More generally, one can think of depth-$p$ quasimodular forms as holomorphic parts of almost holomorphic modular forms of the shape

$$F(\tau) = \sum_{r=0}^{p} \frac{f_r(\tau)}{\mathrm{Im}(\tau)^r}.$$

This viewpoint explains why quasimodular forms behave so much like modular forms even though they are not quite the same.

For the purposes of this paper, the key point is the chain of inclusions and operations

$$M_* \subset QM_*, \qquad D : QM_k \to QM_{k+2}, \qquad T_m : QM_k \to QM_k.$$

Thus the target of the $q$-bracket is not merely larger than the modular world; it still retains much of its algebraic structure.

# 4 Partitions and Shifted Symmetric Functions

We now move to the partition side.

**Definition 3.** A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers with finite sum

$$|\lambda| = \sum_i \lambda_i.$$

We usually represent $\lambda$ by its Young diagram.

The algebra relevant to the Bloch–Okounkov theorem is the algebra of shifted symmetric functions, denoted $\Lambda^*$. It is a shifted analogue of the ordinary ring of symmetric functions and is the natural home for polynomial statistics on partitions.

Why is a shifted version needed? Ordinary symmetric functions are built to capture invariance under permuting variables, but the combinatorics of partitions is not governed by raw coordinates alone. The parts of a partition come with their positions, and many naturally occurring formulas depend not just on $\lambda_j$ but on shifted quantities like $\lambda_j - j$. The theory of shifted symmetric functions is designed precisely to absorb that extra combinatorial offset.

One useful way to describe this algebra is through generators $Q_k(\lambda)$. For a partition $\lambda$, define

$$X_\lambda = \left\{ \lambda_j - j + \frac{1}{2} : j \geq 1 \right\} \subset \mathbb{Z} + \frac{1}{2}.$$

Then form the generating series

$$W_\lambda(z) = \sum_{x \in X_\lambda} e^{xz}.$$

Expanding appropriately near $z = 0$ defines functions $Q_k(\lambda)$ by

$$W_\lambda(z) = \sum_{k \geq 0} Q_k(\lambda) z^{k-1}.$$

The first few values already show that these functions encode basic partition data:

$$Q_0(\lambda) = 1, \qquad Q_1(\lambda) = 0, \qquad Q_2(\lambda) = |\lambda| - \frac{1}{24}.$$

The algebra they generate is

$$\Lambda^* \cong \mathbb{Q}[Q_2, Q_3, Q_4, \ldots], \qquad \deg(Q_k) = k.$$

Thus there is a graded decomposition

$$\Lambda^* = \bigoplus_{k \geq 0} \Lambda^*_k.$$

This grading is crucial because the Bloch–Okounkov theorem respects it.

The point is that $\Lambda^*$ is large enough to contain many interesting statistics but rigid enough to support theorems. One can summarize the situation as

$$\text{partitions } \lambda \longmapsto \{Q_k(\lambda)\}_{k \geq 2} \longmapsto f(Q_2, Q_3, \ldots) \in \Lambda^*.$$

So the partition side is already organized algebraically before the $q$-bracket ever appears.

# 5 The $q$-Bracket and First Examples

Given any function $f$ on partitions, define its $q$-bracket by

$$\langle f \rangle_q = \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}}.$$

This is a normalized average over all partitions, weighted by size. The normalization matters: instead of just forming a raw generating function, one is averaging $f$ with respect to the natural measure for which a partition $\lambda$ has weight proportional to $q^{|\lambda|}$. That makes the construction behave more like an expectation value than like a simple counting series.

The denominator is the partition function

$$Z(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} = \prod_{m \geq 1} (1 - q^m)^{-1} = q^{1/24} \eta(\tau)^{-1}.$$

Thus the $q$-bracket can be written compactly as

$$\langle f \rangle_q = Z(q)^{-1} \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}.$$

It is plainly linear:

$$\langle af + bg \rangle_q = a\langle f \rangle_q + b\langle g \rangle_q,$$

but it is not obviously multiplicative, and this failure will matter later.

Even the simplest examples already point toward quasimodularity. If $f \equiv 1$, then of course

$$\langle 1 \rangle_q = 1.$$

The first nontrivial example is $f(\lambda) = |\lambda|$. Since

$$\sum_{\lambda \in \mathcal{P}} |\lambda| q^{|\lambda|} = q \frac{d}{dq} Z(q),$$

we get

$$\langle |\lambda| \rangle_q = q \frac{d}{dq} \log Z(q).$$

Using $Z(q) = q^{1/24} \eta(\tau)^{-1}$ and the standard logarithmic derivative of $\eta$, one finds

$$\langle |\lambda| \rangle_q = \frac{1 - E_2(\tau)}{24}.$$

Since $Q_2(\lambda) = |\lambda| - \frac{1}{24}$, it follows immediately that

$$\langle Q_2 \rangle_q = -\frac{E_2(\tau)}{24}.$$

So the first genuinely interesting generator of the partition algebra already lands in a classical quasimodular form.

This example is important not only because it is concrete, but because it captures the philosophy of the subject. A simple partition statistic becomes, after normalization, a function with modular-type transformation behavior. The appearance of $E_2$ is especially telling: the answer is not modular, but quasimodular, exactly as one should expect from the general theory.

The basic formal properties may be summarized as

$$\langle \cdot \rangle_q : \Lambda_k^* \longrightarrow QM_k, \qquad \langle 1 \rangle_q = 1, \qquad \langle Q_2 \rangle_q = -\frac{E_2}{24}.$$

These are the first hints that the map preserves much more than mere linearity.

# 6 The Bloch–Okounkov Theorem

The general theorem is the following.

**Theorem 1** (Bloch–Okounkov)**.** *If $f \in \Lambda^*$ is homogeneous of weight $k$, then $\langle f \rangle_q$ is a quasimodular form of weight $k$.*

Equivalently,

$$\langle \Lambda_k^* \rangle_q \subseteq QM_k \qquad (k \geq 0).$$

This theorem is striking because it builds a systematic bridge between partition combinatorics and automorphic objects. It is not just that a few lucky examples happen to be quasimodular. There is an entire graded algebra on the partition side whose image lies inside the graded ring of quasimodular forms.

The theorem changes the way one thinks about partitions. Before Bloch and Okounkov, it was already known that the partition generating series had modular significance. But this result says something much stronger: not just the generating series itself, but a large family of weighted averages of partition statistics belong to a modular-type world. In other words, the connection is not confined to one famous product formula. It is built into the algebraic structure of partition statistics.

There are at least three reasons why the theorem is so compelling.

First, it is genuinely structural. The source is an algebra, the target is an algebra, and the construction respects weight. Symbolically, one has

$$(\Lambda^*, \deg) \xrightarrow{\langle\cdot\rangle_q} (QM_*, \text{weight}).$$

That makes it natural to search for further compatibilities beyond the theorem itself.

Second, the theorem is conceptually surprising. The source is combinatorial, built from statistics on Young diagrams; the target is analytic, built from functions on the upper half-plane with transformation laws under $\mathrm{SL}_2(\mathbb{Z})$. A theorem connecting these worlds is already remarkable. A theorem connecting them in a graded and algebraic way is even more so.

Third, the theorem immediately raises representation-theoretic and operator-theoretic questions. Whenever a map sends one rich algebraic structure into another, one wants to know how much more compatibility is hiding behind the statement. Does it interact with derivations? filtrations? products? correspondences? The Hecke lifting problem is one of the clearest versions of that broader impulse.

# 7 Zagier's Proof and Why It Matters

One reason Zagier's paper is so influential is that it makes the Bloch–Okounkov theorem feel conceptually inevitable rather than mysterious. Instead of relying on a long explicit formula, Zagier introduces a derivation on the polynomial algebra generated by the $Q_k$ and proves a compact operator identity involving the Jacobi theta series. That identity leads to a recursion expressing the $q$-bracket of a given shifted symmetric function in terms of lower-weight ones. The proof then goes through by induction.

The exact details are not the main point here, but the method matters. It shows that the theorem is not simply a surprising output of brute-force computation. Rather, the partition algebra already supports operations that line up with familiar structures on the quasimodular side. The $q$-bracket is therefore not merely a convenient averaging map. It is a map that interacts with meaningful operators.

At a schematic level, Zagier's argument produces the pattern

$$\delta : \Lambda^*_k \to \Lambda^*_{k+2}, \qquad \langle \delta f \rangle_q \sim D\langle f \rangle_q + \text{lower-weight terms},$$

where $D = q\,d/dq$ is the quasimodular derivative. The precise coefficients are not the point here; the point is that there is operator-level compatibility rather than mere coincidence of outputs.

This is the first major reason the Hecke lifting problem feels natural. If one had only the bare statement that $\langle f \rangle_q$ happened to be quasimodular for every $f \in \Lambda^*$, one might reasonably suspect coincidence or computational luck. Zagier's proof argues against that interpretation. It suggests that the partition side has its own internal geometry, and that the quasimodular image is reflecting that geometry rather than inventing it.

There is also a philosophical point here. In mathematics, a theorem often feels deeper when one can explain *why* the target space is the right one. Zagier's proof helps explain why quasimodular forms appear at all. The appearance of quasimodularity is not accidental decoration; it is part of a system of operators, recursion relations, and generating-function identities. Once that is visible, it becomes much more plausible that Hecke symmetry might have a shadow on the partition side as well.

Finally, Zagier's treatment places the story near representation-theoretic ideas, including structures related to an $\mathfrak{sl}_2$-action. One may summarize the heuristic picture as

$$\Lambda^* \quad \text{with operators} \quad (\delta, \text{grading}, \dots) \xrightarrow{\langle\cdot\rangle_q} QM_* \quad \text{with} \quad (D, \text{weight}, \dots).$$

This again points in the same direction: there is more happening here than a single map between two rings.

# 8 Hecke Operators on Quasimodular Forms

Before formulating the open problem carefully, it is important to note that Hecke theory does not stop at modular forms. Quasimodular forms also admit a natural Hecke action.

There are two ways to think about this. One is conceptual: quasimodular forms correspond to almost holomorphic modular forms, and the classical Hecke action preserves that larger space. The other is more direct: the usual averaging formulas defining Hecke operators are compatible with the polynomial error terms that appear in the transformation law of a quasimodular form. Either way, the conclusion is that Hecke operators still make sense on $QM_*$.

For weight $k$, the same averaging formula may be written formally as

$$(T_m f)(\tau) = m^{k-1} \sum_{\substack{ad=m \\ d>0}} \sum_{b \,(\mathrm{mod}\, d)} d^{-k} f\left(\frac{a\tau + b}{d}\right), \qquad f \in QM_k,$$

and one checks that the result is still quasimodular of weight $k$. Thus

$$T_m : QM_k \to QM_k.$$

This fact is easy to state but essential for the paper's central question. If the $q$-bracket landed only in a space with no natural Hecke structure, there would be nothing to lift. But because quasimodular forms inherit Hecke symmetry from the modular world, the question becomes unavoidable: can this symmetry be seen before applying the $q$-bracket, not only after?

At the same time, one should not underestimate the subtlety here. Quasimodular forms are more flexible than modular forms because of depth. A Hecke operator must preserve not only weight but also the controlled shape of the nonmodular correction terms. So even though the Hecke action exists, it is living in a space that is structurally richer and slightly less rigid than the purely modular one. This suggests that any lift to the partition algebra may need to reflect more than just weight preservation.

# 9 The Open Problem

We can now state the motivating problem more precisely.

**Question 1** (Hecke lifting problem)**.** Does there exist, for each positive integer $m$, a natural operator

$$\widetilde{T}_m : \Lambda^* \longrightarrow \Lambda^*$$

such that

$$\langle \widetilde{T}_m f \rangle_q = T_m \langle f \rangle_q$$

for all $f \in \Lambda^*$?

In words, the question asks whether the usual Hecke action on quasimodular forms can be lifted back through the $q$-bracket to the partition algebra itself.

One can think of the problem as asking for a commuting square

$$\begin{array}{ccc} \Lambda^* & \xrightarrow{\widetilde{T}_m} & \Lambda^* \\ \Big\downarrow \langle\cdot\rangle_q & & \Big\downarrow \langle\cdot\rangle_q \\ QM_* & \xrightarrow{T_m} & QM_* \end{array}$$

with the vertical arrows given by the $q$-bracket. If such operators $\widetilde{T}_m$ exist, then the partition algebra would not merely be a source of quasimodular forms. It would carry a genuine shadow of Hecke symmetry.

One may also separate strong and weak versions of the question:

$$\text{strong: } \widetilde{T}_m : \Lambda^* \to \Lambda^*, \qquad \text{weak: } \widetilde{T}_m : \Lambda^*/\ker\langle\cdot\rangle_q \to \Lambda^*/\ker\langle\cdot\rangle_q.$$

This distinction will matter when we discuss noninjectivity below.

This is the open problem around which the whole paper is organized. It is not an arbitrary extension question. It comes directly from the fact that the $q$-bracket already turns partition statistics into quasimodular forms in a highly structured way. Once a map with that much structure exists, it becomes natural to ask how far the analogy can be pushed.

# 10 Why the Problem Matters

There are several reasons this problem is worth asking.

First, it would give a much richer interpretation of the partition algebra. Right now, $\Lambda^*$ is already important because of the Bloch–Okounkov theorem, but a Hecke action would show that it carries not just combinatorial structure but arithmetic symmetry as well. That would significantly change the conceptual status of the algebra. It would no longer be merely a convenient source of partition statistics whose generating functions happen to be quasimodular; it would become an arithmetic object in its own right.

Second, a solution would deepen the relationship between partitions and automorphic forms. The $q$-bracket already tells us that partition statistics produce quasimodular forms. A Hecke lift would say something stronger: that the symmetries seen on the automorphic side are not merely visible after passing through the $q$-bracket, but are already present in some form on the combinatorial side. That would make the bridge between the two theories far more robust.

Third, the problem matters because it could clarify the structure of quasimodular forms themselves. Hecke theory is one of the main tools for understanding modular forms. If a partition-level version exists, it might offer a new way to organize, compute, or conceptualize quasimodular phenomena. Sometimes a structure becomes clearer when one sees it reflected in a different category. A combinatorial realization of the Hecke action might reveal patterns that are less visible in the analytic formulation.

These motivations can be compressed symbolically as

$$\text{Hecke lift on } \Lambda^* \Longrightarrow \begin{cases} \text{arithmetic structure on partition statistics,} \\ \text{stronger compatibility } \Lambda^* \leftrightarrow QM_*, \\ \text{new viewpoint on Hecke theory for quasimodular forms.} \end{cases}$$

Fourth, the problem sits at an intersection of several beautiful theories: modular forms, quasimodular forms, partition combinatorics, and representation-theoretic ideas surrounding shifted symmetric functions. Even partial progress would likely teach us something about how these theories fit together. In this sense the problem is valuable not only because a positive answer would be interesting, but also because the attempt to answer it naturally pushes one to compare structures that are usually studied separately.

Finally, the problem has a strong heuristic appeal. Mathematics often progresses by noticing that one theorem has the shape of a larger hidden pattern. The Bloch–Okounkov theorem feels like that kind of result. The existence of a graded map from partition statistics to quasimodular forms, together with Zagier's proof showing operator-level compatibility, invites the suspicion that the visible theorem is only part of a broader correspondence.

# 11 What We Know So Far

Although the lifting problem is open, the surrounding theory already gives several important pieces of evidence and several serious obstacles. These are worth collecting in one place because they show that the problem is neither random nor easy.

## 11.1 Evidence that a lift might exist

**(1) The target already has Hecke theory.** Since Hecke operators act naturally on quasimodular forms, the right-hand side of the desired identity

$$\langle \widetilde{T}_m f \rangle_q = T_m \langle f \rangle_q$$

is perfectly well defined. So the question is not blocked at the target level. There is a real and established structure that one could hope to mirror.

**(2) The $q$-bracket preserves weight.** If $f \in \Lambda^*$ is homogeneous of weight $k$, then $\langle f \rangle_q$ is quasimodular of weight $k$. Hecke operators also preserve weight. This means the most basic grading is already compatible with the idea of a lift. That kind of compatibility is not enough by itself, but it is exactly the sort of first sign one wants:

$$\Lambda_k^* \xrightarrow{\langle\cdot\rangle_q} QM_k \xrightarrow{T_m} QM_k.$$

**(3) The partition algebra already carries meaningful operators.** Zagier's proof shows that the $q$-bracket interacts with differential operators on $\Lambda^*$ in a nontrivial way. So the partition algebra is not just a passive source of examples; it already supports operator-theoretic structure. Once one knows that some operators survive through the $q$-bracket, it is reasonable to ask whether Hecke operators might also survive.

**(4) There is representation-theoretic evidence.** The Bloch–Okounkov theorem is tied to the infinite wedge picture, and Zagier's exposition places the story near structures related to $\mathfrak{sl}_2$. These are exactly the kinds of contexts in which hidden symmetries often appear. This does not prove a Hecke lift exists, but it makes the idea feel much less implausible.

**(5) Low-weight examples suggest the bridge is not accidental.** The fact that very basic partition statistics already recover standard quasimodular forms, beginning with $E_2$, suggests that the connection between the two worlds is structural rather than coincidental. When the first few generators already land in canonical objects on the quasimodular side, it becomes harder to dismiss the correspondence as a collection of isolated miracles.

A compact summary of the evidence is

$$\big(\text{graded map}\big)+\big(\text{operator compatibility}\big)+\big(\text{canonical low-weight outputs}\big) \Longrightarrow \text{a lift is at least plausible.}$$

## 11.2 Obstacles that make the problem hard

**(1) The $q$-bracket is not injective.** Even if $T_m\langle f\rangle_q$ is known, there may be many different elements of $\Lambda^*$ with the same $q$-bracket. So a lift, if it exists, may not be unique without extra conditions. In symbols,

$$\langle f\rangle_q = \langle g\rangle_q \not\Rightarrow f = g.$$

This is a serious conceptual difficulty: the target does not determine the source uniquely.

**(2) The $q$-bracket is not a ring homomorphism.** In general one does *not* have

$$\langle fg\rangle_q = \langle f\rangle_q \, \langle g\rangle_q.$$

This means one cannot simply transport the Hecke action back to the partition algebra by formal algebraic arguments. If the map preserved all the relevant algebraic structure, one might try to define the lift abstractly. But because it does not, any lift would need genuinely new input.

**(3) Hecke operators are global, while partition statistics are local.** Classical Hecke operators come from averaging over lattices or double cosets, which are global arithmetic constructions. By contrast, shifted symmetric functions are defined as statistics of individual partitions. It is not obvious what combinatorial process should play the role of a Hecke correspondence. This may be the deepest conceptual obstacle: the two theories organize their data in fundamentally different ways.

**(4) Quasimodular forms have depth, but $\Lambda^*$ is graded mainly by weight.** Since depth plays a role on the quasimodular side, a successful lift may require some additional filtration or structure on the partition side that is not yet understood. Symbolically, the target remembers

$$(k,\, p) = \big(\text{weight},\ \text{depth}\big),$$

while the source is naturally organized mainly by $k$.

**(5) Even the right formulation may need adjustment.** It may turn out that the correct lifting statement holds only modulo the kernel of the $q$-bracket, or only on a well-chosen subspace of $\Lambda^*$, rather than on the whole algebra. In other words, the right conjecture may be weaker or more refined than the most naive one.

Taken together, these obstacles explain why the problem remains open. The idea is natural, but the available structures do not align in a way that makes the lift automatic. Any solution would have to explain not only why a Hecke shadow exists on the partition side, but also what form it should take and how much ambiguity one must allow.

# 12 Most Reasonable Next Steps

Based on what is known so far, a full solution probably does not begin by trying to define Hecke operators on all of $\Lambda^*$ at once. A more realistic approach would be to proceed in stages.

A first step would be to test the problem on low-weight generators such as $Q_2$, $Q_3$, and $Q_4$, looking for partition statistics whose $q$-brackets match the Hecke transforms of the corresponding quasimodular forms. Even a few explicit examples could reveal the right pattern. If one can identify what should play the role of $\widetilde{T}_p(Q_2)$ for a prime $p$, that might already suggest a general combinatorial mechanism. Symbolically, one would like to compute

$$T_p\langle Q_2\rangle_q = T_p\left(-\frac{E_2}{24}\right)$$

and then search for some $F_p \in \Lambda^*$ such that

$$\langle F_p\rangle_q = T_p\langle Q_2\rangle_q.$$

A second step would be to work modulo the kernel of the $q$-bracket. Since noninjectivity is one of the main obstructions, the quotient $\Lambda^*/\ker\langle\cdot\rangle_q$ may be a more natural place to search for a lifted action. Even if a canonical lift does not exist on the entire partition algebra, it might exist on the quotient where the ambiguity has been removed.

A third step would be to require compatibility with the differential operators already visible in Zagier's proof. If a Hecke lift exists, it should almost certainly interact coherently with the structures that are already known to survive through the $q$-bracket. One possible compatibility condition would resemble

$$\langle \delta \widetilde{T}_m f\rangle_q \stackrel{?}{=} D\, T_m\langle f\rangle_q + \cdots .$$

In many areas of mathematics, the right object is characterized not by one property in isolation but by the way several structures coexist. This may be one of those situations.

A fourth step would be to ask whether the lift should preserve some additional filtration on $\Lambda^*$. Because quasimodular forms carry both weight and depth, the partition side may need a second grading or filtration that has not yet been isolated clearly enough. Finding the correct notion of depth on the partition side could be part of the problem rather than a detail to be addressed afterward.

Finally, it may be necessary to weaken the goal. Instead of a perfectly defined Hecke action on the entire partition algebra, one might first look for a partial lift, a filtered lift, or a lift on a distinguished subspace. Even such a result would already be meaningful. In fact, partial results of this sort often reveal the true shape of a problem more clearly than an all-at-once conjecture.

## 13 Conclusion

The main point of this paper is that the Hecke lifting problem is not artificial. It arises naturally once one puts together three facts:

$$\boxed{\text{modular forms carry Hecke operators}} \quad + \quad \boxed{\text{quasimodular forms are Hecke-stable and derivative-stable}}$$

Together these suggest the possibility of a partition-side shadow of Hecke symmetry.

Zagier's treatment of the Bloch–Okounkov theorem makes this picture especially compelling because it reveals that the partition algebra already carries nontrivial structure that survives through the $q$-bracket. Once that is visible, it becomes reasonable to ask whether Hecke symmetry might survive too.

At the moment, the answer is not known. The surrounding theory gives real evidence that some lift could exist, but it also shows clearly why the problem is difficult: the $q$-bracket is not injective, the algebraic structures on the two sides are not identical, and the global arithmetic nature of Hecke operators has no obvious partition-theoretic model.

Still, that tension is exactly what makes the problem interesting. A successful lift would not simply extend an existing theory; it would reveal new arithmetic structure inside the partition algebra and strengthen the conceptual bridge between partitions and automorphic forms. Even if the final answer turns out to be weaker than a full lift, understanding why would already tell us something important about the limits and shape of the Bloch–Okounkov correspondence.

More broadly, the problem illustrates a recurring theme in modern mathematics: when two theories are connected by a map that preserves more structure than expected, that map often points beyond itself. The Bloch–Okounkov theorem is already a beautiful result. The Hecke lifting problem asks whether it is also the visible edge of a larger and deeper symmetry.

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