Bailey–Borwein–Plouffe Formulas

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1 Introduction

In 1995, Simon Plouffe, David Bailey, and Peter Borwein discovered and proved the following formula for π [BBP97]:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{1}$$

The formula also allows us to compute the nth hexadecimal (base-16) digit of π directly, without computing any of the preceding $n-1$ digits, which was an important discovery.

The formula belongs to a broader family of BBP type formulas, which express mathematical constants as series of a specific form.

2 Proof of the BBP Formula for π

Our proof follows the approach of Adamchik and Wagon [AW97]. The idea is to express both π and the BBP series as the same definite integral. We define the following family of integrals.

Definition 2.1. For a positive integer i, define

$$g(i) = 2^{i/2} \int_0^{1/\sqrt{2}} \frac{x^{i-1}}{1-x^8}\, dx. \tag{2}$$

Lemma 2.2. *For each positive integer i,*

$$g(i) = \sum_{k=0}^{\infty} \frac{1}{16^k(8k+i)}.$$

Proof. Since $0 \le x < 1/\sqrt{2} < 1$, we have $|x^8| < 1$, so the geometric series $\frac{1}{1-x^8} = \sum_{k=0}^{\infty} x^{8k}$

converges uniformly on $[0, 1/\sqrt{2}]$. Therefore

$$\begin{aligned}
g(i) &= 2^{i/2} \int_0^{1/\sqrt{2}} x^{i-1} \sum_{k=0}^{\infty} x^{8k}\, dx = 2^{i/2} \sum_{k=0}^{\infty} \int_0^{1/\sqrt{2}} x^{8k+i-1}\, dx \\
&= 2^{i/2} \sum_{k=0}^{\infty} \frac{(1/\sqrt{2})^{8k+i}}{8k+i} = 2^{i/2} \sum_{k=0}^{\infty} \frac{2^{-(8k+i)/2}}{8k+i} \\
&= \sum_{k=0}^{\infty} \frac{2^{-4k}}{8k+i} = \sum_{k=0}^{\infty} \frac{1}{16^k(8k+i)}. \square
\end{aligned}$$

Theorem 2.3.

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

Proof. By Lemma 2.2, the right-hand side equals $4\,g(1) - 2\,g(4) - g(5) - g(6)$, which by the definition of $g(i)$ equals

$$I = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}\,x^4 - 8x^5}{1 - x^8}\, dx.$$

Substitute $x = t/\sqrt{2}$, so $dx = dt/\sqrt{2}$ and $x^8 = t^8/16$. The limits become 0 to 1, and $1 - x^8 = (16 - t^8)/16$. Collecting the factors of $\sqrt{2}$ in the numerator:

$$I = \int_0^1 \frac{16(4 - 2t^3 - t^4 - t^5)}{16 - t^8}\, dt.$$

We now factor the numerator polynomial. Observe that $t = 1$ is a root of $t^5 + t^4 + 2t^3 - 4$, and polynomial division gives

$$t^5 + t^4 + 2t^3 - 4 = (t-1)(t^4 + 2t^3 + 4t^2 + 4t + 4).$$

The quartic factor splits as $(t^2 + 2)(t^2 + 2t + 2)$. Therefore

$$4 - 2t^3 - t^4 - t^5 = (1-t)(t^2+2)(t^2+2t+2).$$

The denominator factors as $16 - t^8 = (2-t^2)(2+t^2)(t^4+4)$, and $t^4 + 4 = (t^2-2t+2)(t^2+2t+2)$. Substituting these factorizations:

$$\frac{16(1-t)(t^2+2)(t^2+2t+2)}{(2-t^2)(2+t^2)(t^2-2t+2)(t^2+2t+2)}.$$

The factors (t^2+2) and (t^2+2t+2) cancel between numerator and denominator, leaving

$$I = \int_0^1 \frac{16(1-t)}{(2-t^2)(t^2-2t+2)}\, dt. \tag{3}$$

We decompose this by partial fractions. Writing $2-t^2 = -(t^2-2)$, we want A, B, C, D such that

$$\frac{-16(1-t)}{(t^2-2)(t^2-2t+2)} = \frac{At+B}{t^2-2} + \frac{Ct+D}{t^2-2t+2}.$$

Multiplying both sides by $(t^2-2)(t^2-2t+2)$ and expanding:

$$-16(1-t) = (At+B)(t^2-2t+2) + (Ct+D)(t^2-2).$$

Comparing coefficients of t^3, t^2, t^1, t^0:

$$\begin{aligned} t^3: &\quad A+C &&= 0,\\ t^2: &\quad B-2A+D &&= 0,\\ t^1: &\quad 2A-2B-2C &&= 16,\\ t^0: &\quad 2B-2D &&= -16. \end{aligned}$$

From the first equation, $C=-A$. Substituting into the third: $4A-2B=16$, so $2A-B=8$. From the fourth: $B-D=-8$, so $D=B+8$. Substituting into the second: $B-2A+B+8=0$, giving $B=A-4$. Then $2A-(A-4)=8$ yields $A=4$. Therefore $B=0$, $C=-4$, $D=8$, and

$$I = \int_0^1 \frac{4t}{t^2-2}\,dt + \int_0^1 \frac{-4t+8}{t^2-2t+2}\,dt.$$

For the first integral:

$$\int_0^1 \frac{4t}{t^2-2}\,dt = 2\Big[\log|t^2-2|\Big]_0^1 = 2(\log 1 - \log 2) = -2\log 2.$$

For the second integral, complete the square: $t^2-2t+2 = (t-1)^2+1$, and write $-4t+8 = -2(2t-2)+4$, so

$$\begin{aligned} \int_0^1 \frac{-4t+8}{t^2-2t+2}\,dt &= -2\int_0^1 \frac{2t-2}{t^2-2t+2}\,dt + 4\int_0^1 \frac{dt}{(t-1)^2+1}\\ &= -2\Big[\log(t^2-2t+2)\Big]_0^1 + 4\Big[\arctan(t-1)\Big]_0^1\\ &= -2(\log 1 - \log 2) + 4(\arctan 0 - \arctan(-1))\\ &= 2\log 2 + 4\cdot\frac{\pi}{4} = 2\log 2 + \pi. \end{aligned}$$

Adding the two integrals:

$$I = -2\log 2 + 2\log 2 + \pi = \pi. \qquad \square$$

3 Polylogarithms and BBP Formulas

The BBP formula for π is motivated by a family of functions called polylogarithms.

Definition 3.1. The *polylogarithm* of order s is

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \qquad |z| \le 1.$$

When $s = 1$, we see why this function is related to logarithms.

$$\mathrm{Li}_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1-z), \qquad |z| < 1.$$

This follows from the Taylor expansion $-\log(1-z) = \sum_{k=1}^{\infty} z^k/k$, which converges for $|z| < 1$. When $s = 2$, the function $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} z^k/k^2$ is the *dilogarithm*. At $z = 1$, the polylogarithm is the Riemann zeta function: $\mathrm{Li}_s(1) = \sum_{k=1}^{\infty} 1/k^s = \zeta(s)$ for $\mathrm{Re}(s) > 1$.

Definition 3.2. The *BBP P-function* is defined as

$$P(s,b,m,A) = \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{m} \frac{a_j}{(mk+j)^s}, \tag{4}$$

where s, b, m are positive integers and $A = (a_1, a_2, \ldots, a_m)$ are rational coefficients.

In this notation, the BBP formula for π is

$$\pi = P(1, 16, 8, (4, 0, 0, -2, -1, -1, 0, 0)).$$

The connection between the P-function and other functions is the *Lerch transcendent*:

Definition 3.3. The *Lerch transcendent* is

$$\Phi(z,s,a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}, \qquad |z| < 1,\ a \neq 0, -1, -2, \ldots$$

This generalizes the polylogarithm: $\Phi(z, s, 1) = \mathrm{Li}_s(z)/z$.

Proposition 3.4. *The BBP P-function can be expressed as*

$$P(s,b,m,A) = \frac{1}{m^s} \sum_{j=1}^{m} a_j \, \Phi\left(\frac{1}{b},\ s,\ \frac{j}{m}\right).$$

Proof. Starting from the definition (4),

$$P(s,b,m,A) = \sum_{j=1}^{m} a_j \sum_{k=0}^{\infty} \frac{1}{b^k (mk+j)^s} = \sum_{j=1}^{m} \frac{a_j}{m^s} \sum_{k=0}^{\infty} \frac{(1/b)^k}{(k+j/m)^s} = \frac{1}{m^s} \sum_{j=1}^{m} a_j \, \Phi\left(\frac{1}{b}, s, \frac{j}{m}\right).$$

□

Furthermore, the Lerch transcendent can itself be decomposed in terms of polylogarithms evaluated at roots of unity:

Proposition 3.5. *Let $\omega = e^{2\pi i/m}$ be a primitive mth root of unity. Then*

$$\Phi(z^m, s, j/m) = \frac{m^s}{m} \sum_{\ell=0}^{m-1} \omega^{-j\ell} \, \mathrm{Li}_s(z\,\omega^{\ell}).$$

The proof follows from a standard computation with roots of unity; see [BBG04]. This identity shows that every BBP-type formula is a statement about polylogarithms evaluated at roots of unity. Such evaluations have been extensively studied; see also [Bro98] for related computations involving polylogarithms and mathematical constants.

4 Other BBP-Type Formulas

Theorem 4.1.

$$\log 2 = \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k}.$$

Proof. The Taylor series $-\log(1-x) = \sum_{k=1}^{\infty} x^k/k$ converges for $|x| < 1$. Setting $x = 1/2$:

$$-\log\left(\frac{1}{2}\right) = \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k}.$$

Since $-\log(1/2) = \log 2$, the result follows. □

Theorem 4.2.

$$8 \arctan \frac{1}{2} = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{4k+1} - \frac{1}{4k+3} \right).$$

Proof. The Taylor series for arctangent,

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k \, x^{2k+1}}{2k+1}, \qquad |x| \le 1,$$

converges absolutely on $[-1, 1]$. Setting $x = 1/2$:

$$\arctan \frac{1}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\, 2^{2k+1}}.$$

We split according to parity of k. For even $k = 2j$, $(-1)^k = 1$ and $2^{2k+1} = 2^{4j+1}$; for odd $k = 2j+1$, $(-1)^k = -1$ and $2^{2k+1} = 2^{4j+3}$. Therefore

$$\begin{aligned} \arctan \frac{1}{2} &= \sum_{j=0}^{\infty} \frac{1}{(4j+1)\, 2^{4j+1}} - \sum_{j=0}^{\infty} \frac{1}{(4j+3)\, 2^{4j+3}} \\ &= \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{16^j (4j+1)} - \frac{1}{8} \sum_{j=0}^{\infty} \frac{1}{16^j (4j+3)} = \sum_{j=0}^{\infty} \frac{1}{16^j} \left(\frac{1}{2(4j+1)} - \frac{1}{8(4j+3)} \right). \end{aligned}$$

Multiplying through by 8 gives the stated formula. □

5 The Digit-Extraction Algorithm

The most important consequence of the BBP formula is that it enables computation of the nth hexadecimal digit of π without computing any of the preceding digits. We now explain how this works.

We first state and prove the algorithm in full generality, then apply it to the BBP formula for π specifically.

Theorem 5.1 (BBP Digit Extraction)**.** *Let $b \geq 2$ and $c \geq 1$ be positive integers, and let $p(k)$ be a polynomial taking positive integer values for all integers $k \geq 0$. Define*

$$S = \sum_{k=0}^{\infty} \frac{1}{p(k)\, b^{ck}}.$$

The nth base-b digit of S after the decimal point can be computed without computing any of the preceding $n-1$ digits. The value is equal to $\lfloor b \cdot \{b^n S\} \rfloor$, where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part.

Proof. Multiplying the series through by b^n:

$$b^n S = \sum_{k=0}^{\infty} \frac{b^{n-ck}}{p(k)}.$$

Set $k^* = \lfloor n/c \rfloor$. For $k \leq k^*$ the exponent $n - ck \geq 0$, so b^{n-ck} is a positive integer; for $k > k^*$ the exponent $n - ck < 0$, so $0 < b^{n-ck} < 1$. Splitting the sum with these rules gives us:

$$b^n S = \underbrace{\sum_{k=0}^{k^*} \frac{b^{n-ck}}{p(k)}}_{\text{finite part}} + \underbrace{\sum_{k=k^*+1}^{\infty} \frac{b^{n-ck}}{p(k)}}_{\text{tail}}.$$

For the finite sum, since integers contribute nothing to the fractional part,

$$\left\{\sum_{k=0}^{k^*} \frac{b^{n-ck}}{p(k)}\right\} = \left\{\sum_{k=0}^{k^*} \frac{b^{n-ck} \bmod p(k)}{p(k)}\right\}.$$

Each $b^{n-ck} \bmod p(k)$ is computed by fast modular exponentiation in $O(\log n)$ multiplications. Summing the $k^* + 1 \leq \lfloor n/c \rfloor + 1$ terms therefore requires $O(n \log n)$ operations in total [BBP97].

For the tail, we write $n = k^* c + r$ with $0 \leq r < c$. For $k \geq k^* + 1$ the exponent satisfies $n - ck \leq r - c < 0$, so the tail is bounded by a convergent geometric series:

$$\sum_{k=k^*+1}^{\infty} \frac{b^{n-ck}}{p(k)} \leq \sum_{\ell=0}^{\infty} b^{r-c-c\ell} = \frac{b^{r-c}}{1-b^{-c}} = \frac{b^r}{b^c - 1} \leq \frac{b^{c-1}}{b^c - 1} < \frac{1}{b-1}.$$

Because the ratio $b^{-c} < 1$, summing a fixed small number of tail terms yields sufficient precision in floating-point arithmetic.

Taking the fractional part of the sum of the finite part and the tail gives $\{b^n S\}$. To see why this yields the nth base-b digit, write S in base b as $S = \lfloor S \rfloor . d_0\, d_1\, d_2 \cdots$, so that

$$\{S\} = \sum_{i=0}^{\infty} d_i\, b^{-(i+1)}, \qquad d_i \in \{0, \ldots, b-1\}.$$

Multiplying by b^n shifts the decimal point n places to the right:

$$b^n S = \underbrace{b^n \lfloor S \rfloor + \sum_{i=0}^{n-1} d_i\, b^{n-1-i}}_{\text{integer}} + \underbrace{\sum_{i=n}^{\infty} d_i\, b^{n-1-i}}_{\text{fractional part}=\{b^n S\}} .$$

Therefore $\{b^n S\} = 0.d_n\, d_{n+1} \cdots$ in base b. Multiplying by b gives $b\{b^n S\} = d_n.d_{n+1} \cdots$, and since the part after the decimal point satisfies

$$0 \le 0.d_{n+1} d_{n+2} \cdots < 1,$$

taking the floor isolates exactly d_n:

$$\lfloor b \cdot \{b^n S\} \rfloor = d_n. \qquad \square$$

We now apply Theorem 5.1 to the BBP formula (1) for π.

Suppose we wish to find the $(n+1)$th hexadecimal digit of π (with $n = 0$ corresponding to the first digit after the hexadecimal point). We can compute the fractional part $\{16^n \pi\}$, i.e., the quantity $16^n \pi - \lfloor 16^n \pi \rfloor$, and then the desired digit is $\lfloor 16 \cdot \{16^n \pi\} \rfloor$.

Multiplying the BBP formula by 16^n:

$$16^n \pi = 4 \underbrace{\sum_{k=0}^{\infty} \frac{16^{n-k}}{8k+1}}_{S_1} - 2 \underbrace{\sum_{k=0}^{\infty} \frac{16^{n-k}}{8k+4}}_{S_4} - \underbrace{\sum_{k=0}^{\infty} \frac{16^{n-k}}{8k+5}}_{S_5} - \underbrace{\sum_{k=0}^{\infty} \frac{16^{n-k}}{8k+6}}_{S_6} .$$

We compute the fractional part of each S_j separately. Split S_j into a *finite* sum (where 16^{n-k} is a large integer) and a *tail* (where $16^{n-k} < 1$):

$$S_j = \underbrace{\sum_{k=0}^{n} \frac{16^{n-k}}{8k+j}}_{\text{finite part}} + \underbrace{\sum_{k=n+1}^{\infty} \frac{16^{n-k}}{8k+j}}_{\text{tail}} .$$

For the finite sum, we only need the fractional part of each term:

$$\left\{ \frac{16^{n-k}}{8k+j} \right\} = \frac{16^{n-k} \bmod (8k+j)}{8k+j}.$$

The tail is negligible for large n, and summing a handful of terms gives sufficient precision.

Combining the four sums:

$$\{16^n \pi\} = \{4\,\{S_1\} - 2\,\{S_4\} - \{S_5\} - \{S_6\}\}.$$

This holds because replacing each S_j by $\{S_j\}$ changes the linear combination $4S_1 - 2S_4 - S_5 - S_6$ by an integer: each difference $S_j - \{S_j\} = \lfloor S_j \rfloor$ is an integer, and the coefficients $4, -2, -1, -1$ are integers, so their contribution to the fractional part is zero. The $(n+1)$th hexadecimal digit of π is then $\lfloor 16 \cdot \{16^n \pi\} \rfloor$.

Example 5.2. We can compute the second hexadecimal digit of π after the hexadecimal point ($n = 1$). Since $\pi = 3.243\text{F}6\text{A}\ldots_{16}$, we should expect the digit 4.

For each $j \in \{1, 4, 5, 6\}$, the finite sum has two terms ($k = 0$ and $k = 1$), and we compute $16^{1-k} \bmod (8k + j)$:

j	$k = 0$: $16 \bmod j$	$k = 1$: $1 \bmod (8 + j)$	$\{S_j\}$ (with tail)
1	$16 \bmod 1 = 0$	$1 \bmod 9 = 1$	0.1150
4	$16 \bmod 4 = 0$	$1 \bmod 12 = 1$	0.0866
5	$16 \bmod 5 = 1$	$1 \bmod 13 = 1$	0.2800
6	$16 \bmod 6 = 4$	$1 \bmod 14 = 1$	0.7411

For instance, $\{S_1\}$ is computed as $\{0/1 + 1/9\} + \text{tail} = 0.1111 + 0.0038 = 0.1150$ (the tail contributes $\sum_{k=2}^{\infty} 16^{1-k}/(8k + 1) \approx 0.0038$). Similarly, $\{S_6\}$ starts with $\{4/6 + 1/14\} = \{0.7381\}$, plus a tail of ≈ 0.0030. Combining:

$$4(0.1150) - 2(0.0866) - 0.2800 - 0.7411 = 0.4598 - 0.1732 - 0.2800 - 0.7411 = -0.7345.$$

The fractional part is $\{-0.7345\} = 0.2655$, and $\lfloor 16 \times 0.2655 \rfloor = \lfloor 4.248 \rfloor = 4$, confirming the expected digit. This algorithm can be extended to the other BBP formulas.

References

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