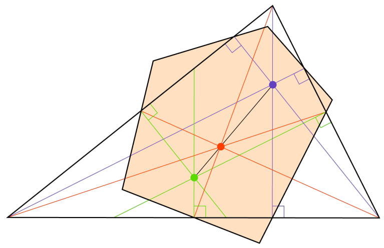

The Euler Pentagon

Euler Circle? More like Euler Pentagon!

Three Increasingly Fun Proofs of Euler's Pentagonal Number Theorem

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1. Summary

This paper is about Euler's Pentagonal Number Theorem. Here's the statement:

Theorem 1.1. *(Euler's Pentagonal Number Theorem.)*

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k\in\mathbb{Z}}(-1)^k x^{\frac{k(3k-1)}{2}}.$$

Remark 1.2. One immediately striking aspect of this theorem is the huge amount of cancellation that occurs in the product. We have every integer power of x in the product form, and somehow very few powers of x emerge in the sum form!

We'll explore three proofs of Euler's Pentagonal Number Theorem, in increasing order of fun.

1. **Euler's Original Proof.** Euler originally proved the theorem by spamming a bunch of algebra (clever algebra, of course, because he's Euler, but nonetheless just a bunch of algebra).

2. **A Combinatorics Side Quest.** Let's go on a side quest to prove a theorem about the integer partition function. Using this side quest, we can show that this implies Euler's Pentagonal Theorem. It also turns out that the sum-of-divisors function follows the exact same recurrence, and we'll prove this too!

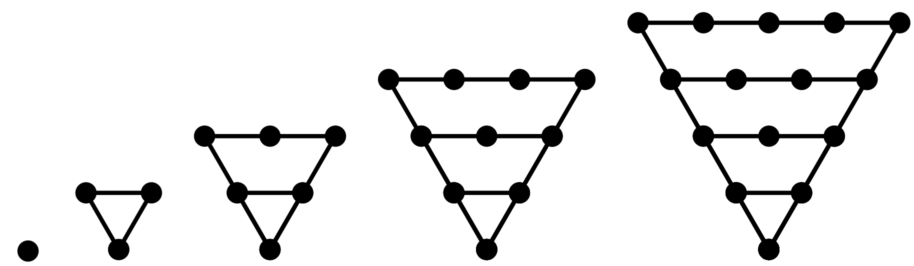
3. **A Tiling Excursion.** This is the least algebraic and most magical proof of all: In 2018, Eichhorn, Nam, & Sohn proved the theorem by enumerating tilings two different ways.

2. This paper is a scam! I was promised pentagons!

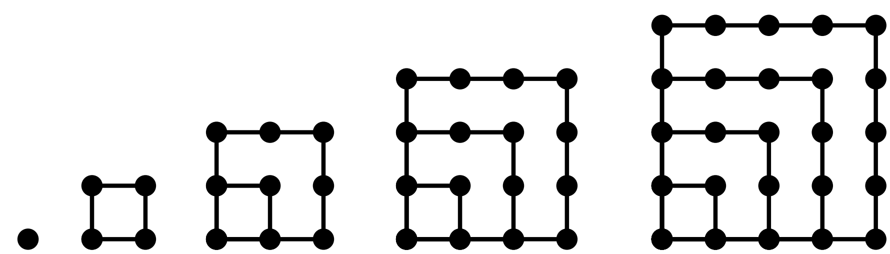
Okay, okay, we're getting to that.

We're already familiar with the triangular numbers $T_n = \frac{n(n+1)}{2}$ and square numbers n^2. How might we generalize these concepts to define pentagonal numbers?

Aptly, we may geometrically visualize these numbers as counting the number of points on n coincident triangles or squares, as shown below.

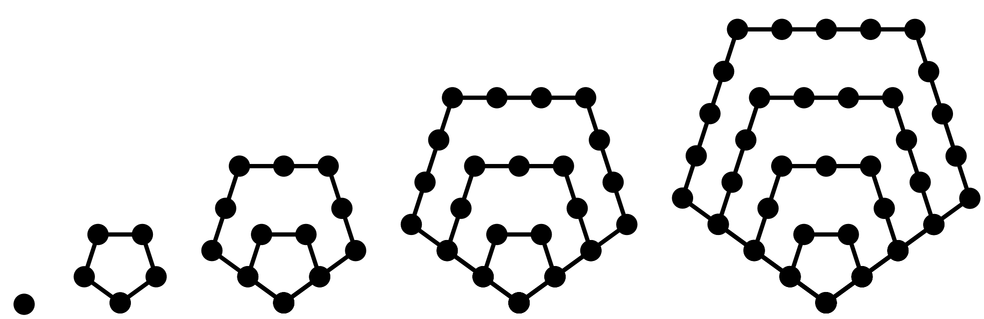


T_1 T_2 T_3 T_4 T_5



1^2 2^2 3^2 4^2 5^2

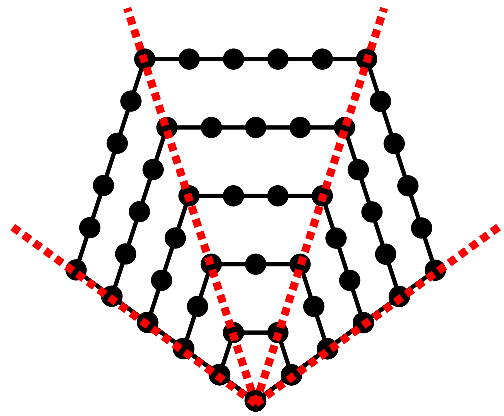
We can now extend this to pentagons!



P_1 P_2 P_3 P_4 P_5

Of course, we do want an algebraic way of counting all these points. Here's one method of counting them: cut up the pentagon along the shown red lines.

Now, each of the 4 red lines contains n points (and the bottommost point is over-counted 3 times). Each of the 3 areas bounded by the red lines is a triangle with $T_{n-2} = \frac{(n-2)(n-1)}{2}$ points. So in total, we have



$$\begin{aligned} P_n &= 3 \cdot \frac{(n-2)(n-1)}{2} + 4n - 3 \\ &= \frac{3n^2 - 9n + 6}{2} + \frac{8n - 6}{2} \\ &= \frac{3n^2 - n}{2} \\ &= \frac{n(3n-1)}{2} \end{aligned}$$

This gives rise to our official definition for the pentagonal numbers.

Definition 2.1 (Pentagonal Number). The *pentagonal numbers* P_n are defined by

$$P_n = \frac{n(3n-1)}{2}.$$

Remark 2.2. Now that our definition is entirely algebraic, we have broken free from our geometric motivation and may plug negative n into the formula.

We can now state Euler's Pentagonal Number Theorem in terms of pentagonal numbers!

Theorem 2.3. *(Euler's **Pentagonal** Number Theorem)*

$$\prod_{n=1}^{\infty}(1 - x^n) = \sum_{k \in \mathbb{Z}} (-1)^k x^{P_k}.$$

3. Euler's Original Proof

Let's go on an algebraic journey, following what Euler did in [2]! We'll start by introducing a surprise tool that will help us later.

Theorem 3.1. *(Surprise Tool.) Let*

$$S_i = \sum_{n=0}^{\infty} \left(x^{in} \prod_{k=i}^{n+i} (1 - x^k) \right).$$

Then, we have

$$S_i = 1 - x^{2i+1} - S_{i+1} x^{3i+2}.$$

Proof: We make a giant equality chain (or, as I like to call it, the "Great Equate"). We have

$$\begin{aligned}
S_i &= \sum_{n=0}^{\infty}\left(x^{in}\prod_{k=i}^{n+i}(1-x^k)\right)\\
&= (1-x^i)\sum_{n=0}^{\infty}\left(x^{in}\prod_{k=i+1}^{n+i}(1-x^k)\right)\\
&= \sum_{n=0}^{\infty}\left(x^{in}\prod_{k=i+1}^{n+i}(1-x^k)\right) - x^i\sum_{n=0}^{\infty}\left(x^{in}\prod_{k=i+1}^{n+i}(1-x^k).\right)\\
&= \sum_{n=0}^{\infty}\left(x^{in}\prod_{k=i+1}^{n+i}(1-x^k)\right) - \sum_{n=0}^{\infty}\left(x^{i(n+1)}\prod_{k=i+1}^{n+i}(1-x^k)\right)\\
&= \sum_{n=0}^{\infty}\left(x^{in}\prod_{k=i+1}^{n+i}(1-x^k)\right) - \sum_{n=1}^{\infty}\left(x^{in}\prod_{k=i+1}^{n+i-1}(1-x^k)\right)\\
&= 1+\sum_{n=1}^{\infty}\left(x^{in}\prod_{k=i+1}^{n+i}(1-x^k)\right) - \sum_{n=1}^{\infty}\left(x^{in}\prod_{k=i+1}^{n+i-1}(1-x^k)\right)\\
&= 1+\sum_{n=1}^{\infty}\left[\left(x^{in}\prod_{k=i+1}^{n+i}(1-x^k)\right) - \left(x^{in}\prod_{k=i+1}^{n+i-1}(1-x^k)\right)\right]\\
&= 1+\sum_{n=1}^{\infty}\left[x^{in}\left(\prod_{k=i+1}^{n+i}(1-x^k) - \prod_{k=i+1}^{n+i-1}(1-x^k)\right)\right]\\
&= 1+\sum_{n=1}^{\infty}\left[x^{in}((1-x^{n+i})-1)\left(\prod_{k=i+1}^{n+i-1}(1-x^k)\right)\right]\\
&= 1+\sum_{n=1}^{\infty}\left[x^{in}(-x^{n+i})\left(\prod_{k=i+1}^{n+i-1}(1-x^k)\right)\right]\\
&= 1-\sum_{n=1}^{\infty}\left[x^{in+n+i}\left(\prod_{k=i+1}^{n+i-1}(1-x^k)\right)\right]
\end{aligned}$$

Intermission. I suggest doing 10 jumping jacks or something.

⋮

Okay, let's continue our Great Equate.

$$\begin{aligned}
S_i &= 1 - \sum_{n=1}^{\infty} \left[x^{in+n+i} \left(\prod_{k=i+1}^{n+i-1} (1-x^k) \right) \right] \\
&= 1 - x^{2i+1} - \sum_{n=2}^{\infty} \left[x^{in+n+i} \left(\prod_{k=i+1}^{n+i-1} (1-x^k) \right) \right] \\
&= 1 - x^{2i+1} - x^{3i+2} \sum_{n=2}^{\infty} \left[x^{in+n-2i-2} \left(\prod_{k=i+1}^{n+i-1} (1-x^k) \right) \right] \\
&= 1 - x^{2i+1} - x^{3i+2} \sum_{n=2}^{\infty} \left[x^{(n-2)+i(n-2)} \left(\prod_{k=i+1}^{(n-2)+i+1} (1-x^k) \right) \right] \\
&= 1 - x^{2i+1} - x^{3i+2} \sum_{n=0}^{\infty} \left[x^{n+in} \left(\prod_{k=i+1}^{n+i+1} (1-x^k) \right) \right] \\
&= 1 - x^{2i+1} - x^{3i+2} \sum_{n=0}^{\infty} \left[x^{(i+1)n} \left(\prod_{k=i+1}^{n+i+1} (1-x^k) \right) \right] \\
&= 1 - x^{2i+1} - x^{3i+2} S_{i+1}.
\end{aligned}$$

Our surprise proof is complete. ■

We need one more identity before proving the main theorem.

Claim 3.2.

$$\prod_{n=1}^{\infty} (1-x^n) = 1 - x - S_1 x^2.$$

Proof: First, let's prove that for all $i \geq 1$,

$$\begin{aligned}
\prod_{n=1}^{i} (1-x^n) &= 1 - x - x^2 \sum_{n=0}^{i-2} \left(x^n \prod_{k=1}^{n+1} (1-x^k) \right) \\
&= 1 - x + \sum_{n=0}^{i-2} \left(-x^{n+2} \prod_{k=1}^{n+1} (1-x^k) \right).
\end{aligned}$$

We proceed by induction on i.

For the base case, when $i = 1$, our equation becomes $1 - i = 1 - i$, which is definitely true.

Now assume inductively that that our equation is true for $i = j - 1$; we want to prove that our equation is true for $i = j$.

We have

$$\begin{aligned}\prod_{n=1}^{j}(1-x^n) &= (1-x^j)\prod_{n=1}^{j-1}(1-x^n)\\ &= \prod_{n=1}^{j-1}(1-x^n) - x^j\prod_{n=1}^{j-1}(1-x^n)\\ &= \prod_{n=1}^{j-1}(1-x^n) - x^{(j-2)+2}\prod_{n=1}^{(j-2)+1}(1-x^n).\end{aligned}$$

We know by the inductive hypothesis that

$$\prod_{n=1}^{j-1}(1-x^n) = 1-x+\sum_{n=0}^{j-3}\left(-x^{n+2}\prod_{k=1}^{n+1}(1-x^k)\right),$$

so then

$$\begin{aligned}\prod_{n=1}^{j}(1-x^n) &= \prod_{n=1}^{j-1}(1-x^n) - x^{(j-2)+2}\prod_{n=1}^{(j-2)+1}(1-x^n)\\ &= 1-x+\sum_{n=0}^{j-3}\left(-x^{n+2}\prod_{k=1}^{n+1}(1-x^k)\right) - x^{(j-2)+2}\prod_{n=1}^{(j-2)+1}(1-x^n)\\ &= 1-x+\sum_{n=0}^{j-2}\left(-x^{n+2}\prod_{k=1}^{n+1}(1-x^k)\right),\end{aligned}$$

which is what we wanted, so our inductive step is complete.

Now that we know that

$$\begin{aligned}\prod_{n=1}^{i}(1-x^n) &= 1-x+\sum_{n=0}^{i-2}\left(-x^{n+2}\prod_{k=1}^{n+1}(1-x^k)\right)\\ &= 1-x-x^2\sum_{n=0}^{i-2}\left(x^n\prod_{k=1}^{n+1}(1-x^k)\right)\end{aligned}$$

for all i, we take the limit as $i \to \infty$ to get

$$\prod_{n=1}^{\infty}(1-x^n) = 1-x-S_1x^2.$$

■

At last, we have all the tools we need to prove the main theorem!

Proof: (of Theorem 2.3) Let's begin where we left off from our claim:

$$\prod_{n=1}^{\infty}(1-x^n) = 1-x-S_1x^2$$

We can use our surprise tool to figure out that

$$S_1x^2 = x^2\left(1 - x^3 - S_2x^5\right) = x^2(1-x^3) - S_2x^7,$$
$$S_2x^7 = x^7\left(1 - x^5 - S_3x^8\right) = x^7(1-x^5) - S_2x^{15},$$
$$S_3x^{15} = x^{15}\left(1 - x^7 - S_4x^{11}\right) = x^{15}(1-x^5) - S_2x^{26},$$

and in general,

$$S_kx^{\sum_{n=0}^{k-1}(3n+2)} = x^{\sum_{n=0}^{k-1}(3n+2)}\left(1 - x^{2k+1}\right) - S_{k+1}x^{\sum_{n=0}^{k}(3n+2)}.$$

Since

$$\sum_{n=0}^{k-1}(3n+2) = \frac{k(2+(3(k-1)+2))}{2} = \frac{3k^2+k}{2},$$

we have

$$S_kx^{\sum_{n=0}^{k-1}(3n+2)} = x^{\frac{3k^2+k}{2}}\left(1 - x^{2k+1}\right) - S_{k+1}x^{\sum_{n=0}^{k}(3n+2)}.$$

Repeatedly substituting this equation into

$$\prod_{n=1}^{\infty}(1-x^n) = 1 - x - S_1x^2$$

gives us that

$$\begin{aligned}
\prod_{n=1}^{\infty}(1-x^n) &= 1 - x + \sum_{k=1}^{\infty} x^{\frac{3k^2+k}{2}}(1-x^{2k+1})(-1)^k \\
&= \sum_{k=0}^{\infty} x^{\frac{3k^2+k}{2}}(1-x^{2k+1})(-1)^k \\
&= \sum_{k=0}^{\infty}\left(x^{\frac{3k^2+k}{2}} - x^{\frac{3k^2+5k+2}{2}}\right)(-1)^k \\
&= \sum_{k=0}^{\infty}\left(x^{\frac{(-k)(3(-k)-1)}{2}} - x^{\frac{(k+1)(3(k+1)-1)}{2}}\right)(-1)^k \\
&= \sum_{k=0}^{\infty}\left(x^{P_{-k}} - x^{P_{k+1}}\right)(-1)^k \\
&= \sum_{k=0}^{\infty} x^{P_{-k}}(-1)^k - \sum_{k=0}^{\infty} x^{P_{k+1}}(-1)^k \\
&= \sum_{k=0}^{\infty} x^{P_{-k}}(-1)^k + \sum_{k=1}^{\infty} x^{P_k}(-1)^k \\
&= \sum_{k\in\mathbb{Z}} x^{P_k}(-1)^k.
\end{aligned}$$

And we are done. Hooray! ∎

4. The Integer Partition Proof

I'm a bit tired of all this algebraic manipulation. Let's take a break, and go on a little combinatorics side quest for a while.

Definition 4.1 (Partition). A *partition* of a nonnegative integer n is a way to write n as a sum of positive integers (up to reordering). For example, $1+1+2+3+5$ is a partition of 12 (and it is considered the same parition as $2+5+1+3+1$). We define the *partition function* $p(n)$ to be the number of partitions of n. As an example, we have that $p(4) = 5$, since $1+1+1+1 = 1+1+2 = 1+3 = 2+2 = 4$.

Theorem 4.2. *(A Little Combinatorics Side Quest.)*

$$p(n) = \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} (-1)^{k-1} p(n-P_k)$$

Proof: First, we create a new function: let $p(n,s)$ be the number of partitions of n with the smallest element being s. For example, $2+2+3+5$ is one of the partitions counted in $p(12,2)$. This then means that

$$p(n) = \sum_{k=1}^{n} p(n,k).$$

We can create a partition of n with smallest element s by starting with a partition of $n-s$ with the smallest element being s or greater, then adding s to it. Thus,

$$p(n,s) = \sum_{k=s}^{n} p(n-s,k).$$

From here, we note that

$$\begin{aligned} p(n,s) + p(n-s,s-1) &= p(n-s,s-1) + \sum_{k=s}^{n} p(n-s,k) \\ &= \sum_{k=s-1}^{n} p(n-s,k) \\ &= \sum_{k=s-1}^{n-1} p(n-s,k), \end{aligned}$$

where the last equality is true because when $k=n$, we have $p(n-s,n) = 0$. Then, we have that

$$\begin{aligned} p(n,s) + p(n-s,s-1) &= \sum_{k=s-1}^{n-1} p(n-s,k) \\ &= \sum_{k=s-1}^{n-1} p(n-1-(s-1),k) \\ &= p(n-1,s-1). \end{aligned}$$

Knowing this, we can express any $p(n,s)$ in terms of sums and differences of $p(n-k,1)$ for different values of k by repeatedly applying $p(n,s) = p(n-1,s-1) - p(n-s,s-1)$. For example, we have that

$$\begin{aligned} p(n,3) &= p(n-1,2) - p(n-3,2) \\ &= (p(n-2,1) - p(n-3,1)) - (p(n-4,1) - p(n-5,1)) \\ &= p(n-2,1) - p(n-3,1) - p(n-4,1) + p(n-5,1). \end{aligned}$$

This then means that we can express

$$p(n) = \sum_{k=1}^{n} p(n,k)$$

in terms of sums and differences of $p(n-k,1)$ for some values of k.

To get an arbitrary $\pm p(n-k,1)$ term from a $p(n,s)$ term in our expansion of $p(n)$, we can expand in the following way.

0. Starting with a $p(n,s)$ term, where $s \geq 2$, we choose to subtract either 1 or s from n to get either $p(n-1,s-1)$ or $-p(n-s,s-1)$.

1. We then have another choice, this time between subtracting either 1 or $s-1$.

   ⋮

i. On step i, we choose between subtracting either 1 or $s-i$.

   ⋮

$(s-1)$. To reach $p(n-k,1)$, we have to repeat this choice $s-1$ times, where our last choice is between subtracting 1 or 2.

The sign of $\pm p(n-k,1)$ is determined by the parity of how many times we chose to subtract $s-j$ for some j (and chose not to subtract 1): an even number of times means a positive sign, and an odd number of times means a negative sign.

We wish to enumerate all possible ways of making such a combination of choices. One convenient way of doing this is by representing each with a binary string $b = b_k b_{k-1} \ldots b_3 b_2$, where $b_i = 1$ represents choosing to subtract i on step $(s-i)$ and $b_i = 0$ represents choosing to subtract 1 on step $(s-i)$.

Thus, if we define the function

$$f(b_k b_{k-1} \ldots b_3 b_2) = \sum_{i=2}^{k} i^{b_i},$$

we have that when we follow the choices corresponding to b, we end up with

$$p(n - f(b), 1)(-1)^{\#\text{ of 1s in } b}.$$

This means that

$$p(n) = p(n,1) + \sum_{\text{all } b} p(n - f(b), 1)(-1)^{\#\text{ of 1s in } b}.$$

Let's take a closer look at some properties of f. If we let x be a binary string of any length, we have that for all nonnegative k,

$$\begin{aligned} f(\underbrace{1\cdots1}_{k\ \text{1s}}x1\underbrace{0\cdots0}_{k\ \text{0s}}) &= f(\underbrace{1\cdots1}_{k\ \text{1s}}x0\underbrace{0\cdots0}_{k\ \text{0s}}) + k + 1 \\ &= f(\underbrace{1\cdots1}_{k\ \text{1s}}0x0\underbrace{0\cdots0}_{k\ \text{0s}}) + k + 1 - k - 1 \\ &= f(\underbrace{1\cdots1}_{k\ \text{1s}}0x0\underbrace{0\cdots0}_{k\ \text{0s}}). \end{aligned}$$

Call binary strings in the form $\underbrace{1\cdots1}_{k\ \text{1s}}x1\underbrace{0\cdots0}_{k\ \text{0s}}$ type-I binary strings and those in the form $\underbrace{1\cdots1}_{k\ \text{1s}}0x0\underbrace{0\cdots0}_{k\ \text{0s}}$ type-II binary strings.

In general, all binary strings fall into two categories: either a string is in the form

$$\underbrace{1\cdots1}_{j\ \text{1s}}0x1\underbrace{0\cdots0}_{k\ \text{0s}}$$

for nonnegative j, k and a binary string x, or in the form

$$\underbrace{1\cdots1}_{j\ \text{1s}}\underbrace{0\cdots0}_{k\ \text{0s}}$$

where at least one j, k is positive. (This is true, because a binary string must either contain a substring of 01 (thus corresponding to the first category) or not (which corresponds to the second category).)

If a string is in the form $\underbrace{1\cdots1}_{j\ \text{1s}}0x1\underbrace{0\cdots0}_{k\ \text{0s}}$ and $j \geq k$, the string is type-I. If $j < k$, the string is type-II. If the string is in the form $\underbrace{1\cdots1}_{j\ \text{1s}}\underbrace{0\cdots0}_{k\ \text{0s}}$ and $j \geq k+1$, we have a type-I string. If $j \leq k-2$, we have a type-II string. This means that if a string is not either type-I or type-II, we must have $j = k-1$ or $j = k$. Call these type-III and type-IV strings, respectively.

Let b_{I} be a type-I string in the form $\underbrace{1\cdots1}_{k\ \text{1s}}x1\underbrace{0\cdots0}_{k\ \text{0s}}$ and b_{II} be the corresponding type-II string in the form $\underbrace{1\cdots1}_{k\ \text{1s}}0x0\underbrace{0\cdots0}_{k\ \text{0s}}$, where x is the same binary substring in both b_{I} and b_{II}. We have that $f(b_{\text{I}}) = f(b_{\text{II}})$ and that b_{I} has one more 1 than b_{II}. Because of this,

$$p(n - f(b_{\text{I}}), 1)(-1)^{\#\text{ of 1s in } b_{\text{I}}} + p(n - f(b_{\text{II}}), 1)(-1)^{\#\text{ of 1s in } b_{\text{II}}} = 0.$$

This implies that if a string is either type-I or type-II, it will cancel out with its corresponding pair in the sum. Thus, we only need to compute our sum for the strings in the form $\underbrace{1\cdots1}_{j\text{ 1s}}\underbrace{0\cdots0}_{k\text{ 0s}}$ where $j = k-1$ or $j = k$.

For type-III strings, when $j = k-1$, where $k \geq 1$, we have

$$\begin{aligned} f(\underbrace{1\cdots1}_{k-1\text{ 1s}}\underbrace{0\cdots0}_{k\text{ 0s}}) &= k + \frac{(k-1)(k+2+2k)}{2} \\ &= \frac{3k^2+k-2}{2} \\ &= \frac{-k(3(-k)-1)}{2} - 1 \\ &= P_{-k} - 1. \end{aligned}$$

For type-IV strings, when $j = k$, we have that, for $k \geq 1$,

$$\begin{aligned} f(\underbrace{1\cdots1}_{k\text{ 1s}}\underbrace{0\cdots0}_{k\text{ 0s}}) &= k + \frac{k(k+2+2k+1)}{2} \\ &= \frac{3k^2+5k}{2} \\ &= \frac{(k+1)(3(k+1)-1)}{2} - 1 \\ &= P_{k+1} - 1. \end{aligned}$$

This means that

$$\begin{aligned} p(n) &= p(n,1) + \sum_{\text{all } b}^{\infty} p(n-f(b),1)(-1)^{\#\text{ of 1s in } b} \\ &= p(n,1) + \sum_{k=1}^{\infty} p(n-(P_{-k}-1),1)(-1)^{k-1} + \sum_{k=1}^{\infty} p(n-(P_{k+1}-1),1)(-1)^{k} \\ &= p(n,1) + \sum_{k=1}^{\infty} p(n-(P_{-k}-1),1)(-1)^{k-1} + \sum_{k=2}^{\infty} p(n-(P_{k}-1),1)(-1)^{k-1}. \end{aligned}$$

Notice that $p(n,1) = p(n-1)$, because we can make a partition of $n-1$ by taking a partition of n involving a 1 and removing the 1. Thus,

$$\begin{aligned} p(n) &= p(n,1) + \sum_{k=1}^{\infty} p(n-(P_{-k}-1),1)(-1)^{k-1} + \sum_{k=2}^{\infty} p(n-(P_{k}-1),1)(-1)^{k-1} \\ &= p(n-1) + \sum_{k=1}^{\infty} p(n-P_{-k})(-1)^{k-1} + \sum_{k=2}^{\infty} p(n-P_{k})(-1)^{k-1} \\ &= \sum_{k=1}^{\infty} p(n-P_{-k})(-1)^{k-1} + \sum_{k=1}^{\infty} p(n-P_{k})(-1)^{k-1} \\ &= \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} p(n-P_k)(-1)^{k-1}, \end{aligned}$$

which is what we wanted. Side quest complete! ■

We did it! Let's go back to the Pentagonal Number Theorem world.

Proof: (of Theorem 2.3) We start by finding the generating function of $p(n)$. We see that

$$\sum_{n=0}^{\infty} p(n)x^n = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots,$$

because to get an x^n term from the product, we need to choose terms from each factor that form a valid partition of n. Choosing a x^k from the first factor corresponds to adding 1 repeatedly k times, choosing x^{2k} from the second factor corresponds to adding 2 repeatedly k times, and so on.

This means that

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)x^n &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots \\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdots \\ &= \prod_{n=1}^{\infty}\frac{1}{1-x^n}.\end{aligned}$$

Since we have

$$\left(\prod_{n=1}^{\infty}(1-x^n)\right)\left(\sum_{n=0}^{\infty}p(n)x^n\right) = \left(\prod_{n=1}^{\infty}(1-x^n)\right)\left(\prod_{n=1}^{\infty}\frac{1}{1-x^n}\right) = 1,$$

we know that for some sequence a_i we have that

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=0}^{\infty} a_k \cdot x^k$$

if and only if

$$\begin{aligned} p(0)a_0 &= 1, \\ p(0)a_1 + p(1)a_0 &= 0, \\ p(0)a_2 + p(1)a_1 + p(2)a_0 &= 0 \\ &\cdots, \end{aligned}$$

and in general, for all $n > 0$,

$$\sum_{k=0}^{n} p(n-k)a_k = 0.$$

From our Combinatorics Side Quest, we know that

$$p(n) = \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} (-1)^{k-1} p(n-P_k),$$

which means

$$p(n) + \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} (-1)^k p(n - P_k) = \sum_{k\in\mathbb{Z}} (-1)^k p(n - P_k) = 0.$$

Using this equation to find a sequence a_i such that

$$\sum_{k=0}^{n} p(n-k)a_k = 0$$

for all n, we find that

$$a_k = \begin{cases} (-1)^i & \text{if } k = P_i \text{ for some } i \in \mathbb{Z} \\ 0 & \text{otherwise,} \end{cases}$$

so we end up with

$$\prod_{n=1}^{\infty} (1 - x^n) = \sum_{k\in\mathbb{Z}} (-1)^k x^{P_k},$$

which is what we want! ■

4.1. Corollary: The Sum-of-Divisors Function

Definition 4.3 (Sum-of-Divisors Function). The *sum-of-divisors function* $\sigma(n)$ is the sum of all positive divisors of n for positive n, and 0 for negative n. For example, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.

One corollary of Euler's Pentagonal Number Theorem is a recurrence for the sum-of-divisors function nearly identical to that of the integer partition function. We'll follow how Euler proved this in [3].

Theorem 4.4.

$$\sigma(n) = \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} (-1)^{k+1} \sigma(n - P_k)$$

where we let $\sigma(n - n) = n$.

Remark 4.5. "But wait," I can hear you say. "This doesn't make any sense! What's the motivation for $\sigma(n - n) = n$? σ isn't even a function anymore!" The answer, dear reader, is that modern mathematicians just wanted to make the Σ-notation look compact. Here's a choice of indexing more akin to how Euler originally put it (and, in fact, we'll be proving this version of the theorem):

Theorem 4.4. If n is not a pentagonal number, then

$$\sigma(n) = \sum_{\substack{k\in\mathbb{Z} \\ 0<P_k<n}} (-1)^{k+1}\sigma(n-P_k).$$

Otherwise, we have $n = P_m$ for some m, and

$$\sigma(n) = (-1)^{m+1}n + \sum_{\substack{k\in\mathbb{Z} \\ 0<P_k<n}} (-1)^{k+1}\sigma(n-P_k).$$

Proof: By Euler's Pentagonal Number Theorem, let

$$f(x) = \prod_{n=1}^{\infty}(1-x^n) = \sum_{k\in\mathbb{Z}}(-1)^k x^{P_k}.$$

Let's compute $-x\frac{d}{dx}\log(f(x))$ in two different ways.

First, we compute directly that

$$\begin{aligned}
-x\frac{d}{dx}\log(f(x)) &= -x\frac{d}{dx}\log\left(\prod_{n=1}^{\infty}(1-x^n)\right) = -x\sum_{n=0}^{\infty}\frac{d}{dx}\log(1-x^n) \\
&= -x\sum_{n=0}^{\infty}\frac{-nx^{n-1}}{1-x^n} = \sum_{n=0}^{\infty}\frac{nx^n}{1-x^n} = \sum_{n=0}^{\infty}nx^n\left(\frac{1}{1-x^n}\right) \\
&= \sum_{n=0}^{\infty}nx^n(1+x^n+x^{2n}+x^{3n}+\dots) \\
&= \sum_{n=0}^{\infty}(nx^n+nx^{2n}+nx^{3n}+nx^{4n}+\dots).
\end{aligned}$$

Now, when we expand out this sum,

$$\begin{array}{cccccccccccccccccc}
x & + & x^2 & + & x^3 & + & x^4 & + & x^5 & + & x^6 & + & x^7 & + & x^8 & + & x^9 & +\dots \\
 & + & 2x^2 & & & + & 2x^4 & & & + & 2x^6 & & & + & 2x^8 & & & +\dots \\
 & & & + & 3x^3 & & & & & + & 3x^6 & & & & & + & 3x^9 & +\dots \\
 & & & & & + & 4x^4 & & & & & & & + & 4x^8 & & & +\dots \\
 & & & & & & & + & 5x^5 & & & & & & & & & +\dots \\
 & & & & & & & & & + & 6x^6 & & & & & & & +\dots \\
 & & & & & & & & & & & + & 7x^7 & & & & & +\dots \\
 & & & & & & & & & & & & & + & 8x^8 & & & +\dots \\
 & & & & & & & & & & & & & & & + & 9x^9 & +\dots \\
 & & & & & & & & & \vdots & & & & & & & &
\end{array}$$

we notice that each coefficient of x^ℓ picks up a term dx^ℓ for each divisor d of ℓ. Thus,

$$-x\frac{d}{dx}\log(f(x)) = \sum_{\ell=1}^{\infty}\sigma(\ell)x^\ell.$$

Second, we compute by implicit differentiation that

$$\begin{aligned}-x\frac{d}{dx}\log(f(x)) = \frac{-xf'(x)}{f(x)} &= \frac{-x\frac{d}{dx}\sum_{j\in\mathbb{Z}}(-1)^j x^{P_j}}{\sum_{k\in\mathbb{Z}}(-1)^k x^{P_k}} \\ &= \frac{-x\sum_{j\in\mathbb{Z}}(-1)^j P_j x^{P_j-1}}{\sum_{k\in\mathbb{Z}}(-1)^k x^{P_k}} \\ &= \frac{\sum_{j\in\mathbb{Z}}(-1)^{j+1} P_j x^{P_j}}{\sum_{k\in\mathbb{Z}}(-1)^k x^{P_k}}.\end{aligned}$$

Equating these, we get

$$\begin{aligned}\sum_{\ell=1}^{\infty}\sigma(\ell)x^\ell &= \frac{\sum_{j\in\mathbb{Z}}(-1)^{j+1} P_j x^{P_j}}{\sum_{k\in\mathbb{Z}}(-1)^k x^{P_k}} \\ \implies \left(\sum_{\ell=1}^{\infty}\sigma(\ell)x^\ell\right)\left(\sum_{k\in\mathbb{Z}}(-1)^k x^{P_k}\right) &= \sum_{j\in\mathbb{Z}}(-1)^{j+1} P_j x^{P_j}.\end{aligned}$$

We now wish to expand the left-hand side. Notice that each coefficient of x^n collects a term $\sigma(\ell)x^\ell \cdot (-1)^k x^{P_k}$ for each ℓ, P_k such that $\ell + P_k = n$. Re-indexing in terms of n (such that $\ell = n - P_k$), we thus have

$$\sum_{n=1}^{\infty}\left(\sum_{\substack{k\in\mathbb{Z}\\P_k<n}}(-1)^k\sigma(n-P_k)\right)x^n = \sum_{j\in\mathbb{Z}}(-1)^{j+1}P_j x^{P_j}.$$

We next equate x^n coefficients by casework on n. If n is not a pentagonal number, equating coefficients yields

$$\begin{aligned}\sum_{\substack{k\in\mathbb{Z}\\P_k<n}}(-1)^k\sigma(n-P_k) &= 0 \\ \implies \sigma(n) + \sum_{\substack{k\in\mathbb{Z}\\0<P_k<n}}(-1)^k\sigma(n-P_k) &= 0 \\ \implies \sigma(n) &= \sum_{\substack{k\in\mathbb{Z}\\0<P_k<n}}(-1)^{k+1}\sigma(n-P_k).\end{aligned}$$

Otherwise, we have $n = P_m$ for some m, and equating coefficients yields

$$\begin{aligned}\sum_{\substack{k\in\mathbb{Z}\\P_k<n}}(-1)^k\sigma(n-P_k) &= (-1)^{m+1}n \\ \implies (-1)^m n + \sum_{\substack{k\in\mathbb{Z}\\P_k<n}}(-1)^k\sigma(n-P_k) &= 0 \\ \implies (-1)^m n + \sigma(n) + \sum_{\substack{k\in\mathbb{Z}\\0<P_k<n}}(-1)^k\sigma(n-P_k) &= 0 \\ \implies \sigma(n) = (-1)^{m+1}n + \sum_{\substack{k\in\mathbb{Z}\\0<P_k<n}}(-1)^{k+1}\sigma(n-P_k).\end{aligned}$$

■

5. The Tiling Proof

5.1. Crash Course in Tilings

It's time for our final proof! Take out your infinitely long paper strip of choice, and draw one square for each natural number. We call a *tiling* an arrangement of a finite number of *tiles*, each of which covers one square, on this strip.

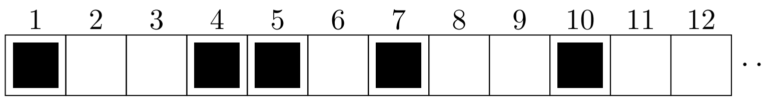


Figure 5.1: An example of a tiling.

Since we eventually want to arrive at the Pentagonal Number Theorem, we need a way to convert tilings into generating functions.

Definition 5.1 (Weight). The *weight* of a tile at square n is $-x^n$. The *weight* of a tiling is the product of the weights of its tiles.

For example, the tiling in Figure 5.1 has weight

$$(-x)(-x^4)(-x^5)(-x^7)(-x^{10}) = -x^{27}.$$

Remark 5.2. If we move a group of m tiles all k squares to the right, we multiply the tiling's weight by x^{mk}.

We next develop some notions by which we'll be categorizing our tilings:

Definition 5.3 (Rank). The *rank* of a non-null tiling is the least m such that there are at most m tiles to the right of square m.

For example, the tiling in Figure 5.1 has rank 4, since there are $3 \leq 4$ tiles to the right of square 4, and since 4 is the smallest number for which this holds.

Definition 5.4 (Type). The rank of a non-null tiling is the unique m such that either (i) there are m tiles to the right of square m, or (ii) square m has a tile, and there are $m - 1$ tiles to the right of square m. We call these *type-I* and *type-II* tilings, respectively, and they are mutually exclusive.

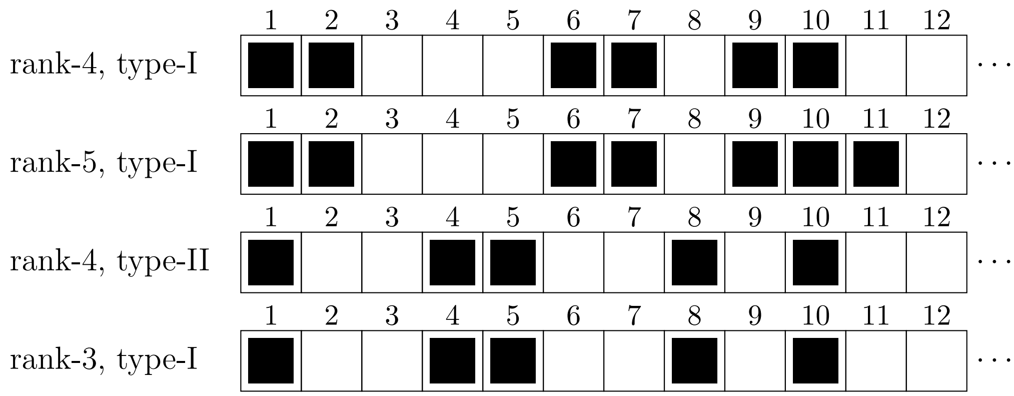


Figure 5.2: Four tilings, of varying ranks and types.

We do need to justify the claim I just asserted.

Claim 5.5. *Every non-null tiling is either type-I or type-II (but not both).*

Proof: Consider a rank-m tiling that isn't type-I; we claim it's type-II. Since the tiling isn't type-I, there are at most $m-1$ tiles to the right of square m. Now consider the number of tiles to the right of square $m-1$: we know the tiling doesn't have rank $m-1$, so there must be at least m of them. Hence, square m must have a tile, and there are exactly $m-1$ tiles to its right. ■

Remark 5.6.

(i) If we take a type-I rank-m tiling and we add or remove tiles on squares $1, 2, \ldots, m$, then the resulting tiling is still type-I and rank-m.

(ii) If we take a type-II rank-m tiling and we add or remove tiles on squares $1, 2, \ldots, m-1$, then the resulting tiling is still type-II and rank-m .

Cautious readers are invited to stare at Figure 5.3 until this remark makes sense.

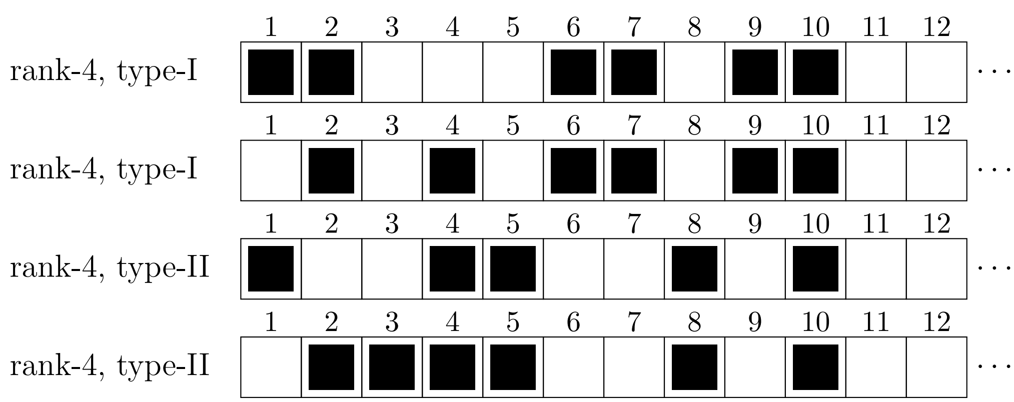


Figure 5.3: Four more tilings, all of rank 4.

5.2. The Proof

We now have all the tools we need to prove Euler's Pentagonal Number Theorem!

Proof: (of Theorem 2.3) Let $f(x)$ denote the sum of the weights of all possible tilings. We compute $f(x)$ in two different ways.

First, we consider that for each tiling, each square n either has a tile (thus multiplying the weight by $-x^n$) or doesn't have a tile (thus multiplying the weight by 1). Thus,

$$f(x) = \prod_{n=1}^{\infty}(1 - x^n).$$

Second, we sum over all possible ranks and types of tilings (and also the null tiling). Let $g_m(x)$ denote the sum of all weights of type-I rank-m tilings, and let $h_m(x)$ denote the sum of all weights of type-II rank-m tilings. Noting that the null tiling has weight 1, we thus have

$$f(x) = 1 + \sum_{m=1}^{\infty}(g_m(x) + h_m(x)).$$

We first compute $g_m(x)$. We will do so by generating each type-I rank-m tiling like so, and considering the contribution each makes to $g_m(x)$.

1. Place tiles on squares $1, 2, \ldots, m$ in all possible ways.

2. Place m tiles on squares $m + 1, m + 2, \ldots, 2m$.

3. Move these m tiles to the right any nonnegative distance; then, move the rightmost $m - 1$ tiles to the right any nonnegative distance; then, move rightmost $m - 2$ tiles to the right any nonnegative distance; so on and so forth.

For step 1, each square n either has a tile or doesn't, thus contributing a factor of

$$\prod_{n=1}^{m}(1 - x^n).$$

For step 2, the m tiles placed on squares $m, m + 1, \ldots, 2m - 1$ have a total weight of

$$(-1)^m x^{(m+1)+(m+2)+\cdots+2m} = (-1)^m x^{\frac{m(3m+1)}{2}}$$

which is their contribution.

For step 3, when we move these m tiles to the right k squares, we multiply the tiling's weight by x^{mk}. Summing over all $k \geq 0$, this step contributes a factor of

$$1 + x^m + x^{2m} + x^{3m} + \cdots = \frac{1}{1 - x^m}$$

to the generating function. Likewise, moving the $m - 1$ squares to the right contributes a factor of $\frac{1}{1-x^{m-1}}$ to the generating function, moving the $m-2$ squares to the right contributes a factor of $\frac{1}{1-x^{m-2}}$ to the generating function, so on and so forth. In total, step 3 contributes a factor of

$$\prod_{n=1}^{m} \frac{1}{1-x^n}.$$

Multiplying these together yields

$$g_m(x) = \left(\prod_{n=1}^{m}(1-x^n)\right)\left((-1)^m x^{\frac{m(3m+1)}{2}}\right)\left(\prod_{n=1}^{m}\frac{1}{1-x^n}\right) = (-1)^m x^{\frac{m(3m+1)}{2}} = (-1)^m x^{\frac{(-m)(3(-m)-1)}{2}}$$
$$= (-1)^m x^{P_{-m}}.$$

We next compute $h_m(x)$. We will again do so by generating each type-II rank-m tiling like so, and considering the contribution each makes to $h_m(x)$.

1. Place tiles on squares $1, 2, \ldots, m-1$ in all possible ways, then place a tile on square m.
2. Place $m-1$ tiles on squares $m+1, m+2 \ldots, 2m-1$.
3. Move these $m-1$ tiles to the right any nonnegative distance; then, move the rightmost $m-2$ tiles to the right any nonnegative distance; so on and so forth.

Using the same logic as before, step 1 contributes a factor of

$$-x^m \prod_{n=1}^{m-1}(1-x^n),$$

step 2 contributes a factor of

$$(-1)^{m-1} x^{(m+1)+(m+2)+\cdots+(2m-1)} = (-1)^{m-1} x^{\frac{3m(m-1)}{2}},$$

and step 3 contributes a factor of

$$\prod_{n=1}^{m-1}\frac{1}{1-x^n}.$$

Multiplying yields

$$h_m(x) = \left(-x^m \prod_{n=1}^{m-1}(1-x^n)\right)\left((-1)^{m-1} x^{\frac{3m(m-1)}{2}}\right)\left(\prod_{n=1}^{m-1}\frac{1}{1-x^n}\right) = (-1)^m x^{\frac{3m(m-1)}{2}+m}$$
$$= (-1)^m x^{\frac{m(3m-1)}{2}} = (-1)^m x^{P_m}.$$

Finally, it's time to bring this all together! We equate our two expressions for $f(x)$ and plug in our formulae for $g_m(x)$ and $h_m(x)$:

$$\begin{aligned}\prod_{n=1}^{\infty}(1-x^n) &= 1+\sum_{m=1}^{\infty}(g_m(x)+h_m(x)) \\ &= 1+\sum_{m=1}^{\infty}\left((-1)^m x^{P_{-m}}+(-1)^m x^{P_m}\right) \\ &= (-1)^0 x^{P_0}+\sum_{m=1}^{\infty}(-1)^{-m}x^{P_{-m}}+\sum_{m=1}^{\infty}(-1)^m x^{P_m} \\ &= \sum_{k\in\mathbb{Z}}(-1)^k x^{P_k}. \qquad \blacksquare\end{aligned}$$

Remark 5.7. This tiling proof strategy can be used to prove generalizations of Euler's Pentagonal Number Theorem and other q-series identities, some of which are covered in [1].

References

[1] Dennis Eichhorn, Hayan Nam, and Jaebum Sohn. "A tiling proof of Euler's Pentagonal Number Theorem and generalizations". In: *The Ramanujan Journal* 54.3 (2021), pp. 613–624.

[2] Leonhard Euler. "Demonstratio theorematis circa ordinem in summis divisorum observatum". In: *Novi commentarii academiae scientiarum Petropolitanae* (1760), pp. 75–83.

[3] Leonhard Euler. "Observatio de summis divisorum". In: *Novi Commentarii academiae scientiarum Petropolitanae* 5 (1760), pp. 59–74.