THE RAMANUJAN FORMULA FOR PI

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1. Introduction

Among the many remarkable discoveries of Srinivasa Ramanujan, his formulas for $\frac{1}{\pi}$ are some of the most celebrated. Found in the early twentieth century, these identities stood out for the unexpected way they connected π with elliptic functions, modular quantities, and hypergeometric series. Furthermore, these series for $\frac{1}{\pi}$ converge at an extremely fast rate. For this reason, Ramanujan's formulas have attracted attention not only for their beauty, but also for their later influence on high-precision computation of π.

The formula considered in this paper is perhaps the most famous example:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 (396)^{4k}}.$$

Its convergence is extraordinarily rapid. Even the first term already gives a very accurate approximation to π, and each successive term improves the approximation dramatically. This makes the identity especially striking: the coefficients are explicit and elementary in appearance, yet the series converges with a speed that seems almost mysterious when first encountered.

Ramanujan's identity does not arises from an interaction between complete elliptic integrals, Jacobi theta functions, singular moduli, and hypergeometric series. The goal of this paper is to present this connection clearly and to provide a proof for this formula.

2. The Ramanujan Pi Formula

Theorem 2.1 (Ramanujan)**.**

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 (396)^{4k}}$$

The convergence of this forumla is very fast. Just taking the first term of this series gives an approximation of $\pi \approx 3.14159273001$, which is already accurate to 7 digits. We will now explain a proof of this formula using elliptic integrals, theta functions, and hypergeometric series. The overall plan is to first derive a more general identity for $1/\pi$, and then plug in a special value where all of the constants simplify in a remarkable way.

From here on, let $k \in [0, 1]$ denote the elliptic modulus.

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Definition 2.2. The complete elliptic integrals of the first and second kind respectively are defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta}\ d\theta$$

The complementary modulus is defined as $k' = \sqrt{1 - k^2}$

These two functions are the main analytic objects in the proof. They appear often in the theory of elliptic functions, and they satisfy several useful identities. Our goal is to use those identities to eventually rewrite $1/\pi$ as a hypergeometric series.

Lemma 2.3. *We have that*

$$\frac{dK}{dk} = \frac{E - (k')^2 K}{k(k')^2}$$

and

$$\frac{dE}{dk} = \frac{E - K}{k}$$

Proof. We will prove the first identity. The other one follows similarly. By using the definition of K, we have

$$K'(k) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial k} \left(\frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \right) d\theta$$

where the result then follows from this. ∎

So the complementary integrals are defined as

$$K'(k) := K(k') \qquad E'(k) := E(k')$$

Besides $K(k)$ and $E(k)$, we also need the corresponding functions attached to the complementary modulus k'. These complementary quantities are closely related to the original ones, and together they satisfy an identity known as Legendre's relation.

Lemma 2.4.

$$E(k)K'(k) + E'(k)K(k) - K(k)K'(k) = \frac{\pi}{2}$$

This identity is one of the central tools in the argument. It is important because it gives a direct formula involving π, and so it will later let us transform an identity involving elliptic integrals into one involving $1/\pi$. We now introduce Jacobi's theta functions. These functions give another way to describe elliptic quantities, but in terms of power series in a new variable q, called the nome. This point of view is very useful because it lets us pass from elliptic integrals to series expansions.

Definition 2.5. The (specialized) Jacobi theta functions are defined as

$$\vartheta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}$$

$$\vartheta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$\vartheta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \vartheta_3(-q)$$

where $|q| < 1$, and is called the nome.

Although these are only special cases of the full theta functions, they are exactly the cases we need here. They contain enough information to describe the modulus k, its complement k', and the elliptic integral $K(k)$ in a very explicit way.

We can also define the nome in terms of the elliptic integrals.

$$q = \exp\left[-\pi\frac{K'(k)}{K(k)}\right]$$

However, we also need to see that k can be defined as a function of q.

Theorem 2.6.

$$k = \frac{\vartheta_2^2(q)}{\vartheta_3^2(q)}$$

$$k' = \frac{\vartheta_4^2(q)}{\vartheta_3^2(q)}$$

$$K(k) = \frac{\pi}{2}\vartheta_3^2(q)$$

These formulas are extremely useful. They show that the modulus and the elliptic integral can both be written in terms of the nome q. In other words, they let us move back and forth between elliptic integrals and q-series, which is an important step in the proof.

3. The g-invariant

We next introduce another quantity used by Ramanujan, called the g-invariant. This quantity may look mysterious at first, but it is chosen very carefully. It packages certain algebraic combinations of k and k' into a form that is especially well suited for the transformations we need later.

Definition 3.1. Define the Ramanujan g-invariant as

$$g = \left(\frac{(k')^2}{2k}\right)^{\frac{1}{12}}$$

From here, we have the following formula for g_n, which was proved by Ramanujan.

Theorem 3.2.

$$\prod_{k=1,3,5\cdots} \left(1 - e^{-k\pi\sqrt{n}}\right) = 2^{1/4} e^{-\frac{\pi\sqrt{n}}{24}} g_n$$

This formula shows that the g-invariant is closely connected with special values of q-products. That is one of the reasons it is so useful. In the final steps of the proof, the g-invariant will allow us to write the argument of the hypergeometric series in a very simple algebraic form.

We now define two special functions depending on a positive real number r. These are called singular value functions. The reason for this name is that, for certain values of r, the corresponding elliptic quantities take especially nice algebraic values. Those special values are exactly what make Ramanujan's formula possible.

Definition 3.3. Let $r > 0$. Let $q_r = e^{-\pi\sqrt{r}}$. Let $k_r \in (0, 1)$ be the unique modulus satisfying

$$\frac{K'(k_r)}{K(k_r)} = \sqrt{r}$$

We call k_r the singular modulus associated to r. Define

$$\alpha(r) = \frac{E'(k_r)}{K(k_r)} - \frac{\pi}{4(K(k_r))^2}$$

to be the singular value function of the second kind.

The definition of $\alpha(r)$ uses the complementary integral $E'(k)$, but for later calculations it is much more convenient to rewrite it in terms of $E(k)$, $K(k)$, and r. This is where Legendre's relation becomes important.

Theorem 3.4.

$$\alpha(r) = \frac{\pi}{4[K(k)]^2} - \sqrt{r}\left[\frac{E(k)}{K(k)} - 1\right]$$

Proof.

$$\alpha(r) = \frac{E'(k)}{K(k)} - \frac{\pi}{4[K(k)]^2} = \frac{4E'(k)K(k) - \pi}{4[K(k)]^2}$$

By Legendre's relation, $4K(k)E'(k) = 2\pi - 4E(k)K'(k) + 4K(k)K'(k)$. Substituting this in gives

$$= \frac{\pi + 4K(k)K'(k) - 4E(k)K'(k)}{4K(k)^2}$$

However, we know that

$$\frac{K'(k_r)}{K(k_r)} = \sqrt{r}$$

Then, when we set k to be the singular modulus k_r, we get the result

$$\alpha(r) = \frac{\pi}{4[K(k)]^2} - \sqrt{r}\left[\frac{E(k)}{K(k)} - 1\right]$$

■

The quantity k_r, called the singular modulus associated to r, is the special elliptic modulus determined by the condition

$$\frac{K'(k_r)}{K(k_r)} = \sqrt{r}.$$

Equivalently, if we write the nome as

$$q = \exp\left(-\pi\frac{K'(k)}{K(k)}\right),$$

then the singular modulus is the modulus corresponding to the special nome

$$q_r = e^{-\pi\sqrt{r}}.$$

Thus, instead of starting with an arbitrary modulus k and computing its nome, we reverse the process and choose a nome of a particularly simple form. The resulting modulus k_r is called singular because, for special values of r, especially positive rational values, it leads to remarkably structured algebraic quantities.

The importance of the singular modulus in Ramanujan's formula is that it turns a general identity into an explicit one. For an arbitrary modulus, the quantities built from k are typically quite complicated. By contrast, when $k = k_r$ is a singular modulus, expressions involving k_r, k_r', and the associated g-invariant often simplify dramatically and can be evaluated in closed algebraic form. This is exactly what makes it possible to choose a value such as $r = 58$ and obtain the constants that appear in Ramanujan's series for $1/\pi$.

We now arrive at the main identity that leads to the series expansion. The point of this formula is that it expresses $1/\pi$ in terms of $K(k)$, its derivative, and the singular value $\alpha(r)$. Once we rewrite $K(k)$ in hypergeometric form, this identity will turn into a series for $1/\pi$.

Theorem 3.5. *Let $k = k_r$. Then*

$$\frac{1}{\pi} = \sqrt{r}k(k')^2\left[\left(\frac{2}{\pi}\right)^2 K(t)\frac{dK}{dk}\right] + [\alpha(r) - \sqrt{r}k^2]\left[\frac{2}{\pi}K(k)\right]^2$$

Proof. By the derivative formula for K,

$$k(k')^2\frac{dK}{dk} = E - (k')^2K.$$

Hence

$$\sqrt{r}\,k(k')^2\left[\left(\frac{2}{\pi}\right)^2 K\frac{dK}{dk}\right] = \left(\frac{2}{\pi}\right)^2\sqrt{r}\,K\bigl(E-(k')^2K\bigr).$$

Adding the second term gives

$$\left(\frac{2}{\pi}\right)^2\Bigl[\sqrt{r}\,KE - \sqrt{r}\,(k')^2K^2 + (\alpha(r) - \sqrt{r}\,k^2)K^2\Bigr].$$

Since $k^2 + (k')^2 = 1$, this becomes

$$\left(\frac{2}{\pi}\right)^2\Bigl[\sqrt{r}\,KE - \sqrt{r}\,K^2 + \alpha(r)K^2\Bigr].$$

Now substitute

$$\alpha(r) = \frac{\pi}{4K^2} - \sqrt{r}\left(\frac{E}{K} - 1\right).$$

Then

$$\alpha(r)K^2 = \frac{\pi}{4} - \sqrt{r}\,KE + \sqrt{r}\,K^2.$$

So the bracket simplifies to $\pi/4$, and therefore the whole expression is

$$\left(\frac{2}{\pi}\right)^2\frac{\pi}{4} = \frac{1}{\pi}.$$

■

4. Hypergeometric Series

The next step is to bring in hypergeometric series. These are exactly the kind of series that appear naturally when elliptic integrals are expanded in a suitable way. After this step, the proof becomes a matter of substituting the right formulas and simplifying the coefficients.

Definition 4.1. Define $(q)_n = \begin{cases} 1 & \text{if } n = 0 \\ q(q+1)\cdots(q+n+-1) & \text{if } n > 0 \end{cases}$

Now, we can define the hypergeometric functions.

Definition 4.2. The hypergeometric series is given by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} $$

and by

$$ {}_3F_2(a,b,c;\alpha,\beta;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(\alpha)_n(\beta)_n}\frac{z^n}{n!} $$

These identities connect the elliptic integral $K(k)$ with hypergeometric functions. This is a crucial transition in the proof. It lets us replace the analytic behavior of $K(k)$ by a power series whose coefficients can be studied directly.

Proposition 4.3.

$$ \frac{2}{\pi}K(k) = {}_2F_1\left(\frac{1}{4},\frac{1}{4};1;(2kk')^2\right) $$

$$ \left[\frac{2}{\pi}K(k)\right]^2 = {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(2kk')^2\right) $$

The following transformation is the specific one that is needed for Ramanujan's formula. It rewrites the square of $K(k)$ in a form whose argument is expressed through the g-invariant.

Corollary 4.4. *For* $0 \le k \le \frac{1}{\sqrt{2}}$*, we have that*

$$ \left[\frac{2}{\pi}K(k)\right]^2 = \frac{1}{k^2+1}\ {}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};1,1;\left(\frac{2}{g^{12}+g^{-12}}\right)^2\right) $$

We can therefore express $\left[\frac{2}{\pi}K(k)\right]^2$ in to form of $m(k)F(\varphi(k)) = mF$, where m and ϕ are algebraic numbers and F is a hypergeometric series given by

$$ F(\varphi(k)) = \sum_{n=0}^{\infty} a_n\varphi^n $$

Now that the square of $K(k)$ has been written in this form, we can differentiate it using the chain rule. This is the step that produces both a constant term and a term involving n, which is exactly the kind of coefficient pattern that appears in Ramanujan-type series.

Differentiating both sides of the equation gives

$$ \left(\frac{2}{\pi}\right)^2 K\frac{dK}{dk} = \frac{1}{2}\left(\frac{dm}{dk}F + m\frac{dF}{d\varphi}\frac{d\varphi}{dk}\right) $$

However, we know that

$$ \frac{dF}{d\varphi} = \sum_{n=0}^{\infty} na_n\varphi^{n-1} = \frac{1}{\varphi}\sum_{n=0}^{\infty}(n+1)a_{n+1}\varphi^{n+1} $$

Substituting this differentiated expression into the earlier identity and then expanding term by term gives a general series for $1/\pi$.

By 3.5,

$$ \frac{1}{\pi} = \sqrt{r}k(k')^2\left[\frac{1}{2}\left(\frac{dm}{dk}F + m\frac{dF}{d\varphi}\frac{d\varphi}{dk}\right)\right] + [\alpha(r) - \sqrt{r}k^2]mF $$

Then, using the previous equations,

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} a_n \left[\frac{1}{2}\sqrt{r}k(k')^2 \frac{dm}{dk} + [\alpha(r) - \sqrt{r}k^2]m + \frac{mn}{2\varphi}\sqrt{r}k(k')^2\frac{d\varphi}{dk} \right] \varphi^n$$

Recall that $a_n = \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3}$, $m(k) = \frac{1}{k^2+1}$ and $\varphi(k) = (\frac{2}{g^{12}+g^{-12}})^2$ We can express the terms of the series as $a_n(A + nB)$ for algebraic A, B.

It is convenient to introduce a shorter notation for the argument of the series. This will simplify the formulas and make the final specialization much easier to see. We therefore write this argument in terms of a new quantity x_N.

Set

$$x_N = \frac{2}{g^{12} + g^{-12}} = \frac{4k_N(k'_N)^2}{(1 + k_N^2)^2}$$

We then get the series

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} \left[\frac{\alpha(N)}{x_N(1 + k_N^2)} - \frac{\sqrt{N}}{4g_N^{12}} + n\sqrt{N} \cdot \frac{g_N^{12} - g_N^{-12}}{2} \right] x_N^{2n+1}$$

This is now the general form of the series that we need. It already has the right shape, but the coefficients still look complicated. The next goal is to simplify the hypergeometric coefficients and then choose a special value of N for which the algebraic quantities can all be evaluated explicitly.

To match Ramanujan's formula, we now rewrite the Pochhammer products in factorial form. This is a direct computation, but it is important because it turns the coefficients into the expression involving $(4n)!$ and $(n!)^4$.

Now, we will obtain an expression for $\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n$.

Lemma 4.5.

$$\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n = \frac{1}{256^n} \frac{(4n)!}{n!}$$

Proof.

$$\left(\frac{1}{4}\right)_n = \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{4^n}$$

$$\left(\frac{1}{2}\right)_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}$$

$$\left(\frac{3}{4}\right)_n = \frac{3 \cdot 7 \cdot 11 \cdots (4n-1)}{4^n}$$

Multiplying these all together gives

$$\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n = \frac{1}{32^n} \cdot \frac{(4n)!}{4^n(2n)!} \cdot \frac{(2n)!}{2^n n!} = \frac{1}{256^n} \frac{(4n)!}{n!}$$

∎

Substituting this into our sum gives

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{1}{256^n} \frac{(4n)!}{(n!)^4} \left[\frac{\alpha(N)}{x_N(1 + k_N^2)} - \frac{\sqrt{N}}{4g_N^{12}} + n\sqrt{N} \cdot \frac{g_N^{12} - g_N^{-12}}{2} \right] x_N^{2n+1}$$

With this simplification, the series now looks much closer to Ramanujan's formula. What remains is to evaluate the algebraic terms in the bracket for a suitable choice of N.

We now choose the special value $N = 58$. This is the value that leads to Ramanujan's famous formula. The reason this choice works is that the associated quantities g_{58}, k_{58}, x_{58}, and $\alpha(58)$ can all be computed explicitly. However, computation is very difficult. Thus, we will not show it here. It turns out that when $N = 58$, we are able to evaluate the g invariant. The evaluation is as such:

$$g_{58} = \sqrt{\frac{5+\sqrt{29}}{2}}$$

We then compute $k_{58} = (\sqrt{2}-1)^6(13\sqrt{58}-99)$. Finally, we also have that $\frac{g_{58}^{12}+g_{58}^{-12}}{2} = 9801$, meaning $x_{58} = \frac{1}{9801}$. We also then get $\alpha(58) = 3g_{58}^6 k_{58}(33\sqrt{29} - 148)$.

Then we have

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{1}{256^n} \frac{(4n)!}{(n!)^4} \left[\frac{\alpha(N)}{x_N(1+k_N^2)} - \frac{\sqrt{N}}{4g_N^{12}} + n\sqrt{N} \cdot \frac{g_N^{12} - g_N^{-12}}{2} \right] x_N^{2n+1}$$

5. Putting the pieces together

We now simplify the three parts of the expression above. First we determine the overall power of x_{58}, since this is what produces the denominator 396^{4n}. Next we simplify the coefficient of n, which will give the number 26390. Finally, we compute the constant term, which will turn out to be 1103. Once these three pieces are identified, the series takes exactly the form stated by Ramanujan.

We begin with the power of x_{58}. This is the term responsible for the denominator in the final series, and it is the source of the factor 396^{4n} that makes Ramanujan's formula converge so rapidly.

We compute

$$\frac{x_{58}^{2n+1}}{256^n} = \frac{1}{9801}\left(\frac{1}{256 \cdot 9801^2}\right)^n = \frac{1}{9801} \cdot \frac{1}{396^{4n}}$$

We next simplify the coefficient of n. This term comes from the part of the general series involving $g_{58}^{12} - g_{58}^{-12}$. After substituting the explicit values and simplifying, this coefficient becomes the number 26390 appearing in Ramanujan's formula. We know $g_{58}^{12} = \left(\sqrt{\frac{5+\sqrt{29}}{2}}\right)^{12} = 9801 + 1820\sqrt{29}$. Therefore $g_{58}^{-12} = 9801 - 1820\sqrt{29}$. So

$$x_{58}\sqrt{58}\frac{g_{58}^{12} - g_{58}^{-12}}{2} = \frac{\sqrt{58}}{9801} \cdot 1820\sqrt{29} = \frac{2\sqrt{2}}{9801} \cdot 26390$$

It remains to compute the constant term. This part is slightly more delicate, since it involves both $\alpha(58)$ and g_{58}^{12}. We therefore simplify the two contributions separately and then combine them at the end. After cancellation, the constant turns out to be 1103. The constant part is

$$x_{58}\left[\frac{\alpha(58)}{x_{58}(1+k_{58}^2)} - \frac{\sqrt{58}}{4g_{58}^{12}}\right]$$

So

$$\frac{\sqrt{58}}{4 \cdot 9801 g_{58}^{12}} = \frac{\sqrt{58}}{4 \cdot 9801}(9801 - 1820\sqrt{29}) = \frac{\sqrt{58}}{4} - \frac{13195\sqrt{2}}{9801}$$

Recall that we have $\alpha(58) = 3g_{58}^6 k_{58}(33\sqrt{29} - 148)$. Thus, we obtain

$$\frac{\alpha(58)}{1 + k_{58}^2} = \frac{\sqrt{58}}{4} - \frac{37\sqrt{2}}{33} = \frac{\sqrt{58}}{4} - \frac{10989\sqrt{2}}{9801}$$

So the full term is

$$\frac{2\sqrt{2}}{9801} 1103$$

Substituting these simplifications back into the series gives exactly the desired expression. The resulting expression is exactly Ramanujan's formula for $\frac{1}{\pi}$.

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1}{396^{4n}} \left[\frac{2\sqrt{2}}{9801}(1103 + 26390n) \right] = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}$$

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