

MULTIPLE ZETA VALUES

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1. INTRODUCTION

A famous result of Euler is the Basel Problem:

Theorem 1.1 (Basel Problem).

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This was a well-known problem at the time, so he was widely regarded for solving it. His work was built on by Bernhard Riemann in his influential 1859 paper “On the Number of Primes Less Than a Given Magnitude.” The Basel Problem is a special value of the Riemann Zeta function introduced in Riemann’s paper.

The Riemann Zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

when $\Re(s) > 1$ and extended analytically to the complex plane. There has been a plethora of research relating this function, and it appears in a million-dollar conjecture [3]:

Conjecture 1.2 (Riemann Hypothesis). *The roots of the $\zeta(s)$ besides the negative even integers all lie on the line $\Re(s) = \frac{1}{2}$.*

One useful representation of the Riemann Zeta function is known as Euler’s Product Representation.

Proposition 1.3. *For any s with $\Re(s) > 1$, we have*

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} = \left(\prod_p \left(1 - \frac{1}{p^s} \right) \right)^{-1}$$

where the product is over all primes.

Proof. If we rewrite

$$\frac{1}{1 - p^{-s}} = 1 + p^{-s} + (p^2)^{-s} + \dots$$

and expand, we get

$$\prod_p \frac{1}{1 - p^{-s}} = \sum_{a_1, a_2, \dots} \frac{1}{(p_1^{a_1} p_2^{a_2} \dots)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

where p_1, p_2, \dots are the primes. This follows from the Fundamental Theorem of Arithmetic, and we can disregard all the terms with infinitely many positive a_i , since those are 0. ■

One nice application of this product formula is the following:

Proposition 1.4. *The asymptotic probability that s natural numbers are all relatively prime is*

$$\frac{1}{\zeta(s)}.$$

Proof. An informal argument follows intuitively from 1.3, noting that the probability that s natural numbers are divisible by a prime p is $\frac{1}{p^s}$. Since divisibility by primes is asymptotically independent, the probability that they are all coprime is

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}.$$

■

One topic of study is related to the values of $\zeta(k)$ for positive integers $k > 1$. For even k this problem was solved by Euler, who was able to generalize his solution to the Basel problem to $\zeta(2k)$ for $k \geq 1$.

Theorem 1.5. *For any positive integer n ,*

$$\zeta(2n) = \frac{B_{2n}(2\pi)^{2n}(-1)^{n+1}}{2(2n)!}.$$

Here, B_{2n} is the $2n$ -th Bernoulli number, which is given by the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} \frac{B_n x^n}{n!}.$$

For example, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, and $B_6 = \frac{1}{42}$ while

$$\begin{aligned}\zeta(2) &= \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945}.\end{aligned}$$

For odd zeta values, however, much less is known. It was proven by Apéry in 1978 that $\zeta(3)$ is irrational, and it is unknown whether or not $\zeta(3)$ is transcendental. For every odd $k \geq 5$, it is unknown whether or not $\zeta(k)$ is rational, but it is believed that they are all transcendental. However, it is known that infinitely many odd zeta values are irrational and that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

The values of ζ at negative integers were also found by Euler, and these are relevant to the study of modular forms.

Definition 1.6. Given a field K , a subfield L of K and a set $S \subseteq K$, we say that S is *algebraically independent* over L if the elements of S satisfy no nontrivial polynomial relation.

It is thought that the following very strong conjecture is true, but there has been little progress towards proving it.

Conjecture 1.7. *The numbers*

$$1, \pi, \zeta(3), \zeta(5), \dots$$

are algebraically independent.

Now we focus on multiple zeta values, which are an extension of the Riemann Zeta function.

Definition 1.8. For any sequence of positive integers $\mathbf{s} = (s_1, s_2, \dots, s_\ell)$, we can define the multiple zeta value

$$\zeta(s_1, s_2, \dots, s_\ell) = \sum_{n_1 > n_2 > \dots > n_\ell \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_\ell^{s_\ell}}$$

as a natural extension of the Riemann Zeta function, provided that this sum converges.

Note that $s_1 > 1$ is necessary for convergence, as

$$\sum_{n_1 > n_2 > \dots > n_\ell \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_\ell^{s_\ell}} > \sum_{n \geq \ell} \frac{1}{n(\ell-1)^{s_2} (\ell-2)^{s_3} \dots 1^{s_\ell}} = \infty.$$

It is also sufficient.

Claim 1.9. *If $(s_1, s_2, \dots, s_\ell)$ is a sequence of positive integers with $s_1 > 1$, then $\zeta(s_1, s_2, \dots, s_\ell)$ is a real number. We call such sequences admissible.*

It suffices to show that $\zeta(2, \underset{\ell-1 \text{ ones}}{1, \dots, 1})$ converges, since $\zeta(s_1, s_2, \dots, s_\ell)$ is bounded above by this value. Note that

$$\begin{aligned} \zeta(2, \underset{\ell-1 \text{ ones}}{1, \dots, 1}) &= \sum_{n_1 > n_2 > \dots > n_\ell} \frac{1}{n_1^2 n_2 \dots n_\ell} \leq \sum_{n \geq 1} \frac{1}{n^2} \left(\sum_{k=1}^{n-1} \frac{1}{k} \right)^{\ell-1} \\ &\leq \sum_{n \geq 1} \frac{(\log(n-1) + 1)^{\ell-1}}{n^2} \end{aligned}$$

which converges.

For any admissible sequence, we say that $s_1 + s_2 + \dots + s_\ell$ is the *weight* of \mathbf{s} and that ℓ is the *length* of \mathbf{s} . We also say that a number x is a Multiple Zeta Value or MZV if there is some admissible sequence \mathbf{s} such that

$$\zeta(\mathbf{s}) = x.$$

In this paper, we will look at some special properties and relations between Multiple Zeta Values. Then we will explore the algebra of MZVs by looking at shuffle and stuffle products and multiple polylogarithms.

2. PROPERTIES, RELATIONS AND SPECIAL CASES

Another identity of Euler is the following:

Proposition 2.1.

$$\zeta(2, 1) = \zeta(3)$$

and

$$\frac{\pi^4}{90} = \zeta(4) = \zeta(2, 1, 1) = \zeta(3, 1) + \zeta(2, 2)$$

These arise as special cases of the following identity, known as the Sum Theorem for Multiple Zeta Values:

$$\zeta(n) = \sum_{\substack{\sum s_i = n \\ s_1 \geq 2, s_i \geq 1}} \zeta(s_1, s_2, \dots, s_\ell).$$

We also have the identity $\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(b, a) + \zeta(a + b)$.

Proof. Note that

$$\begin{aligned} \zeta(a)\zeta(b) &= \left(\sum_{n \geq 1} \frac{1}{n^a} \right) \left(\sum_{n \geq 1} \frac{1}{n^b} \right) = \sum_{m, n \geq 1} \frac{1}{m^a n^b} \\ &= \sum_{m > n \geq 1} \frac{1}{m^a n^b} + \sum_{m < n \geq 1} \frac{1}{m^a n^b} + \sum_{m = n \geq 1} \frac{1}{m^a n^b} \\ &= \zeta(a, b) + \zeta(b, a) + \zeta(a + b). \end{aligned}$$

■

This can be extended to products of more than 2 zeta values, which we will see in the next section.

We can also derive the following, which comes from the product expansion of $\sin(x)$:

Proposition 2.2.

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n + 1)!}.$$

Proof. The product expansion of $\sin(x)$ is

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 k^2} \right),$$

and we also know that

$$\sin(\pi x) = \sum_{k=0}^{\infty} \frac{x^{2k+1} (-1)^k}{(2k + 1)!},$$

so

$$[x^{2n}] \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 k^2} \right) = [x^{2n}] \sum_{k=0}^{\infty} \frac{x^{2k} (-1)^k}{(2k + 1)!} = \frac{(-1)^n}{(2n + 1)!}.$$

However, the LHS is equal to

$$\frac{(-1)^n \zeta(\{2\}^n)}{\pi^{2n}}$$

by expansion, so

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n + 1)!}.$$

■

3. ALGEBRA OF MULTIPLE ZETA VALUES

We have the following integral identity:

$$\begin{aligned}
\int_0^1 \frac{1}{x_1} dx_1 \int_0^{x_1} \frac{1}{1-x_2} dx_2 &= \int_0^1 \frac{1}{x_1} dx_1 \int_0^{x_1} \left(\sum_{n=0}^{\infty} x_2^n \right) dx_2 \\
&= \int_0^1 \frac{1}{x_1} dx_1 \sum_{n=0}^{\infty} \frac{x_1^{n+1}}{n+1} \\
&= \sum_{n=0}^{\infty} \int_0^1 \frac{x_1^n}{n+1} dx_1 \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \zeta(2).
\end{aligned}$$

In general, a similar representation holds for Multiple Zeta Values, or

$$\zeta(s_1, s_2, \dots, s_\ell) = \underbrace{\int_0^1 \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \dots \int_0^{x_{s_1-1}} \frac{dx_{s_1}}{1-x_{s_1}} \dots \int_0^{x_{s_1+s_2+\dots+s_\ell-1}} \frac{dx_{s_1+s_2+\dots+s_\ell}}{1-x_{s_1+s_2+\dots+s_\ell}}}_{s_1-1}.$$

We will prove this by considering MZVs as special cases of multiple polylogarithms.

3.1. Multiple Polylogarithms. The polylogarithm function is a generalization of the Riemann Zeta function, defined as

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

Note that $\text{Li}_s(1) = \zeta(s)$. Now we can generalize further to the multiple polylogarithm.

Definition 3.1. We define the multiple polylogarithm as

$$\text{Li}_{s_1, s_2, \dots, s_\ell}(z) = \sum_{n_1 > \dots > n_\ell \geq 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \dots n_\ell^{s_\ell}}.$$

This series always converges when $|z| < 1$, and converges for $z = \pm 1$ when \mathbf{s} is admissible.

These have the integral representation

$$\text{Li}_{s_1, s_2, \dots, s_\ell}(z) = \underbrace{\int_0^z \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \dots \int_0^{x_{s_1-1}} \frac{dx_{s_1}}{1-x_{s_1}} \dots \int_0^{x_{s_1+s_2+\dots+s_\ell-1}} \frac{dx_{s_1+s_2+\dots+s_\ell}}{1-x_{s_1+s_2+\dots+s_\ell}}}_{s_1-1}.$$

For convenience, we may write $\omega_x(x_k)$ in place of $\frac{dx_k}{x_k}$ and $\omega_y(x_k)$ in place of $\frac{dx_k}{1-x_k}$ in the integral expansions of multiple polylogarithms and MZVs. For example, the iterated integral for $\zeta(2)$ can be written as

$$\int_{1 > x_1 > x_2 > 0} \omega_x(x_1) \omega_y(x_2).$$

We will introduce the spaces \mathfrak{H} , \mathfrak{H}^1 , and \mathfrak{H}^0 to prove the iterated integral expansion and the shuffle and stuffle products.

3.2. Shuffle and Stuffle Products. We can now look at these Polylogarithms and MZVs in terms of words with two letters.

Definition 3.2. Let a *word* be some sequence of x s and y s. Let

$$\mathfrak{H} = \mathbb{Q}\langle x, y \rangle$$

be the free algebra over \mathbb{Q} with the two noncommutative variables x and y . Equivalently, \mathfrak{H} is formed by \mathbb{Q} -linear combinations of words. Similarly, let $\mathfrak{H}^1 = \mathbb{Q}\langle y, xy, xxy, \dots \rangle$, or the space spanned by words ending in y . We also let $\mathfrak{H}^0 = \mathbb{Q}\langle xy, xxy, xxxy, \dots \rangle$. Note that \mathfrak{H}^0 is also the space spanned by admissible sequences, when viewing x s and y s as corresponding to $\frac{1}{x_i}$ and $\frac{1}{1-x_i}$ in the integral representation. For example,

$$\mathfrak{H}^0 \ni xxxxyxyxy \leftrightarrow (3, 2, 3).$$

We use $\mathbf{1}$ to refer to the empty word in \mathfrak{H} . We can also say that for an admissible sequence $\mathbf{s} = (s_1, s_2, \dots, s_\ell)$,

$$\zeta(\mathbf{s}) = \zeta(x^{s_1-1}y \dots x^{s_\ell-1}y)$$

and we view $\zeta(w)$, $w \in \mathfrak{H}$ as a linear combination of the zeta values for these words. We also think of Li as a linear map from \mathfrak{H}^1 to the space of functions over $(0, 1)$. Then, we write $\text{Li}(w)$ as $\text{Li}_w(z)$ for any $w \in \mathfrak{H}^1$.

Lemma 3.3. *If $w = au = (s_1, s_2, \dots, s_k) = \mathbf{s}$ is a word in \mathfrak{H}^1 when a is a letter and u is a word, then*

$$\frac{d}{dz} \text{Li}_w(z) = \begin{cases} \frac{1}{z} \text{Li}_u(z) & \text{if } a = x \\ \frac{1}{1-z} \text{Li}_u(z) & \text{if } a = y \end{cases}.$$

Proof. Note that

$$\frac{d}{dz} \text{Li}_w(z) = \sum_{n_1 > \dots > n_\ell \geq 1} \frac{z^{n_1-1}}{n_1^{s_1-1} n_2^{s_2} \dots n_\ell^{s_\ell}}.$$

If $a = x$ then $s_1 > 1$ and

$$\sum_{n_1 > \dots > n_\ell \geq 1} \frac{z^{n_1-1}}{n_1^{s_1-1} n_2^{s_2} \dots n_\ell^{s_\ell}} = \frac{1}{z} \text{Li}_u(z).$$

If $a = y$ then $s_1 = 1$ and

$$\begin{aligned} \sum_{n_1 > \dots > n_\ell \geq 1} \frac{z^{n_1-1}}{n_1^{s_1-1} n_2^{s_2} \dots n_\ell^{s_\ell}} &= \sum_{n_1 > \dots > n_\ell \geq 1} \frac{z^{n_1-1}}{n_2^{s_2} \dots n_\ell^{s_\ell}} \\ &= \sum_{n_2 > \dots > n_\ell \geq 1} \frac{1}{n_2^{s_2} \dots n_\ell^{s_\ell}} \sum_{n_1 > n_2} z^{n_1-1} \\ &= \sum_{n_1 > \dots > n_\ell \geq 1} \frac{1}{1-z} \frac{z^{n_2}}{n_2^{s_2} \dots n_\ell^{s_\ell}} = \frac{1}{1-z} \text{Li}_u(z). \end{aligned}$$

■

Now we can prove the integral representation of Li .

Theorem 3.4. For any word $\mathfrak{H}^1 \ni w = a_1 a_2 \dots a_k$ where a_i are letters and $z \in [0, 1)$,

$$Li_w(z) = \int_0^z \omega_{a_1}(x_1) \int_0^{x_1} \omega_{a_2}(x_2) \cdots \int_0^{x_{k-1}} \omega_{a_k}(x_k).$$

Proof. The proof is by induction on k , where $k = 1$ follows because

$$\int_0^z \omega_y(x) = \int_0^z \frac{1}{1-x} dx = \int_0^z \sum_{n=0}^{\infty} x^n dx = \sum_{n=1}^{\infty} \frac{z^n}{n} = Li_y(z).$$

Then, the induction step follows by 3.3 as

$$\begin{aligned} \frac{d}{dz} Li_{xu} &= \frac{1}{z} Li_u(z) \\ \implies Li_{xu}(z) &= C + \int_0^z \omega_x(x_1) Li_u(x_1) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dz} Li_{yu} &= \frac{1}{1-z} Li_u(z) \\ \implies Li_{yu}(z) &= C + \int_0^z \omega_y(x_1) Li_u(x_1) \end{aligned}$$

but $Li(0) = 0$, so C is 0 for any u, z and we are done. ■

Now we can define the shuffle and stuffle products of words in \mathfrak{H}^1 .

Definition 3.5. The shuffle product over \mathfrak{H} , or \sqcup , is defined by the rules $w \sqcup \mathbf{1} = \mathbf{1} \sqcup w = w$ for any word w and

$$au \sqcup bv = a(u \sqcup bv) + b(au \sqcup v)$$

where a and b are letters and u and v are words.

Definition 3.6. The stuffle product over \mathfrak{H}^1 , or $*$, is defined by the rules $w * \mathbf{1} = \mathbf{1} * w = w$ and

$$x_j u * x_k v = x_j(u * x_k v) + x_k(x_j u * v) + x_{j+k}(u * v)$$

where $x_m = x^{m-1}y$ and u and v are words. We can extend the stuffle product to \mathfrak{H} by letting $x^j * w = w * x^j = wx_j$.

These two operations are extended linearly over \mathfrak{H} . One can see by induction on the length of words that the stuffle and shuffle operations are commutative and associative.

Let's look at some examples of these products. Note that

$$\begin{aligned} xy \sqcup xy &= x(y \sqcup xy) + x(xy \sqcup y) \\ &= x(y(xy) + x(y \sqcup y)) + x(x(y \sqcup y) + yxy) \\ &= xyxy + 2xxyy + 2xxyy + xyxy \\ &= 2xyxy + 4xxyy. \end{aligned}$$

As an example of a stuffle product,

$$\begin{aligned} xy * xy &= xy(xy) + xy(xy) + xxy \\ &= 2xyxy + xxy. \end{aligned}$$

These products give rise to many relationships between multiple zeta values. This is because we have the two beautiful relations

$$\zeta(w \sqcup v) = \zeta(w)\zeta(v) \quad (3.1)$$

and

$$\zeta(w * v) = \zeta(w)\zeta(v) \quad (3.2)$$

for any $w, v \in \mathfrak{H}^0$. These are called the shuffle relations. In our previous examples, this tells us that

$$\zeta(2)^2 = 2\zeta(2, 2) + 4\zeta(3, 1) = 2\zeta(2, 2) + \zeta(4).$$

We can prove 3.1 from the iterated integral representations. In our previous example,

$$\zeta(2) = \int_{1 > x_1 > x_2 > 0} \omega_x(x_1)\omega_y(x_2)$$

so

$$\zeta(2)^2 = \left(\int_{1 > x_1 > x_2 > 0} \omega_x(x_1)\omega_y(x_2) \right) \left(\int_{1 > y_1 > y_2 > 0} \omega_x(y_1)\omega_y(y_2) \right).$$

We can rewrite this as an integral over the region $1 > x_1 > x_2 > 0$ and $1 > y_1 > y_2 > 0$ and casework on the order gives

$$\begin{aligned} \zeta(2)^2 &= \left(\int_{1 > y_1 > y_2 > x_1 > x_2 > 0} + \int_{1 > y_1 > x_1 > y_2 > x_2 > 0} + \int_{1 > y_1 > x_1 > x_2 > y_2 > 0} \right. \\ &\quad \left. + \int_{1 > x_1 > y_1 > y_2 > x_2 > 0} + \int_{1 > x_1 > y_1 > x_2 > y_2 > 0} + \int_{1 > x_1 > x_2 > y_1 > y_2 > 0} \right) \omega_x(x_1)\omega_y(x_2)\omega_x(y_1)\omega_y(y_2) \\ &= \zeta(xyxy) + \zeta(xxyy) + \zeta(xxyy) + \zeta(xxyy) + \zeta(xxyy) + \zeta(xyxy) \\ &= 2\zeta(2, 2) + 4\zeta(3, 1) \end{aligned}$$

which is exactly what we found with 3.1.

In fact, 3.1 holds for multiple polylogarithms.

Theorem 3.7. *For any $w, v \in \mathfrak{H}^0$,*

$$\text{Li}_w(z)\text{Li}_v(z) = \text{Li}_{w \sqcup v}(z).$$

The proof follows from 3.3, and can be found in more detail in [1] (50).

The result in 3.2 does not follow from the iterated integral expansions, but instead from truncated Multiple Zeta Values, defined as

$$\zeta_M(s_1, \dots, s_\ell) = \sum_{M > n_1 > \dots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \dots n_\ell^{s_\ell}}.$$

Theorem 3.8. *For any M and $w, u \in \mathfrak{H}^0$, $\zeta_M(w)\zeta_M(u) = \zeta_M(w * u)$.*

Proof. The proof can be found in [2] (53-54). ■

Taking the limit as $M \rightarrow \infty$ proves 3.2.

Note that the shuffle relations only give linear dependencies of values of the same weight. However, these are not the only linear dependencies among MZVs of the same weight. If we let \mathcal{Z} be the \mathbb{Q} -vector space spanned by MZVs of elements of \mathfrak{H}^0 and \mathcal{Z}_k to be the \mathbb{Q} -vector space spanned by MZVs with weight k , then there have been a number of results and conjectures involving \mathcal{Z} and \mathcal{Z}_k .

For example, an upper bound is known on $\dim \mathcal{Z}_k$.

Theorem 3.9. *We have $\dim \mathcal{Z}_k \leq d_k$ where d_k is defined by $d_0 = 1, d_1 = 0, d_2 = 1$ and $d_k = d_{k-2} + d_{k-3}$ for $k \geq 3$.*

It is conjectured that equality holds [2] (4).

Conjecture 3.10. *For all k , $\dim \mathcal{Z}_k = d_k$.*

It was also proven that \mathcal{Z}_k is spanned by indices with just 2s and 3s.

Theorem 3.11. *The set*

$$\{\zeta(s_1, \dots, s_\ell) \mid s_1, \dots, s_\ell \in \{2, 3\}\}$$

generates the \mathbb{Q} -subspace \mathcal{Z} .

It is conjectured that this set is a basis of \mathcal{Z} , which would imply 3.10.

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