Fourier Analysis through the Lens of Linear Algebra

Etienne Pema La Fleur

Abstract

Fourier analysis rests on the claim that functions can be expressed as sums of sines and cosines. This paper argues that the Fourier coefficient formulas become natural when viewed through the lens of linear algebra. By interpreting the definite integral of a product of functions as an inner product, trigonometric functions form an orthogonal system analogous to an orthogonal basis in $\mathbb{R}^n$. The Fourier coefficients then arise from orthogonal projection.

1 Introduction

Fourier analysis studies how functions can be expressed as sums of trigonometric functions. In particular,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

At first glance, this representation may seem surprising. Why should combinations of simple sine and cosine functions capture the behavior of more complicated functions?

In this paper, we show that this idea becomes natural when viewed through the lens of linear algebra. By defining an inner product using a definite integral, functions behave like vectors in an infinite-dimensional space. The Fourier coefficients arise from projection onto orthogonal functions.

We develop this perspective by first reviewing orthogonal projection in $\mathbb{R}^n$, then extending these ideas to functions. Using this framework, we derive the Fourier coefficient formulas and interpret results such as Parseval's Theorem geometrically.

Questions about convergence and which functions admit Fourier series lie beyond the scope of this paper; see [1].

2 Projection of Orthogonal Basis

Definition 1. *Let $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in $\mathbb{R}^n$. The dot product is*

$$u \cdot v = \sum_{i=1}^{n} u_i v_i.$$

If

$$u \cdot v = 0,$$

then u and v are orthogonal.

If $\{e_1, \dots, e_n\}$ is an orthogonal basis, then

$$v = \sum_{i=1}^{n} c_i e_i, \quad c_k = \frac{v \cdot e_k}{e_k \cdot e_k}.$$

3 Extension to Functions

Define

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx.$$

This inner product is linear, symmetric, and positive definite.

The trigonometric functions satisfy:

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0.$$

Using identities:

$$\sin(mx)\sin(nx) = \tfrac{1}{2}[\cos((m-n)x) - \cos((m+n)x)]$$

$$\cos(mx)\cos(nx) = \tfrac{1}{2}[\cos((m-n)x) + \cos((m+n)x)]$$

$$\sin(mx)\cos(nx) = \tfrac{1}{2}[\sin((m+n)x) + \sin((m-n)x)]$$

4 Bridging the Two Concepts

We express

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

Using projection:

$$a_n = \frac{\langle f, \cos(nx)\rangle}{\langle \cos(nx), \cos(nx)\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

5 Energy and Further Results

Definition 2. *The energy of a function is*

$$\|f\|^2 = \int_{-\pi}^{\pi} |f(x)|^2\,dx.$$

Parseval's Theorem

If

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

then

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx = \pi \left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right).$$

This is analogous to the Pythagorean theorem.

Plancherel's Theorem

For functions f and g,

$$\langle f, g\rangle = \pi \left(\frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} (a_n c_n + b_n d_n) \right),$$

where the coefficients correspond to f and g.

Poisson Summation Formula

For suitable functions,

$$\sum_{k=-\infty}^{\infty} f(k) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

6 Conclusion

Fourier analysis becomes natural when viewed through linear algebra. Functions behave like vectors, and trigonometric functions form an orthogonal system. Fourier coefficients arise from projection, and results such as Parseval's Theorem reflect geometric structure.

References

[1] Gilbert Strang. *Introduction to Linear Algebra*. Wellesley-Cambridge Press, 5 edition, 2016.