Rational Approximations and Periodicity: An Exploration of Continued Fractions

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Abstract

In this paper, I explore the theory of simple continued fractions, focusing on how their convergents provide strong rational approximations of irrational numbers. I connect these approximations to the criterion for irrationality discussed in Chapter 4 of our book. I also show how the size of the partial quotients controls the quality of approximation, which explains some well-known examples. Finally, I investigate the periodic nature of continued fractions, culminating in a proof of the forward direction of Lagrange's Theorem regarding quadratic irrationals.

1 Introduction

Throughout our study of infinite series, we have relied on partial sums to approximate the values of infinite sums, testing for convergence as the number of terms approaches infinity. A parallel concept exists in number theory and analysis, namely continued fractions. Instead of adding a sequence of terms, a continued fraction represents a number through a sequence of nested denominators.

In Chapter 4 of our class text, *Infinite Series* [1], we encountered the Criterion for Irrationality (Theorem 4.1), which essentially states that a number is irrational if it can be approximated "too well" by a sequence of rational numbers. This raises a natural question of how we find especially good rational approximations. We can find them in the convergents of continued fractions. By analyzing the mechanics of continued fractions, we get a systematic way to generate rational approximations that satisfy these irrationality criteria. Furthermore, by restricting the sequence of terms in the fraction, we can classify exactly what kinds of numbers we are dealing with.

In this paper, I will first formalize the definition of continued fractions and their convergents. I will then prove the core recurrence relations that generate them, showing how the convergents provide especially strong rational approximations. I will also show how the size of each partial quotient controls how well the preceding convergent approximates the number, and use this to explain a famous example involving π. Finally, I will focus on periodic continued fractions and prove that they always converge to quadratic irrationals.

2 The Mechanics of Continued Fractions

We begin by defining what a continued fraction is and establishing the notation we will use throughout the paper.

Definition 1. *A* ***simple continued fraction*** *is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}},$$

where a_0 is an integer, and $a_1, a_2, a_3, \dots$ are positive integers.

Because the traditional fraction notation takes up too much vertical space, we typically denote a simple continued fraction using the bracket notation $[a_0, a_1, a_2, a_3, \dots]$. The numbers a_i are called the *partial quotients*. Just as we evaluate an infinite series by looking at the limit of its partial sums, we evaluate an infinite continued fraction by looking at the limit of its *convergents*.

Definition 2. *The n-th* **convergent** *of the continued fraction $x = [a_0, a_1, a_2, \dots]$, denoted C_n, is the finite continued fraction formed by cutting off the expansion after a_n. That is, $C_n = [a_0, a_1, a_2, \dots, a_n]$.*

To compute the value of C_n, there is a recursive method. Let us define two sequences, p_n and q_n, representing the numerators and denominators of the convergents.

Theorem 3. *Define sequences p_n and q_n recursively as follows:*

$$\begin{aligned} p_{-2} &= 0, & p_{-1} &= 1, & p_n &= a_n p_{n-1} + p_{n-2} \text{ for } n \geq 0, \\ q_{-2} &= 1, & q_{-1} &= 0, & q_n &= a_n q_{n-1} + q_{n-2} \text{ for } n \geq 0. \end{aligned}$$

Then for all $n \geq 0$, the n-th convergent is given by $C_n = \dfrac{p_n}{q_n}$.

Proof. We proceed by induction on n.

Base Cases: For $n = 0$, $p_0 = a_0(1) + 0 = a_0$ and $q_0 = a_0(0) + 1 = 1$. So $C_0 = \frac{p_0}{q_0} = a_0$, which is correct. For $n = 1$, $p_1 = a_1 a_0 + 1$ and $q_1 = a_1(1) + 0 = a_1$. So

$$C_1 = \frac{a_1 a_0 + 1}{a_1} = a_0 + \frac{1}{a_1},$$

which matches $[a_0, a_1]$.

Inductive Step: Assume that for some arbitrary $k \geq 1$, the theorem holds for any continued fraction. Consider the $(k+1)$-th convergent $C_{k+1} = [a_0, a_1, \dots, a_k, a_{k+1}]$. We can think of this as a k-th convergent where the final term a_k is replaced by $\left(a_k + \frac{1}{a_{k+1}}\right)$. By our inductive hypothesis, the numerator and denominator of this modified k-th convergent are

$$\text{Numerator} = \left(a_k + \frac{1}{a_{k+1}}\right) p_{k-1} + p_{k-2},$$
$$\text{Denominator} = \left(a_k + \frac{1}{a_{k+1}}\right) q_{k-1} + q_{k-2}.$$

To find C_{k+1}, we divide the numerator by the denominator and multiply the top and bottom by a_{k+1} to clear the internal fraction:

$$C_{k+1} = \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}}.$$

The terms in parentheses are exactly p_k and q_k by the recurrence relation, so substituting gives

$$C_{k+1} = \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}} = \frac{p_{k+1}}{q_{k+1}},$$

completing the induction. □

3 Rational Approximations and Irrationality

Now that we have a mechanism for generating the convergents p_n/q_n, we can examine why they are so useful for rational approximation, and connect this back to the themes of Chapter 4. A key property of convergents is the determinant identity.

Theorem 4 (Determinant Identity)**.** *For any $n \geq -1$,*

$$p_{n+1} q_n - p_n q_{n+1} = (-1)^n.$$

Proof. We substitute the recurrence relations from Theorem 3:

$$\begin{aligned} p_{n+1} q_n - p_n q_{n+1} &= (a_{n+1} p_n + p_{n-1}) q_n - p_n (a_{n+1} q_n + q_{n-1}) \\ &= a_{n+1} p_n q_n + p_{n-1} q_n - a_{n+1} p_n q_n - p_n q_{n-1} \\ &= -(p_n q_{n-1} - p_{n-1} q_n). \end{aligned}$$

This shows the quantity $(p_{n+1} q_n - p_n q_{n+1})$ flips sign with each increment of n. Since for $n = -1$ we have

$$p_0 q_{-1} - p_{-1} q_0 = a_0(0) - 1(1) = -1 = (-1)^{-1},$$

the identity follows by induction. □

Dividing the determinant identity by $q_n q_{n+1}$ gives a useful result about the difference between consecutive convergents:

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n+1}}.$$

Because the true value of the continued fraction x always lies strictly between consecutive convergents, the error in approximating x by p_n/q_n is bounded above by the gap to the next convergent:

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

3.1 A Sharper Bound

The estimate above can be improved by using the recurrence relation for the denominators. Since

$$q_{n+1} = a_{n+1} q_n + q_{n-1} \geq a_{n+1} q_n,$$

the bound

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

implies

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2}.$$

So the quality of the approximation depends not only on the denominator q_n but also on the next partial quotient a_{n+1}. When a_{n+1} is large, the convergent p_n/q_n is especially close to x. When the partial quotients stay small, the improvement is more gradual.

A well-known example is

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \ldots].$$

The convergent just before the term 292 is 355/113. Since the next partial quotient is 292, our bound gives

$$\left| \pi - \frac{355}{113} \right| < \frac{1}{292 \cdot 113^2} = \frac{1}{3{,}728{,}548} \approx 2.7 \times 10^{-7}.$$

Indeed, $355/113 \approx 3.14159292\ldots$, which agrees with π to six decimal places. The presence of 292 as a partial quotient is the reason this particular approximation is so good relative to its denominator.

3.2 Convergents and Irrationality

The convergents of a simple continued fraction do more than provide a sequence of approximations. Since

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2},$$

they produce infinitely many rational approximations that satisfy the kind of inequality used in the Criterion for Irrationality from Chapter 4. Therefore, if a real number has an infinite simple continued fraction expansion, it must be irrational. In this way, continued fractions give a direct and systematic source of rational approximations strong enough to detect irrationality.

4 Periodic Continued Fractions and Quadratic Irrationals

We have established that infinite simple continued fractions represent irrational numbers. Just as the decimal expansions of rational numbers can be repeating, continued fractions exhibit their own form of periodicity.

Definition 5. *A continued fraction is* ***eventually periodic*** *if there exist positive integers N and k such that for all $n \geq N$, the partial quotients satisfy $a_n = a_{n+k}$. We denote the repeating block with an overline, for example,*

$$x = [a_0, a_1, \ldots, a_{N-1}, \overline{b_1, \ldots, b_k}].$$

A major result in the theory of continued fractions identifies exactly which numbers have eventually periodic continued fractions.

Theorem 6 (Lagrange's Theorem)**.** *A real number x can be represented as an eventually periodic simple continued fraction if and only if x is a quadratic irrational, that is, a root of a quadratic equation with integer coefficients that is not rational.*

Proving the full theorem requires a lengthy argument for the reverse direction that proved to be too difficult for me. However, the forward direction, which states that every eventually periodic continued fraction evaluates to a quadratic irrational, follows nicely from the recurrence relations we established in Section 2.

Proof of the Forward Direction. Let x be a purely periodic continued fraction, meaning $x = [\overline{a_0, a_1, \ldots, a_k}]$. Because it is purely periodic, we can write

$$x = [a_0, a_1, \ldots, a_k, x].$$

We treat this as evaluating the $(k+1)$-th convergent of a sequence, but instead of the last term being an integer, the last term is the exact value x itself. By the inductive step logic from the proof of Theorem 3,

$$x = \frac{xp_k + p_{k-1}}{xq_k + q_{k-1}}.$$

Multiplying both sides by the denominator yields

$$x(xq_k + q_{k-1}) = xp_k + p_{k-1},$$

so

$$q_k x^2 + (q_{k-1} - p_k)x - p_{k-1} = 0.$$

Since all a_i are integers, the sequences p_n and q_n consist entirely of integers. Therefore $A = q_k$, $B = (q_{k-1} - p_k)$, and $C = -p_{k-1}$ are all integers, so x is a root of the integer-coefficient quadratic

$$Ax^2 + Bx + C = 0.$$

Since x is defined by an infinite continued fraction, it is irrational, and therefore it is a quadratic irrational.

If the fraction is not purely periodic but eventually periodic, say

$$y = [c_0, c_1, \ldots, c_m, x],$$

where x is the purely periodic part, then x is a quadratic irrational by the above. Since

$$y = \frac{xp_m + p_{m-1}}{xq_m + q_{m-1}},$$

applying rational operations to a quadratic irrational gives another quadratic irrational, completing the proof. □

Example 7. *Consider the simplest periodic continued fraction, the Golden Ratio* $\phi = [\overline{1}]$. *Setting* $\phi = [1, \phi]$ *gives* $\phi = 1 + 1/\phi$, *so* $\phi^2 = \phi + 1$, *or* $\phi^2 - \phi - 1 = 0$. *Using the quadratic formula and knowing* $\phi > 0$, *we find* $\phi = \frac{1+\sqrt{5}}{2}$, *which is a quadratic irrational.*

5 Conclusion

Continued fractions serve as a useful tool in number theory. As we have seen, the convergents generated by the recurrence relations give especially strong rational approximations to irrational numbers. The size of the next partial quotient tells us how good a given convergent is, and this explains why 355/113 is such an accurate approximation of π. At the same time, the structural properties of continued fractions let us categorize numbers by the pattern of their partial quotients, identifying quadratic irrationals by checking for periodicity. While what we have learned in infinite series has mostly focused on sums to evaluate constants and functions, continued fractions offer a complementary approach that is central to rational approximation theory and Diophantine analysis.

References

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