The Mystery of the Flint Hills Series

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March 25 2026

Abstract

In this paper, we will talk about the Flint Hills Series and the mystery of whether it converges. We will then explore the closely related *irrationality exponent function* $\mu(x)$, reviewing some of its basic properties and then demonstrating why has the connection it does to the Flint Hills series. We will talk about continued fractions, and how they can give some information on the value of $\mu(x)$ for a particular x. At the end, we will also touch on other irrationality measures and how they relate to $\mu(x)$.

1 Introduction

The *Flint Hills series* is defined as the following infinite sum:

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}.$$

Whether this series converges is an open problem. This seems bizzare; the terms of the sum have an elementary representation, and it's generally easy to figure out whether such a sum converges.

The reason this is an open problem is that the values of $\csc^2(n)$ for positive integers n are erratic. Looking at a graph of $\csc^2(x)$, one can see that it diverges to ∞ near the integer multiples of π. Therefore, whenever a positive integer n happens to be close to an integer multiple of π, the value of $\csc^2(n)$ can get quite large. The problem is, such n show up sporadically, so there's no formulaic method for dealing with the corresponding terms in the summation. This makes it difficult for us to use our usual tricks for figuring out whether a sum converges; we need something new.

Whether the Flint Hills series converges or not comes down to how often positive integers n get really close to integer multiples of π. If they don't get close too often, then the $\frac{1}{n^3}$ takes over and the series converges. But if some of them end up getting abnormally close, then the corresponding terms in the sum will build up and the series will diverge. Now, notice that the expression $n \approx k\pi$ can be rewritten as $\frac{n}{k} \approx \pi$. This suggests that we need some sort of measure for how well π is approximated by rational numbers.

Definition 1.1. Let $x \in \mathbb{R}$. Then, let S be the set of all $\mu \in (0, \infty)$ for which there are infinitely many coprime integer pairs (p, q) with $q > 0$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}.$$

The *irrationality exponent* or *Liouville–Roth irrationality measure* of x, denoted $\mu(x)$, is the supremum of the set S. If there is no upper bound on the set S, we say that $\mu(x) = \infty$.

Remark 1.2. If S is empty, we may as well say $\mu(x) = 0$. However, we'll soon see that this case can't happen.

Now, it turns out that the value of $\mu(\pi)$ has an intimate connection with the Flint Hills series. More precisely:

Theorem 1.3 ([1] and [6]). *If $\mu(\pi) < 2.5$, then the Flint Hills series converges. If $\mu(\pi) > 2.5$, it diverges.*

Note that if $\mu(\pi) = 2.5$, this theorem doesn't tell us anything.

If we could somehow prove the convergence of the Flint Hills series, by this theorem we would know that $\mu(\pi) \leq 2.5$. For context, that's much better then the best currently known upper bound, which is $7.103205\ldots$ (see [8]). We may not know the value of $\mu(x)$ at $x = \pi$, but let's look at what we do know about μ.

2 What We Know

Let us start with the rational numbers: We want to determine how well rational numbers are approximated by other rational numbers.

Theorem 2.1. *For any $x \in \mathbb{Q}$, we have $\mu(x) = 1$.*

Proof. Let $\frac{a}{b}$ be a rational number written so that $\gcd(a, b) = 1$ and $b > 0$. Let

$$\frac{p_n}{q_n} = \frac{an+1}{bn},$$

where p_n and q_n are coprime integers and $n \geq 2$. It's easy to see that n is coprime to $an+1$, which means that factor is still contained in q_n, and $q_n > 1$. Also, all the fractions $\frac{p_n}{q_n}$ are distinct numbers, because

$$\frac{an+1}{bn} = \frac{a}{b} + \frac{1}{bn}$$

changes as n grows. Next, we have that

$$\left|\frac{a}{b} - \frac{an+1}{bn}\right| = \frac{1}{bn},$$

and that

$$0 < \frac{1}{bn} \leq \frac{1}{q_n}.$$

In particular, if $\varepsilon > 0$, then

$$\frac{1}{q_n} < \frac{1}{q_n^{1-\varepsilon}}$$

since $q_n > 1$. Thus we have that

$$0 < \left|\frac{a}{b} - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1-\varepsilon}}$$

for all $n \geq 2$ and $\varepsilon > 0$. These are infinitely many *distinct* coprime pairs satisfying the inequality for all $\mu < 1$, so S (defined as in Definition 1.1) contains everything in the interval $(0, 1)$.

Next, we have to show that no element of $(1, \infty)$ is in S. Stated differently, we want to show that for all $\varepsilon > 0$, there are only finitely many coprime pairs (p, q) such that

$$0 < \left|\frac{a}{b} - \frac{p}{q}\right| < \frac{1}{q^{1+\varepsilon}}.$$

If the above inequality is true, then in particular

$$\left|\frac{a}{b} - \frac{p}{q}\right| < 1.$$

Thus for each q, there are only finitely many possibilities for what p can be. Therefore (since $q > 0$) it suffices to give an upper bound N on the value of q.

We can rearrange the original inequality:

$$\begin{aligned}
\left|\frac{a}{b} - \frac{p}{q}\right| &< \frac{1}{q^{1+\varepsilon}} \\
\frac{|aq - bp|}{bq} &< \frac{1}{q^{1+\varepsilon}} \\
|aq - bp| &< \frac{bq}{q^{1+\varepsilon}} \\
|aq - bp| &< \frac{b}{q^{\varepsilon}}
\end{aligned}$$

Since we assumed $\left|\frac{a}{b}-\frac{p}{q}\right| \neq 0$, we know that $|aq-bp| \neq 0$, so $|aq-bp| \geq 1$ since it's a positive integer. Combining this with the previous inequality gives $1 < \frac{b}{q^\varepsilon}$. Thus $q^\varepsilon < b$, or $q < b^{\frac{1}{\varepsilon}}$. Setting $N \geq b^{\frac{1}{\varepsilon}}$ gives an upper bound on q for this particular ε. We have now shown that when $\mu < 1$, $\mu \in S$, and when $\mu > 1$, $\mu \notin S$. Thus, the supremum of S is 1, and

$$\mu\left(\frac{a}{b}\right) = 1.$$

□

Next, we want to study the irrational case. To start, we'll need *Dirichlet's Approximation Theorem*:

Theorem 2.2 (Dirichlet's Approximation Theorem)**.** *Let $\alpha \in \mathbb{R}$, and let N be a positive integer. Then there exist coprime integers p, q with $1 \leq q \leq N$ such that*

$$|q\alpha - p| < \frac{1}{N}.$$

Proof. It is sufficient to find p, q not necessarily coprime; one can divide both by $\gcd(p,q)$ and the result will still hold. For $0 \leq i \leq N$, let $a_i = \alpha i - \lfloor \alpha i \rfloor \in [0,1)$. Consider the intervals

$$I_k = \left[\frac{k-1}{N}, \frac{k}{N}\right)$$

for $1 \leq k \leq N$. These N intervals partition $[0,1)$, and the a_i's are $n+1$ numbers all in $[0,1)$, so by the Pigeonhole Principle there exist a_i and a_j (say with $i > j$) that are in the same interval I_k. Thus

$$|a_i - a_j| = |i\alpha - \lfloor i\alpha \rfloor - j\alpha + \lfloor j\alpha \rfloor| < \frac{1}{N}.$$

Letting $q = i - j$ and $p = \lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor$ gives the desired result. □

Using this theorem, we can prove the following irrationality criterion:

Theorem 2.3. *Let $\alpha \in \mathbb{R}$. Then, α is irrational if and only if, for any $\varepsilon > 0$, there exist integers p and q such that*

$$0 < |q\alpha - p| < \varepsilon.$$

Proof. Suppose that $\alpha \notin \mathbb{Q}$, and let $\varepsilon > 0$. Choose N such that $\frac{1}{N} < \varepsilon$. By the Dirichlet Approximation Theorem, there exist integers p, q with $1 \leq q \leq N$ such that $|q\alpha - p| < \frac{1}{n} < \varepsilon$. Furthermore, $0 < |q\alpha - p|$ because $\alpha \notin \mathbb{Q}$.

Now suppose that $\alpha \in \mathbb{Q}$, say with $\alpha = \frac{r}{s}$. If $\frac{p}{q} \neq \frac{r}{s}$, then we have

$$\left|\frac{r}{s} - \frac{p}{q}\right| \geq \frac{1}{qs},$$

or

$$|q\alpha - p| \geq \frac{1}{s}.$$

Thus if we choose $N \geq s$, there do not exist integers p and q with $1 \leq q \leq N$ such that $0 < |q\alpha - p| < \frac{1}{N}$. □

The above theorem is super useful when it comes to testing a number's irrationality. For example, as we'll see below, the hard question of determining whether e is irrational is made much easier with this criterion.

Corollary 2.4. *e is irrational.*

Proof. Recall the series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Let N be a positive integer such that $\frac{1}{N} < \varepsilon$. Let $q = N!$, and let p be such that

$$\frac{p}{q} = \frac{p}{N!} = \sum_{n=0}^{N} \frac{1}{n!}.$$

Note that p is an integer. We then have

$$\begin{aligned}
|qe - p| &= \left| N!e - \sum_{n=0}^{N} \frac{N!}{n!} \right| \\
&= \sum_{n=N+1}^{\infty} \frac{N!}{n!} \\
&= \sum_{n=N+1}^{\infty} \frac{1}{(N+1)(N+2)\cdots n} \\
&\leq \sum_{n=N+1}^{\infty} \frac{1}{(N+1)^{n-N}} \\
&= \frac{1/(N+1)}{1 - 1/(N+1)} \\
&= \frac{1}{N} \\
&< \varepsilon.
\end{aligned}$$

Since clearly $|qe - p| \neq 0$, we can conclude that e is irrational. ☐

Theorem 2.2 is also useful when it comes to the irrationality measure μ.

Corollary 2.5. *If $\alpha \in \mathbb{R}$ is irrational, then $\mu(\alpha) \geq 2$.*

Proof. We know that for any positive integer i, there exist coprime integers p_i, q_i with $1 \leq q_i \leq i$ such that

$$0 < |q_i\alpha - p_i| < \frac{1}{i} \leq \frac{1}{q_i}.$$

Thus

$$0 < \left| \alpha - \frac{p_i}{q_i} \right| < \frac{1}{q_i^2}.$$

These (p_i, q_i) pairs are such that $\frac{p_i}{q_i}$ gets arbitrarily close to α since, in particular,

$$\left| \alpha - \frac{p_i}{q_i} \right| < \frac{1}{i}.$$

Thus there are infinitely many distinct pairs (p_i, q_i), and we're done. ☐

As it turns out, the vast majority of real numbers have irrationality exponent exactly 2. For the readers among you who know measure theory, it turns out that *almost all* real numbers have irrationality exponent 2:

Theorem 2.6 ([2])**.** *Let T be the set of all $x \in \mathbb{R}$ such that $\mu(x) \neq 2$. Then, T has Lebesgue measure* 0.

In fact, all algebraic irrational numbers have irrationality exponent 2.

Theorem 2.7 ([4])**.** *Let $\alpha \in \mathbb{R}$ be an algebraic irrational number. Then, $\mu(\alpha) = 2$.*

This allows you to prove that some specific numbers are transcendental by showing that they have irrationality exponent > 2. However, only a tiny fraction of all numbers satisfy this property, so this is a far from general criterion. For instance, e is a transcendental number that nonetheless has irrationality exponent 2.

Theorem 2.8. $\mu(e) = 2$.

We will prove this theorem when we look at *continued fractions*.

In general, though, it's much harder to find the irrationality exponent of known constants, and π is an example of that. Next, let us look at how $\mu(\pi)$ ties into the convergence or divergence of the Flint Hills Series.

3 Determining Convergence of the Flint Hills Series

The Flint Hills Series easily generalizes to series of the form

$$F(u,v) := \sum_{n=1}^{\infty} \frac{1}{n^u|\sin(n)|^v}$$

for $u, v > 0$, with $F(3,2)$ being the original series. Theorem 1.3 is simply one example of a much larger result connecting the convergence or divergence of $F(u,v)$ to a certain bound on u and v. First, we need a lemma:

Lemma 3.1 ([1]). *For $x \in \mathbb{R}$, we have*

$$|\sin(x)| \leq |x|.$$

Furthermore, if $|x| \leq \frac{\pi}{2}$, then

$$|\sin(x)| \geq \frac{2}{\pi} \cdot |x|.$$

Proof. For the former bound, we have that

$$\begin{aligned}|\sin(x)| &= \left|\int_0^x \cos(t)dt\right| \\ &\leq \int_0^{|x|} |\cos(t)|dt \\ &\leq \int_0^{|x|} 1dt \\ &= |x|.\end{aligned}$$

For the second bound, notice that when $|x| \leq \frac{\pi}{2}$, we have $|\sin(x)| = \sin(|x|)$. Thus without loss of generality we can assume that $0 \leq x \leq \frac{\pi}{2}$. Let $x_0 = \arccos\left(\frac{2}{\pi}\right)$. Then, for $x \leq x_0$, we have $\cos(x) \geq \frac{2}{\pi}$ and thus

$$\begin{aligned}\sin(x) &= \int_0^x \cos(t)dt \\ &\geq \int_0^x \frac{2}{\pi}dt \\ &= \frac{2}{\pi}x.\end{aligned}$$

For $x \geq x_0$, we have $\cos(x) \leq \frac{2}{\pi}$, and thus

$$\begin{aligned}\sin(x) &= 1 - \int_x^{\frac{\pi}{2}} \cos(t)dt \\ &\geq 1 - \int_x^{\frac{\pi}{2}} \frac{2}{\pi}dt \\ &= 1 - \frac{2}{\pi}\left(\frac{\pi}{2} - x\right) \\ &= \frac{2}{\pi}x.\end{aligned}$$

□

Now, for the next theorem we're going to look at, the proof in [1] had a subtle flaw that we'll need a lemma to fix:

Lemma 3.2. *Let $n \geq 4$ be an integer, and denote m for the value of $\frac{n}{\pi}$ rounded to the nearest integer. Then for any positive integer B, there exists a positive integer N such that whenever $n \geq N$,*

$$\frac{n}{\gcd(n,m)} > B.$$

(Note that $n \geq 4$ so that m is positive.)

Proof. Since $\frac{1}{\pi}$ is irrational, by Theorem 2.3 there exist integers p, q such that

$$0 < \left|\frac{q}{\pi} - p\right| < \frac{1}{2B}.$$

In particular, letting

$$k = \frac{1}{2}\left|\frac{q}{\pi} - p\right|,$$

we have

$$k < \left|\frac{q}{\pi} - p\right| < \frac{1}{2B}$$

for k a positive constant independent of n. Next, we multiply out by n to get

$$kn < \left|q \cdot \frac{n}{\pi} - pn\right| < \frac{n}{2B}.$$

By definition of m, we know that

$$\left|m - \frac{n}{\pi}\right| \leq \frac{1}{2}.$$

Thus

$$|qm - pn| \leq \left|qm - q \cdot \frac{n}{\pi}\right| + \left|q \cdot \frac{n}{\pi} - pn\right| < \frac{q}{2} + \frac{n}{2B}.$$

Since q is independent of n, we can choose N_1 large enough so that if $n \geq N_1$, then

$$\frac{q}{2} + \frac{n}{2B} < \frac{n}{B}.$$

Therefore for $n \geq N_1$,

$$|qm - pn| < \frac{n}{B}.$$

Next, we know that

$$\left|q \cdot \frac{n}{\pi} - pn\right| \leq \left|q \cdot \frac{n}{\pi} - qm\right| + |qm - pn|,$$

so that

$$\left|q \cdot \frac{n}{\pi} - pn\right| - \left|q \cdot \frac{n}{\pi} - qm\right| \leq |qm - pn|.$$

Thus

$$|qm - pn| \geq \left|q \cdot \frac{n}{\pi} - pn\right| - \left|q \cdot \frac{n}{\pi} - qm\right| > kn - \frac{q}{2}.$$

We can choose N_2 large enough so that if $n \geq N_2$ then

$$kn - \frac{q}{2} > 0,$$

and

$$|qm - pn| > 0.$$

Let $N = \max(N_1, N_2)$. What we've seen so far is that if $n \geq N$, then

$$0 < |qm - pn| < \frac{n}{B}.$$

However, notice that $|qm - pn|$ is an integer multiple of $\gcd(n, m)$. We've now shown that it is in fact a positive integer multiple of $\gcd(n, m)$, so $\gcd(n, m) \leq |qm - pn|$. Thus

$$\gcd(n, m) < \frac{n}{B}.$$

Rearranging this inequality gives

$$\frac{n}{\gcd(n, m)} > B,$$

as desired. $\square$

Next, before we look at the limit of the series $F(u, v)$, we need to know the limiting behavior of the terms in the series.

Theorem 3.3 ([1])**.** *For positive real numbers u, v, we have that for any $\varepsilon > 0$,*

$$\frac{1}{n^u |\sin(n)|^v} = O\left(\frac{1}{n^{u-(\mu(\pi)-1)v-\varepsilon}}\right).$$

Furthermore, if $\mu(\pi) < 1 + \frac{u}{v}$, the sequence

$$\frac{1}{n^u |\sin(n)|^v}$$

converges to 0 as $n \to \infty$, whereas if $\mu(\pi) > 1 + \frac{u}{v}$, it diverges.

Remark 3.4. As before, the case $\mu(\pi) = 1 + \frac{u}{v}$ is inconclusive.

Proof. Let $\varepsilon > 0$ and $k = \mu(\pi) + \frac{\varepsilon}{v}$. Then the inequality

$$\left|\pi - \frac{p}{q}\right| < \frac{1}{q^k} \tag{1}$$

holds only for a finite number of co-prime positive integers p and q. For an integer $n \geq 4$, denote m as in Lemma 3.2, so that

$$\left|\frac{n}{\pi} - m\right| \leq \frac{1}{2}$$

and thus

$$|n - m \cdot \pi| \leq \frac{\pi}{2}.$$

Thus by Lemma 3.1,

$$|\sin(n)| = |\sin(n - m\pi)| \geq \frac{2}{\pi} \cdot |n - m\pi| = \frac{2}{\pi} \cdot m \cdot \left|\frac{n}{m} - \pi\right|.$$

However, for large enough n and m, we have

$$\left|\frac{n}{m} - \pi\right| \geq \frac{1}{m^k},$$

implying that

$$|\sin(n)| \geq \frac{2}{\pi} \cdot m \cdot \left|\frac{n}{m} - \pi\right| \geq \frac{2}{\pi} \cdot \frac{1}{m^{k-1}} \geq c \cdot \frac{1}{n^{k-1}}$$

for some constant $c > 0$ depending on k but not on n, since $\frac{n}{m}$ tends to π as n grows. By construction of k, we can choose a positive integer B big enough so that for any coprime pair (p, q) with $p \geq B$ and $q > 0$,

$$\left|\pi - \frac{p}{q}\right| < \frac{1}{q^k}.$$

By Lemma 3.2, we can choose an N so that for all $n \geq N$, $\frac{n}{\gcd(n,m)} \geq B$ and

$$\left|\frac{n}{m} - \pi\right| = \left|\pi - \frac{n/\gcd(n, m)}{m/\gcd(n, m)}\right| < \frac{1}{q^k}.$$

Therefore, for all $n \geq N$, we have that

$$\frac{1}{n^u \sin(n)^v} \leq \frac{1}{c^v \cdot n^{u-(k-1)v}} = O\left(\frac{1}{n^{u-(\mu(\pi)-1)v-\varepsilon}}\right).$$

Now suppose that $\mu(\pi) < 1 + \frac{u}{v}$. Rearranging this inequality gives

$$u - (\mu(\pi) - 1)v > 0.$$

In particular, we can take $\varepsilon < u - (\mu(\pi) - 1)v$ in the O bound, so that

$$\frac{1}{n^u \sin(n)^v} = O\left(\frac{1}{n^{u-(\mu(\pi)-1)v-\varepsilon}}\right) = o(1).$$

Statement 1 of the theorem now follows.

Next, let's prove statement 2. If $\mu(\pi) > 1 + \frac{u}{v}$, then for $k = 1 + \frac{u}{v}$ the inequality (1) holds for infinitely many co-prime positive integer pairs (p,q) with $q > 0$. In particular, there exists an infinite sequence of distinct pairs (p_i, q_i) such that

$$|p_i - \pi q_i| < \frac{1}{q_i^{k-1}}.$$

Then

$$|\sin(p_i)| = |\sin(p_i - q_i\pi)| \leq |p_i - q_i\pi| < \frac{1}{q_i^{k-1}} < C \cdot \frac{1}{p_i^{k-1}}$$

for some constant $C > 0$ depending only on k. Therefore, for $n = p_i$ we have

$$\frac{1}{n^u \cdot |\sin(n)|^v} > C^{-v} \cdot n^{v(k-1)-u} = C^{-v}.$$

Thus this subsequence is bounded below by a positive constant, so the original sequence does not converge to 0 when $\mu(\pi) > 1 + \frac{u}{v}$. In fact, we know that as $i \to \infty$,

$$|\sin(1 + p_i)| = |\sin(1 + p_i - q_i\pi)| \to \sin(1),$$

so that

$$\frac{1}{(1 + p_i)^u \cdot |\sin(1 + p_i)|^v} \to 0.$$

We conclude that the sequence $\frac{1}{n^u \cdot |\sin(n)|^v}$ diverges, since it contains two subsequences one of which is bounded below by a positive constant, while the other converges to 0. ☐

Now, we can say something about the series $F(u, v)$ itself based on the limit of its terms, but this only works in one direction.

Corollary 3.5 ([1])**.** *If $\mu(\pi) > 1 + \frac{u}{v}$, then the series $F(u, v)$ diverges.*

Proof. If $\mu(\pi) > 1 + \frac{u}{v}$, by Theorem 3.3 we have that

$$\lim_{n\to\infty} \frac{1}{n^u |\sin(n)|^v}$$

is undefined. If

$$\sum_{n=1}^{\infty} \frac{1}{n^u |\sin(n)|^v}$$

were to converge, that would mean that

$$\lim_{n\to\infty} \frac{1}{n^u |\sin(n)|^v} = 0.$$

Thus $F(u, v)$ must diverge. ☐

Remark 3.6. In the case $(u, v) = (3, 2)$, we have that if $\mu(\pi) > 1 + \frac{3}{2} = 2.5$, then the Flint Hills series $F(3, 2)$ diverges.

The unfortunate thing is that this argument doesn't work in reverse, since there are sequences a_n for which

$$\lim_{n\to\infty} a_n = 0$$

but

$$\sum_{n=1}^{\infty} a_n$$

diverges. The prototypical example of this is the Harmonic Series.

Example 3.7. The Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges to ∞.

Proof. We know that

$$\frac{1}{3}, \frac{1}{4} \geq \frac{1}{4}.$$

We also know that

$$\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8} \geq \frac{1}{8},$$

and so on. Thus

$$\begin{aligned}\frac{1}{1} + \frac{1}{2} + \frac{1}{3} &\geq \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{16} \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\ &= \infty.\end{aligned}$$

What we've just shown is that the Harmonic Series is bounded below by a series that diverges to ∞, so it must also diverge to ∞ as well. ☐

Thus, we need a new argument to go the other direction. In [1], an argument is given for the following fact:

Theorem 3.8. *If $\mu(\pi) < 1 + \frac{u-1}{v}$, then $F(u, v)$ converges.*

Proof. The inequality $\mu(\pi) < 1 + \frac{u-1}{v}$ implies that $u - v(\mu(\pi) - 1) > 1$. Then there exists $\varepsilon > 0$ such that $w := u - v(\mu(\pi) - 1) - \varepsilon > 1$. By Theorem 3.3,

$$\frac{1}{n^u \cdot |\sin(n)|^v} = O\left(\frac{1}{n^w}\right),$$

so that

$$\sum_{n=1}^{N} \frac{1}{n^u \cdot |\sin(n)|^v} = O(\zeta(w)) = O(1)$$

as $N \to \infty$. Thus the series $F(u, v)$ is bounded above by a constant. Since all the terms are nonnegative, that implies convergence. ☐

Again plugging in $(u, v) = (3, 2)$, this theorem says that if $\mu(\pi) < 1 + \frac{3-1}{2} = 2$, then the Flint Hills series converges. However, we know by Corollary 2.5 that $\mu(\pi)$ cannot be less than 2, so this does not prove anything nontrivial.

Since then, a tighter upper bound has been found.

Theorem 3.9 ([6])**.** *If $\mu(\pi) < 1 + \frac{u}{v}$, then $F(u, v)$ converges.*

This is a near-converse to Corollary 3.5, with $\mu(\pi) = 1 + \frac{u}{v}$ an inconclusive case. Thus, Theorem 1.3 is proven.

Not only do these theorems help us with the original Flint Hills Series, they also help us determine convergence of many other values of $F(u, v)$. For example, [8] gives us the bound $\mu(\pi) < 8$ in particular. Using Corollary 3.8, this implies that $F(8, 1)$, $F(16, 2)$, $F(24, 3)$, etc. converge. However, using the stronger bound given by Theorem 3.9, we can now also prove that things like $F(7, 1), F(14, 2), F(21, 3)$, etc. converge, when before we couldn't.

Going the other direction, if we're somehow able to use new tactics to determine convergence or divergence of some $F(u, v)$, then that will give us bounds on $\mu(\pi)$. By far the most likely case scenario is that $\mu(\pi) = 2$, so that if $u < v$, then $F(u, v)$ diverges and if $u > v$, then $F(u, v)$ converges.

4 Continued Fractions

Now, let's talk about continued fractions:

Definition 4.1. A *continued fraction* is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots}}},$$

which may terminate or may go on forever.

Example 4.2. An example of a *finite* (or *terminating*) continued fraction is

$$1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7}}}.$$

You can evaluate this to give the rational number $\frac{233}{151}$.

Example 4.3. The prototypical example of an infinite continued fraction is

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}.$$

In order to evaluate this, start by calling it x. Then, notice that

$$1 + \frac{1}{x} = 1 + \cfrac{1}{\left(1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}\right)} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} = x.$$

Rearranging this equation gives

$$x^2 - x - 1 = 0.$$

Using the quadratic formula, we have

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

Since the continued fraction must have a nonnegative value, we know that

$$x = \frac{1 + \sqrt{5}}{2}.$$

This number is known as the *golden ratio*.

Remark 4.4. Formally, we define the value of a continued fraction

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots}}}$$

to be

$$\lim_{n\to\infty}\left(a_0+\cfrac{b_0}{a_1+\cfrac{b_1}{a_2+\cfrac{b_2}{\ddots+\cfrac{b_{n-1}}{a_n}}}}\right).$$

Thus, you would need a more rigorous method than in the above example to show that something like this even converges.

Typically, the values a_n, b_n will be integers, or functions, as in

$$f(x)=x+\cfrac{1}{(x+1)+\cfrac{1}{(x+2)+\cfrac{1}{(x+3)+\cdots}}}.$$

For our purposes, though, we will only look at *standard continued fractions*:

Definition 4.5. A *standard continued fraction* is a continued fraction of the form

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}}$$

(terminating or infinite), such that the a_n's are all integers with $a_n>0$ for $n\geq 1$ (a_0 can be nonpositive).

It turns out that all rational numbers have almost unique finite continued fraction expansions, with the exception of manipulations of the form

$$1+\cfrac{1}{2+\cfrac{1}{3}}=1+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{1}}}.$$

Since $a_n=(a_n-1)+\frac{1}{1}$, one can tweak the very bottom term in the above manner. However, since irrational numbers have infinite continued fraction expansions, theirs are truly unique since there's no bottom term.

Theorem 4.6 ([5])**.** *Every irrational number has a unique standard continued fraction, and it's infinite. Every rational number has exactly two standard continued fractions, of the form*

$$a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n}}}=a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{(a_n-1)+\cfrac{1}{1}}}}$$

with $a_n\geq 2$ *if* $n\geq 1$.

Remark 4.7. The above result is analogous to how (positive) rational numbers of the form $\frac{a}{10^b}$ have almost unique decimal expansions, up to manipulations of the form

$$n_0\cdots n_k.d_1\cdots d_n000\cdots=n_0\cdots n_k.d_1\cdots(d_n-1)999\cdots,$$

while other numbers have truly unique decimal expansions.

From now on, we will write

$$[a_0; a_1, a_2, \ldots]$$

for the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}},$$

and

$$[a_0; a_1, a_2, \ldots, a_n]$$

for the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}.$$

Note that if $[a_0; a_1, a_2, \ldots]$ is a standard continued fraction, then all the expressions of the form $[a_0; a_1, a_2, \ldots, a_n]$ will be rational numbers. Thus there exist coprime integers p_n, q_n with $q_n > 0$ such that

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n].$$

Definition 4.8. Let

$$\alpha = [a_0; a_1, a_2, \ldots].$$

Then,

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$$

as defined above is called the *nth convergent* of α. The *canonical form* of the nth convergent is the expression $\frac{p_n}{q_n}$ for which $\gcd(p_n, q_n) = 1$ and $q_n > 0$.

These pairs (p_n, q_n) are especially important because they give rise to a formula for the irrationality exponent of a number:

Theorem 4.9 ([7])**.** *Let*

$$\alpha = [a_0; a_1, a_2, \ldots]$$

be irrational, with $\frac{p_n}{q_n}$ *the canonical form of the* n*th convergent of* α*. Then*

$$\mu(\alpha) = 1 + \limsup_{n\to\infty} \frac{\ln q_{n+1}}{\ln q_n} = 2 + \limsup_{n\to\infty} \frac{\ln a_{n+1}}{\ln q_n}.$$

In particular, $\mu(\alpha) = \infty$ *iff the sequences* $\frac{\ln q_{n+1}}{\ln q_n}$ *and* $\frac{\ln a_{n+1}}{\ln q_n}$ *are unbounded from above.*

One particularly nice corollary of this is the following:

Corollary 4.10 ([7])**.** *If* $\ln(a_n) = o(n)$*, then* $\mu(\alpha) = 2$*.*

In other words, *subexponential growth* of the a_n's implies $\mu(\alpha) = 2$.

As an example, we can now prove Theorem 2.8:

Proof of Theorem 2.8. Let $\alpha = e$. It is famously true that the standard continued fraction for e is

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots].$$

(See [3]). In particular, we have that for $n \geq 1$,

$$a_{3n-1} = 2n.$$

Since $2n \leq 3n - 1$ for n sufficiently large, and $1 \leq k$ for k sufficiently large, we have that

$$a_k \leq k$$

for large k. Thus

$$\ln(a_k) \leq \ln(k)$$

for large k, and in particular

$$\ln(a_n) = O(\ln(n)) = o(n).$$

Thus $\mu(e) = 2$. ☐

This is an example of how if the a_n's grow polynomially, they must grow subexponentially as well, so Corollary 4.10 applies. However, there are functions like $e^{\sqrt{n}}$ that grow subexponentially but not polynomially, meaning that Corollary 4.10 is somewhat stronger than the polynomial case.

5 Other Irrationality Measures

There are many functions besides μ that measure how well a number is approximated by rational numbers. The general formula that produces such functions is the following:

Definition 5.1. Let

$$f : [1, \infty) \times (0, \infty) \to (0, \infty)$$

be a two-variable function that is strictly decreasing in both variables. Now consider some $x \in \mathbb{R}$. Let S be the set of all $\lambda \in (0, \infty)$ for which there are infinitely many coprime integer pairs (p, q) with $q > 0$ such that

$$0 < \left| x - \frac{p}{q} \right| < f(q, \lambda).$$

Then, $\lambda(x) = \sup S$ is called an *irrationality measure of x with regard to λ*. As before, if S is empty $\lambda(x) = 0$, and if S is unbounded $\lambda(x) = \infty$.

In particular, if you let $f(t, \lambda) = \frac{1}{t^\lambda}$, you get the irrationality exponent function $\mu(x)$. There's one other important irrationality measure known as the *irrationality base*:

Definition 5.2. The *irrationality base* or *Sondow irrationality measure* $\beta(x)$ is the irrationality measure obtained by setting

$$f(t, \lambda) = \frac{1}{\lambda^t}.$$

This function is especially useful for distinguishing different *Liouville numbers*:

Definition 5.3. A *Liouville number* is a number $x \in \mathbb{R}$ such that $\mu(x) = \infty$.

Theorem 5.4 ([7])**.** *For all $\alpha \in \mathbb{R}$, $\beta(\alpha) \geq 1$. If $\mu(\alpha) < \infty$, then $\beta(\alpha) = 1$.*

Proof. Take some number $0 < c < 1$. Taking $p = 1$ and $q > 0$ sufficiently large so that

$$|\alpha| + 1 < \frac{1}{c^q},$$

we have that

$$\left| \alpha - \frac{p}{q} \right| \leq |\alpha| + 1 < \frac{1}{c^q}.$$

Thus there are infinitely many (p, q) in the case of c, and $\beta(\alpha) \geq 1$. If $\mu = \mu(\alpha)$ is finite, then for any $\varepsilon > 0$ we have

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{q^{\mu+\varepsilon}} > \frac{1}{(1+\varepsilon)^q}$$

for all integers p, q with $q > 0$ sufficiently large, so $\beta(\alpha) = 1$ in this case. ☐

Thus, only Liouville numbers can have interesting irrationality bases. Not all of them do, however; there are some with irrationality base equal to 1.

Example 5.5. The number

$$\alpha = [0; 2^{1!}, 2^{2!}, 2^{3!}, \ldots]$$

has irrationality exponent ∞ and irrationality base 1. (See [7] for a proof.)

There are some numbers x with $\beta(x) = \infty$ in addition to $\mu(x) = \infty$. These numbers are called *super Liouville numbers.*

Definition 5.6. A *super Liouville number* is a number x such that $\beta(x) = \infty$.

Example 5.7. The number

$$\alpha = [0; 1, 2^2, 3^{3^3}, \ldots]$$

is a super Liouville number (see [7]).

Much like irrationality exponents, there's also a formula for irrationality bases using continued fractions.

Theorem 5.8 ([7])**.** *Let*

$$\alpha = [a_0, a_1, a_2, \ldots]$$

be irrational, with $\frac{p_n}{q_n}$ *the nth convergent of* α*. Then*

$$\ln \beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n} = \limsup_{n \to \infty} \frac{\ln b_{n+1}}{q_n}.$$

In general, there are many different irrationality measures out there, with unique spins on how to approximate irrational numbers. There are many ongoing efforts to bound the irrationality exponents of known constants, including π, ln(2), $\zeta(3)$, and more. For now, the convergence or divergence of the Flint Hills series remains a mystery.

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