EULER'S PENTAGONAL NUMBER THEOREM

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Abstract. This paper presents an exposition of Euler's Pentagonal Number Theorem and its connection to integer partitions. We introduce the generating function for the partition numbers and explain how Euler's theorem leads to a recurrence for the partition function. Lastly, we will look at Franklin's proof on Euler's Pentagonal Number Theorem.

1. Integer Partitions, Generating Functions and Pentagonal Numbers

To understand Euler's Pentagonal Number Theorem, we begin with the concept of integer partitions.

Definition 1.1. A *partition* of a positive integer n is a way of writing n as a sum of positive integers where the order of the sum does not matter. We denote by $p(n)$ the number of partitions of n.

Although there is no simple closed formula for $p(n)$, generating functions allow us to derive partitions indirectly.

Theorem 1.2. *The generating function for the partition numbers is*

$$(1+x+x^2+x^3+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)(\dots)=\sum p(n)x^n$$

Proof. Let us observe the product

$$(1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+x^9+\cdots)\cdots.$$

When we expand the infinite product, let's observe each common terms. To find the coefficient of x^n after expansion, we can choose x^{a_1} from the first column, x^{2a_2} from the second column, x^{3a_3} from the third column, and so on, such that $n=a_1+2a_2+3a_3+\cdots+ka_k$. Then, each common term can expressed like the following:

$$\sum_{a_1,a_2,\dots} x^{a_1+2a_2+3a_3+\cdots+ka_k}=\sum_{a_1,a_2,\dots} x^n.$$

Then, the coefficient of x^n would be the number of ways we could write $n=a_1+2a_2+3a_3+\cdots+ka_k$, which is equivalent to partition as it is the number of ways of representing n in terms of a_1 1's, a_2 2's, a_3 3's, and so on. □

Finally, we will define pentagonal numbers.

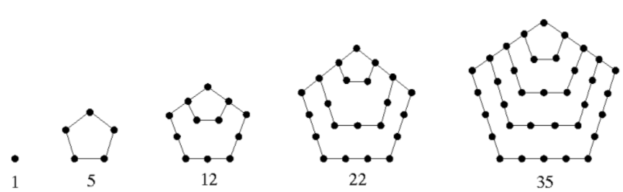

Figure 1. First five pentagonal numbers

Theorem 1.3. *The nth pentagonal number p_n can be obtained through the formula*

$$p_n = \frac{3n(n-1)}{2} = \binom{n}{1} + 3\binom{n}{2}, \quad n \in \mathbb{Z}.$$

As shown in Figure 3, the nth pentagonal number, p_n can be obtained by counting the number of distinct dots, which consists the outline of pentagons that are overlaid so that they share one vertex.

Proof. A simple way to prove this formula is to divide the pentagon into 3 triangles that would each consist $\frac{(n-1)(n-2)}{2}$ dots inside. Adding all the dots inside the 3 triangle and around the lines that separate the 3 triangles, we can get p_n as

$$p_n = 3 \cdot \frac{(n-1)(n-2)}{2} + 4n - 3 = \frac{3n^2 - n}{2} = \frac{n(3n-1)}{2} = \binom{n}{1} + 3\binom{n}{2}$$

□

2. Euler's Pentagonal Number Theorem

As shown in the theorem 1.2, we can use this theorem to represent the inverse of the generating function.

Lemma 2.1.

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{n=0}^{\infty} p(n)x^n$$

Proof. From the theorem 1.2, we know that

$$(1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)(\dots) = \sum p(n)x^n.$$

Using the sum of geometric series where

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$$

we can correspondingly find the inverses for each column where

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$
$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots$$
$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots.$$

Doing this to all of the columns, we get

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{n=0}^{\infty} p(n)x^n.$$

□

Now, let us observe Euler's Pentagonal Number Theorem.

Theorem 2.2. *Euler's Pentagonal Number Theorem*

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k-1)}{2}} = 1 + \sum_{k=1}^{\infty}(-1)^k \left(x^{\frac{k(3k+1)}{2}} + x^{\frac{k(3k-1)}{2}}\right).$$

Euler originally proved this theorem algebraically.

Lemma 2.3.

$$\prod_{n=1}^{\infty}(1-x^n) = 1 - x$$
$$- (1-x)x^2$$
$$- (1-x)(1-x^2)x^3$$
$$- (1-x)(1-x^2)(1-x^3)x^4$$
$$- (1-x)(1-x^2)(1-x^3)(1-x^4)x^5 - \cdots .$$

Proof. This identity follows by expanding the product

$$(1-x)(1-x^2)(1-x^3)\cdots$$

and grouping terms according to the first factor from which the term $-x^k$ is chosen. □

Proof. Now, we intend to prove theorem 2.2 by using the lemma 2.3. We intend to prove theorem 2.2 algebraically.

Let

$$s = s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\cdots$$

Since

$$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\cdots$$

it follows that

$$s = 1 - x - x^2(1-x) - x^3(1-x)(1-x^2) - x^4(1-x)(1-x^2)(1-x^3) - \cdots$$

Set $s = 1 - x - Ax^2$, so that

$$A = 1 - x^3 - x^5(1-x^2) - x^7(1-x^2)(1-x^3) - x^9(1-x^2)(1-x^3)(1-x^4) - \cdots$$

Set $A = 1 - x^3 - Bx^5$, so that

$$B = 1 - x^5 - x^8(1-x^3) - x^{11}(1-x^3)(1-x^4) - x^{14}(1-x^3)(1-x^4)(1-x^5) - \cdots$$

Set $B = 1 - x^5 - Cx^8$, so that

$$C = 1 - x^7 - x^{11}(1-x^4) - x^{15}(1-x^4)(1-x^5) - x^{19}(1-x^4)(1-x^5)(1-x^6) - \cdots$$

Set $C = 1 - x^7 - Dx^{11}$, so that

$$D = 1 - x^9 - x^{14}(1-x^5) - x^{19}(1-x^5)(1-x^6) - x^{24}(1-x^5)(1-x^6)(1-x^7) - \cdots$$

Setting $D = 1 - x^9 - Ex^{14}$, it follows similarly that

$$E = 1 - x^{11} - Fx^{17}$$

and continuing,

$$F = 1 - x^{13} - Gx^{20}, \quad G = 1 - x^{15} - Hx^{23}, \quad H = 1 - x^{17} - Ix^{26} \quad \cdots$$

Substituting these values successively:

$$s = 1 - x - Ax^2,$$
$$Ax^2 = x^2(1-x^3) - Bx^7,$$
$$Bx^7 = x^7(1-x^5) - Cx^{15},$$
$$Cx^{15} = x^{15}(1-x^7) - Dx^{26},$$
$$Dx^{26} = x^{26}(1-x^9) - Ex^{40}$$
$$\vdots$$

Telescoping these substitutions gives

$$s = 1 - x - x^2(1 - x^3) + x^7(1 - x^5) - x^{15}(1 - x^7) + x^{26}(1 - x^9) - x^{40}(1 - x^{11}) + \cdots$$

or equivalently,

$$\boxed{s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \cdots}$$

where the exponents of the expansion are pentagonal numbers. □

Hence, from both lemma 2.1 and theorem 2.2, this would give us the following:

$$\sum_{n=0}^{\infty} p(n)x^n \cdot \prod_{k=1}^{\infty}(1 - x^k) = 1.$$

In other words,

$$(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \ldots)(1 + p(1)x + p(2)x^2 + p(3)x^3 + \ldots) = 1.$$

Since the product of the two polynomials result in 1, we can notice that the coefficients of each term should be zero. In other words, all the coefficients of each x^n term has to be zero. Through this observation, we can express partitions in an recursive form, which allows us to indirectly calculate the nth partition.

Theorem 2.4. *The recurrent expression of $p(n)$, which expresses the number of partitions of n is*

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-15) - p(n-22) - \cdots \\ &= \sum_{k=1}^{\infty}(-1)^{k+1}\left[p\left(n - \frac{k(3k-1)}{2}\right) + p\left(n - \frac{k(3k+1)}{2}\right)\right] \end{aligned}$$

Proof. We can prove this theorem using the similar logic when we prove **Theorem 1.2.** Notice that we can simply expand each terms to combine all the like terms of x^n. Therefore sum of all x^ns would look like

$$1 \cdot x^n + (-x) \cdot p(n-1)x^{n-1} + (-x^2) \cdot p(n-2)x^{n-2} + \cdots.$$

Hence, this would be represented as

$$\sum \left(\text{coefficient of } x^k \text{ in } \prod_{k=1}^{\infty}(1 - x^k)\right) p(n-k).$$

In turn, this summation would give us

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-15) + p(n-22) + \cdots = 0.$$

Therefore, this gives us

$$p(n) = \sum_{k=1}^{\infty}(-1)^{k+1}\left[p\left(n - \frac{k(3k-1)}{2}\right) + p\left(n - \frac{k(3k+1)}{2}\right)\right]$$

as desired. □

3. Franklin's proof on Euler's Pentagonal Number Theory

In contrast to Euler's proof which required a lot of algebraic manipulation, Franklin provides a algebra-free proof of Euler's Pentagonal Number Theory through a combinatorics perspective.

The motivation for Franklin's proof stems from how when expanding Euler's Pentagonal Number Theorem, the coefficients of x^n is usually zero, where each coefficient of the expansion could be interpreted as the distinct number of partitions of n with even number of parts minus the partitions with odd number of parts.

Theorem 3.1.

$$\prod_{k=1}^{\infty}(1-x^k)=\sum_{n=0}^{\infty}a_nx^n$$

where

$$a_n = (\textit{number of distinct partitions of } n \textit{ with even number of parts}) - (\textit{with odd number of parts})$$

Proof. Notice that the expansion of the pentagonal number theory would be the following:

$$(1-x)(1-x^2)(1-x^3)(\dots)=\sum_{b_1,b_2,\dots}(-1)^{b_1+b_2+b_3+\dots+b_k}x^{b_1+2b_2+3b_3+\dots+kb_k}.$$

Notice how from this expansion, unlike theorem 1.2, we now distinguish partitions with even and odd numbers of terms, where we now count partitions with even terms as positive and odd terms as negative.

Hence, this shows how the coefficient of the expansion would be simply the sum of those two, which would be number of distinct partitions of n with even number of parts minus number of distinct partitions of n with odd number of parts.

□

Now with this set, we can observe the odd and even partitions through the Ferrers diagram of any partition of n into distinct parts. In particular, consider the Ferrers diagram below.

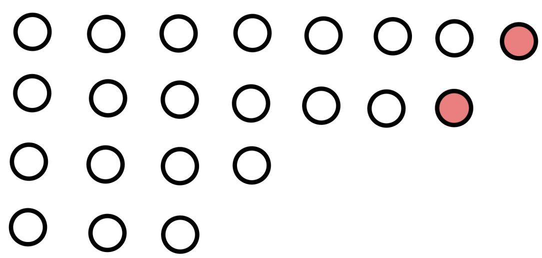

Figure 2. Ferrers diagram for distinct parts.

Here, this diagram represents when $n=22$ and the partition $22=8+7+4+3$. If we let a be the number of dots in the smallest row and b be the number of dots in the rightmost diagonal, we can notice that if $a>b$ (like $3>2$ in this example), we can move the dots in the diagonal and put it below, making a new for just like in Figure 3:

Now, notice that if it is the reverse, which is when $a\le b$ like our changed Ferrers diagram ($2\le 2$), we can also reverse the process by moving the dots in the bottom row to the rightmost diagonal, which would be our initial diagram.

Therefore, this process would indicate that we can pair each odd partitions with even partitions, which would result in a net coefficient of 0 for the x^n. However, this doesn't work for two cases:

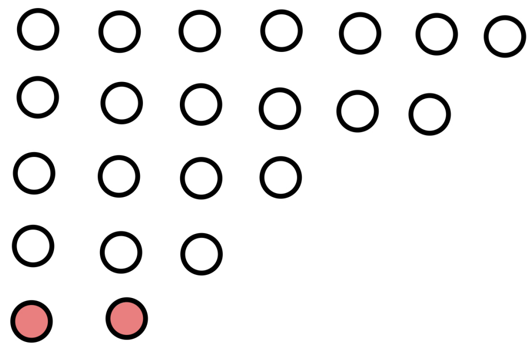

FIGURE 3. Changed Ferrers diagram

Case 1: When $a = b$, which would mean the most bottom row and the rightmost diagonal meeting.

We can quickly notice that in this case, unlike the previous cases, it wouldn't be reversible, which would then affect the parity. Therefore, we can then notice that

$$n = a + (a+1) + \cdots + (2a-1) = \frac{a(3a-1)}{2} = \frac{k(3k-1)}{2}$$

where $k = a$ in this case.

Case 2: When $a = b + 1$ where also the most bottom row and the rightmost diagonal meet.

Likewise, this would result in

$$n = a + (a+1) + \cdots + (2a-2) = \frac{(a-1)(3a-2)}{2} = \frac{k(3k-1)}{2}$$

where here, this would imply $k = 1 - a$.

Hence, we've shown the connection between the Ferrers diagram and Euler's pentagonal number theory!

References

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