Infinite Series Paper

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ABSTRACT. In this paper, we will discuss the proof and results of Euler's Pentagonal Number Theorem.

1 Introduction

Euler's pentagonal number theorem states that the generating function $\prod_{i=1}^{\infty}(1-x^i) = \sum_{-\infty}^{\infty}(-1)^k x^{k(3k-1)/2} = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} \cdots$.

In this research paper, we will prove Euler's Pentagonal Number Theorem in multiple ways and explore its connection to the partition function. A main result of this theorem is a proof that the number of partitions of n with an even number of parts is equivalent to the number of partitions of n with an odd number of parts when n is not pentagonal.

2 Definitions

Definition 2.1. The pentagonal number sequence is defined $P_n = \frac{n(3n-1)}{2}$. Euler's Pentagonal Number Theorem explicitly involves the sequence of pentagonal numbers that also includes *negative* n and include 0, so our pentagonal number sequence will begin $0, 1, 2, 5, 7 \cdots$ instead of $1, 5, 12, \cdots$, and corresponding to $0, 1, -1, 2, -2 \cdots$. This is very important in Euler's Pentagonal Number Theorem, since we are dealing with generating functions.

Theorem 2.1 The amount of dots in a pentaganal grid with sides of length k is the $k+1$-th pentagonal number. We can prove this by induction. First, for $k = 1$, the amount of dots in a set of points defined by a $0x0x0x0x0$ pentagon is clearly 1.

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Figure 1: A single point defined by a pentagon of length 0

Now, assume that the pentagon of dimensions $kxkxkxkxk$ has $\frac{(k+1)(3k+2)}{2}$ dots in the grid. Then, the pentagon of dimensions $(k+1)x(k+1)x(k+1)x(k+1)x(k+1)$ will include all the dots in the pentagon of side length k, in addition to $2k+1$ extra points from the perimeter of the large pentagon. Therefore, the next pentagonal number is $\frac{3k^2+5k+2}{2} + (3k+2) = \frac{3k^2+11k+6}{2} = \frac{(k+2)(3k+5)}{2}$, which follows the recursive formula.

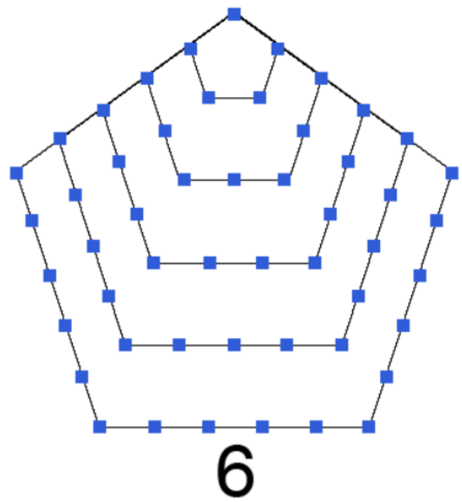

Figure 2: The 7th pentagonal number (pentagon with length 6) containing the 6th pentagonal number (side length 5)

Definition 2.2. The partition function is the amount of distinct ways to express an integer as the sum of (not necessarily distinct) integers. For example, $5, 1+1+3$, and $2+2+1$ are all partitions of 5, and the partitions $2+2+1$ and $2+1+2$ are identical (not counted twice). The generating function for the partition function is $\prod_{k=1}^{\infty} \frac{1}{1-x^k}$.

Theorem 2.2. We can write $\frac{1}{1-x^k}$ as $1 + x^k + x^{2k} + \cdots$, so we can write our infinite product as $(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(\cdots)$. The x^k term in this generating function can be written as the sum of any amount of 1's, which is represented by $(1 + x + x^2 + \cdots)$, added to the sum of any amount of 2's, represented by $(1 + x^2 + x^4 + \cdots)$, and so on, which happen to be the terms of the partition function. Therefore, $\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{k=0}^{\infty} x^k p(k)$

3 The Proofs

Proof 1 (Numerical Bijection).

Let us define $P(x) = \prod_{n=1}^{\infty}(1-x^n)$. Every term in the expansion corresponds to choosing either 1 or $-x^n$ from each factor, so every term corresponds to selecting a subset of positive integers, so all terms can be written as $(-1)^k x^{n_1+n_2+\cdots+n_k}$ where the integers n_i are distinct. This means that $P(x)$ is the generating function for partitions into distinct parts, with sign depending on the parity of the number of parts. This is because if there are an odd number of distinct parts, then $(-1)^k$ would be -1, contributing to a $(-1)(x^{n_1+n_2+\cdots+n_k})$ to the final generating function, and if there are an even number of parts, $(-1)^k = 1$, adding $x^{n_1+n_2+\cdots+n_k}$ to the sum.

Let $P(x) = \sum_{m=0}^{\infty} a_m x^m$, where a_m can be defined as

$$a_m = \sum_{\text{partitions with distinct parts of} m} (-1)^{\text{number of parts}}.$$

This means that a_m measures the (non-absolute) difference between the amount of partitions of m with an even number of all distinct parts and the amount of partitions of m with an odd number of all distinct parts. Note that almost all terms cancel in pairs. For example, the partitions of 8 with an odd number of all distinct parts is $8, 1+2+5, 1+3+4$, and the partitions of 8 with an even number of all distinct parts is $1+7, 2+6, 3+5$. For most integers m, there is a complete bijection between partitions with an even number of distinct parts and partitions with an odd number of distinct parts, so $a_m = 0$ for almost all m. We will prove that the only values where cancellation does not happen is when $m = \frac{k(3k-1)}{2}$ for some integer k.

Let us explain the mapping (from a partition with an even number of distinct parts to a partition with an odd number of distinct parts). Consider a partition of $m = a_1 + a_2 + \cdots a_n$ into n distinct parts, so that $a_1 < a_2 \cdots < a_n$. Let r be the greatest integer such that $a_r \neq a_{r+1} - 1$.

If $n-r < a_1$, then we can subtract 1 from each of the largest $n-r$ parts, and then let $a_{n+1} = n-r$ to the partition, so that $a_{n+1} < a_1 < a_2 < \cdots a_r < a_{r+1} - 1 < \cdots < a_n - 1$.. If $n-r \geq a_1$, then we can remove a_1 from the partition and add 1 to the a_1 greatest elements, giving us $a_2 < a_3 < \cdots a_{n-a_1+1} + 1 < \cdots a_n + 1$. Notice that this second mapping is a reverse of the first mapping; any partition p_1 that gets sent to a partition p_2 by this first mapping will be transformed back into the original partition p_1 by the second mapping, and vice versa. Now that we have proven the mapping, we can finish our proof. When m equals a generalized pentagonal number, one partition is left unmatched: for $m = \frac{k(3k-1)}{2}$, the partition $k + (k+1) + \cdots (2k-1) = m$ is unmatched, since all the elements are consecutive, and the smallest element, k is not greater than the total amount of elements, which is also k.

Since it has k parts, $a_m = (-1)^k$ in this case, meaning that the sign of each $x^{k(3k-1)/2}$ alternates through each pentagonal number. Therefore, $P(x) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots$, where the exponents are the generalized pentagonal numbers.

Here's a visual representation of the mappings:

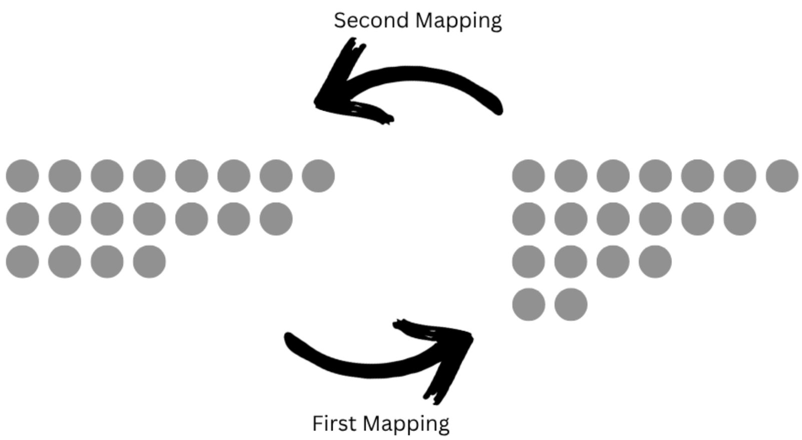

Figure 3: Shown are mappings between $4 + 7 + 8$ and $2 + 4 + 6 + 7$

References:

Wikimedia Foundation, "Pentagonal Number Theorem." *Wikipedia: The Free Encyclopedia*, 21 February 2026, `https://en.wikipedia.org/wiki/Pentagonal_number_theorem`