On Fourier Series

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Abstract

In this paper, we will show methods of using Fourier Series to evaluate infinite sums, starting from Orthogonality Relations to applications in other fields.

1 Introduction

The Fourier series can be thought of as an analogue to the Taylor series for periodic functions. Suppose we have a periodic function f, for simplicity of period 2π so that $f(x+2\pi) = f(x)$. We want to decompose f into the sum of two trigonometric functions.

Concretely, given a 2π periodic function f, we wish to express it as

$$f(x) = \sum_{m=0}^{\infty} a_m \cos(mx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Recall that if we have a Taylor series, then finding the coefficients is relatively straightforward. For instance, if $f(x) = \sum a_n x^n$, then $a_n = \frac{d^n f}{dx^n}|_{x=0}$. We would like to find the coefficients a_m and b_n in terms of f. We can do so through the following orthogonality relations:

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \begin{cases} 2\pi & m = n = 0 \\ \pi & m = n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \begin{cases} \pi & m = n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0$$

Beginning with our $f(x)$, we multiply by $\cos(kx)$ to get:

$$f(x)\cos(kx) = \sum_{m=0}^{\infty} a_m \cos(mx)\cos(kx) + \sum_{n=1}^{\infty} b_n \sin(nx)\cos(kx)$$

We now integrate over $-\pi$ to π. Notice that because of the orthogonality relations, most of the terms go to 0 except the $a_k \cos(kx)$ terms.

$$\int_{-\pi}^{\pi} f(x)\cos(kx)dx = a_k \int_{-\pi}^{\pi} \cos^2(kx)dx = \begin{cases} \pi a_k & k > 0 \\ 2\pi a_0 & k = 0 \end{cases}$$

Therefore, the general a_k is $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx$, Moreover, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$.

For the b_n terms, we multiply by $\sin(kx)$ and integrate from $-\pi$ to π. Similarly, most of the terms go to 0 except those of the form $b_k \sin(kx)$:

$$\int_{-\pi}^{\pi} f(x)\sin(kx)dx = b_k \int_{-\pi}^{\pi} \sin^2(kx)dx = \pi b_k$$

which gives us $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$.

We can also express our periodic function f in exponential form because $\sin(x)$ and $\cos(x)$ are very easily related to the exponential e^{ix}. Specifically, $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}$.

We multiply this form by e^{-ikx} and note the exponential setting of the orthogonality relation:

$$\int_{-\pi}^{\pi} e^{imx}e^{inx}dx = \begin{cases} 2\pi & m = -n \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = c_k \int_{-\pi}^{\pi} e^{ikx}e^{-ikx}dx = c_k 2\pi$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

We might wonder as to what the conditions under which the Fourier series of f converges to f. However, by the following theorem from Dirichlet, everything is fine if f is piecewise differentiable.

Theorem 1.1. *If f is piecewise differentiable, then for every x, the Fourier series of f converges to $\frac{f(x_+)+f(x_-)}{2}$, the average of the right and left limits of f at x.*

Let us now consider an example of what Fourier Series can do for us.

Example. Consider the sawtooth function of $f(x) = kx$ for $k \in \mathbb{R}$ that has period L over $-\frac{L}{2} < x < \frac{L}{2}$. Find the fourier series corresponding to this function.

A pictorial representation of the sawtooth function for $k = 2$ over $L = 2$ is shown below:

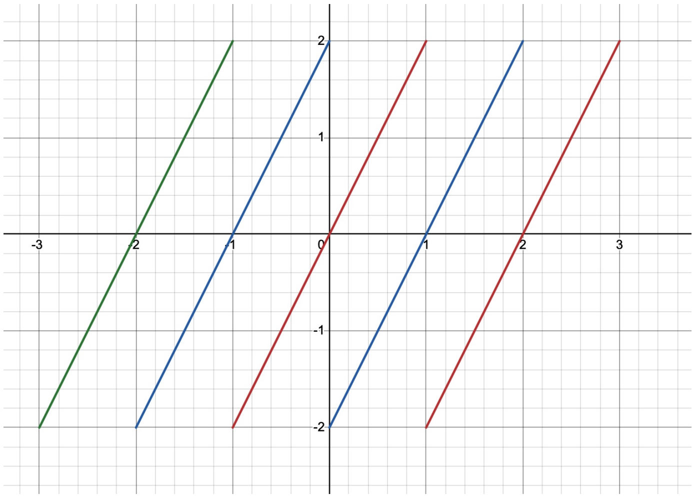

Figure 1: Sawtooth function for $k = 2$ over $L = 2$

We begin with a general fourier series

$$f(x) = \sum_{m=0}^{\infty} a_m \cos(\frac{2\pi m x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{2\pi n x}{L})$$

where the inputs to the trigonometric functions are slightly changed for the period. Recall that $a_k = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos(\frac{2\pi m x}{L})dx$ and $a_0 = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)dx$. Since $f(x)$ is an odd function, both of these are 0. Hence, we only have to work on the b_n coefficients.

$$b_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} kx\sin(\frac{2\pi n x}{L})dx$$

Let $A = \frac{2\pi n}{L}$. By integration by parts, we find that

$$\int x\sin(Ax)dx = x\frac{-\cos(Ax)}{A} + \int \frac{\cos(Ax)}{A}dx = \frac{\sin(Ax)}{A^2} - \frac{x\cos(Ax)}{A}$$

Making use of this, we have

$$b_n = \frac{2k}{L}[\frac{L^2}{4\pi^2 n^2}\sin(\frac{2\pi n x}{L}) - \frac{Lx}{2\pi n}\cos(\frac{2\pi n x}{L})]_{-\frac{L}{2}}^{\frac{L}{2}} =$$

$$= \frac{2k}{L}((\frac{L^2}{4\pi^2 n^2}\sin(\pi n) - \frac{L^2}{4\pi n}\cos(\pi n)) - (\frac{L^2}{4\pi^2 n^2}\sin(-\pi n) + \frac{L^2}{4\pi n}\cos(-\pi n))$$

Notice that $\sin(\pi n) = 0$ for all n, we have

$$b_n = \frac{2k}{L}(-\frac{L^2}{4\pi n}\cos(\pi n) - \frac{L^2}{4\pi n}\cos(-\pi n))$$

Notice that $\cos(x)$ is even, so we have:

$$b_n = -\frac{kL}{\pi n}\cos(\pi n) = (-1)^{n+1}\frac{kL}{\pi n}$$

Therefore, our Fourier Series is

$$f(x) = \frac{kL}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(\frac{2\pi nx}{L})$$

We can go a bit further by writing out the series in full:

$$f(x) = \frac{kL}{\pi}[\sin(\frac{2\pi x}{L}) - \frac{1}{2}\sin(\frac{4\pi x}{L}) + \frac{1}{3}\sin(\frac{6\pi x}{L}) - \cdots]$$

The beauty of the Fourier series is that we can now substitute values for x and L to make an infinite series. For example, substituting $L = 2\pi$ and $x = \frac{\pi}{2}$ gives:

$$f(x) = k\frac{\pi}{2} = \frac{k\cdot 2\pi}{\pi}[\sin(\frac{\pi}{2}) - \frac{1}{2}\sin(\pi) + \frac{1}{3}\sin(\frac{3\pi}{2}) - \frac{1}{4}\sin(2\pi) + \frac{1}{5}\sin(\frac{5\pi}{2}) - \cdots]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

which gives the Leibniz formula for $\frac{\pi}{4}$. We can also substitute more "exotic" chances as the following with $L = 3$ and $x = 1$:

$$f(x) = k = \frac{3k}{\pi}[\sin(\frac{2\pi}{3}) - \frac{1}{2}\sin(\frac{4\pi}{3}) + \frac{1}{3}\sin(\frac{6\pi}{3}) - \frac{1}{4}\sin(\frac{8\pi}{3}) + \frac{1}{5}\sin(\frac{10\pi}{3})\cdots]$$

$$\frac{2\pi\sqrt{3}}{9} = \frac{1}{1} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \cdots$$

2 More Infinite Series

It turns out that we can do much more with Fourier series in making infinite series. The following theorem is particularly useful.

Theorem 2.1. *Suppose that the Fourier series of f is*

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Proof. We can prove this by simple substitution of $f(x)$ into the integral and use of the orthogonality relations.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)))^2 dx =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} a_0^2+2a_0\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))+(\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx)))^2dx$$

Consider the middle term $\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nxdx$:

$$\sum_{n=1}^{\infty}[a_n\frac{\sin(nx)}{n}-b_n\frac{\cos(nx)}{n}]_{-\pi}^{\pi}=\sum_{n=1}^{\infty}[a_n\frac{\sin(n\pi)}{n}-b_n\frac{\cos(n\pi)}{n}-a_n\frac{\sin(-n\pi)}{n}+b_n\frac{\cos(-n\pi)}{n}]=0$$

For the squared summation, we consider transforming $\int_{-\pi}^{\pi}(\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx)))^2dx$ into a double sum.

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_{-\pi}^{\pi}a_na_m\cos(nx)\cos(mx)+a_nb_m\cos(nx)\sin(mx)+a_mb_n\cos(mx)\sin(nx)+b_nb_m\sin(nx)\sin(mx)dx$$

Using the orthogonality relation between $\sin(nx)\cos(mx)$ the two middle term are 0. For $\int_{-\pi}^{\pi}a_na_m\cos(nx)\cos(mx)dx$, this is only nonzero when $n=m$, in which case

$$\int_{-\pi}^{\pi}a_na_m\cos(nx)\cos(mx)dx=\pi a_n^2$$

Similarly, using the orthogonality relation between $\sin(nx)$ and $\sin(mx)$, the last term is nonzero for $n=m$ in which

$$\int_{-\pi}^{\pi}b_nb_m\sin(nx)\sin(mx)dx=\pi b_n^2$$

Considering the entire summation, we find that:

$$\int_{-\pi}^{\pi}(\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx)))^2dx=\pi\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

Putting everything together yields:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)^2dx=2a_0^2+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

■

An analogue of Parseval's Theorem for an Exponential Fourier Series is given below. The proof is quite similar, so we will omit it.

Theorem 2.2. *Suppose that the Fourier series of f is given as*

$$f(x)=\sum_{n=-\infty}^{\infty}c_ne^{inx}$$

Then

$$\frac{1}{2\pi}\int_{0}^{2\pi}|f(x)|^2dx=\sum_{n=-\infty}^{\infty}|c_n|^2$$

Parseval's theorem is incredibly useful in generating ζ values. We can find that

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \cdot \frac{4}{n^2} \cos(nx)$$

Simply substituting this into Parseval's Theorem gives us

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = 2\frac{\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\frac{2\pi^4}{5} = 2\frac{\pi^4}{9} + 16\zeta(4)$$

$$\zeta(4) = \frac{\pi^4}{90}$$

We can find larger values of $\zeta(2x)$. For instance, we can find that

$$x^3 = 0 + \sum_{n=1}^{\infty} 2 \cdot (-1)^n (\frac{6}{n^3} - \frac{\pi^2}{n}) \sin(nx)$$

Applying Parseval's Theorem directly gives us:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^6 dx = \sum_{n=1}^{\infty} 4(\frac{6}{n^3} - \frac{\pi^2}{n})^2 = \sum_{n=1}^{\infty} 4(\frac{36}{n^6} - \frac{12\pi^2}{n^4} + \frac{\pi^4}{n^2})$$

$$\frac{2\pi^6}{7} = 144\zeta(6) - 48\pi^2\zeta(4) + 4\pi^4\zeta(2)$$

$$\frac{2\pi^6}{7} = 144\zeta(6) - \frac{48\pi^6}{90} + \frac{4\pi^6}{6}$$

$$144\zeta(6) = \frac{2\pi^6}{7} + \frac{48\pi^6}{90} - \frac{4\pi^6}{6} = \frac{\pi^6}{945}$$

This method easily generalizes to $\zeta(2x)$ values through a recursive sequence. Overall, it makes sense as to why we would expect being able to get even zeta terms since Parseval's Theorem incorporates squaring, which would show up as $\frac{1}{n^{2x}}$ terms.

However, Parseval's Theorem isn't just used for zeta values. Consider the following example.

Example. Consider the function $f(x) = e^{ax}$ made periodic over the intervale $(0, 2\pi)$. We work with the exponential form of the Fourier Series for simplicity. It is possible to do this with the trigonometric form, but it makes it needlessly complicated.

This means that we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

and we want to find the c_n coefficients. We do so using the following fact found in the Introduction.

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{ax}e^{-inx}dx = \frac{1}{2\pi}\int_0^{2\pi} e^{x(a-in)}dx =$$

$$= \frac{1}{2\pi}[\frac{e^{(a-in)x}}{a-in}]_0^{2\pi} = \frac{e^{2\pi a}-1}{2\pi(a-in)}$$

We then apply Parseval's Theorem in exponential form:

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|$$

We consider the left-hand-side first.

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2ax}dx = \frac{1}{2\pi}[\frac{e^{2ax}}{2a}]_0^{2\pi} = \frac{1}{2\pi}\frac{e^{4\pi a}-1}{2a}$$

We then consider the right-hand-side.

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=-\infty}^{\infty} \frac{(e^{2\pi a}-1)^2}{4\pi^2(a^2+n^2)}$$

We then combine these two pieces:

$$\frac{1}{2\pi}\frac{e^{4\pi a}-1}{2a} = \frac{(e^{2\pi a}-1)^2}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = \frac{(e^{2\pi a}-1)^2}{4\pi^2}[\frac{1}{a^2}+2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}]$$

$$\frac{e^{2\pi a}+1}{4a\pi} = \frac{e^{2\pi a}-1}{4\pi^2}[\frac{1}{a^2}+2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}] \implies$$

$$\frac{\pi}{a}\frac{e^{2\pi a}+1}{e^{2\pi a}-1} = \frac{1}{a^2}+2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}$$

Recall that $\frac{e^{2\pi a}+1}{e^{2\pi a}-1} = \coth(\pi a)$:

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{1}{2}(\frac{\pi}{a}\coth(\pi a)-\frac{1}{a^2})$$

Thus, we can solve for any infinite series of the form $\sum\frac{1}{n^2+a^2}$. For instance, we have the following:

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1} = \frac{1}{2}(\pi\coth(\pi)-1) = \frac{1}{2}[\frac{\pi e^{2\pi}+\pi}{e^{2\pi}-1}-1]$$

3 Fourier Transform

The Fourier Series we have been looking at relate to periodic functions. However, is it possible for us to extend this to an aperiodic function? Is it possible to decompose an aperiodic function into trigonometric components. It turns out that the answer is yes which we justify by examining the function over a certain period, and then extending the period to ∞. This makes sense because an aperiodic function has a period of ∞ in an intuitive description. It is cumbersome to use sin and cos components, so we refer to the exponential form. Our main theorem is the one following:

Theorem 3.1. *Let a function* $f : \mathbb{R} \to \mathbb{R}$ *be aperiodic. We can express this function as*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk$$

where

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

Proof. In the previous section, we saw that a periodic function can be expressed as $\sum_{-\infty}^{\infty} c_n e^{\frac{2\pi inx}{L}}$ since the period is L taking $\frac{2\pi}{L}$ in the power. For simplicity, we call $k_n = \frac{2\pi n}{L}$. The derivative is $dk_n = 2\pi\frac{dn}{L}$. Since we have a summation, we work over a discrete sum of integers, thus the rate of change $\frac{d}{dn} = 1$ in going from $n \to n + 1$. Since we have an aperiodic function, we take $L \to \infty$ in the limit of dk_n:

$$\lim_{L\to\infty} dk_N = \lim_{L\to\infty} \frac{2\pi}{L} = 0$$

using the fact that $dn = 1$. Again using that fact, we can write $f(x)$'s summation with dn to attempt to turn it into an integral.

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{ik_n x} d_n$$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{ik_n x} (\frac{L}{2\pi} dk_n)$$

Recall that the limit of dk_n as $L \to \infty$ is 0. Thus taking the limit of $f(x)$ as $L \to \infty$, we get 0. This implies that we can write $f(x)$ as a continuous sum, an integral.

$$f(x) = \int_{-\infty}^{\infty} c_n (\frac{L}{2\pi}) e^{ik_n x} dk_n$$

At this point, the indices are no longer needed and we denote $\hat{f}(k) = c_n\frac{L}{2\pi}$. Hence, we have:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k)(\frac{L}{2\pi}) e^{ikx} dk_n$$

Recall the formula for c_n presented towards the end of the introduction. We slightly alter the exponent due to a different period, but we now find:

$$\hat{f}(k) = \frac{L}{2\pi}c_n = \frac{L}{2\pi} \cdot \frac{1}{L}\int_0^L f(x)e^{-\frac{2\pi inx}{L}}dx$$

$$\implies \hat{f}(k) = \frac{1}{2\pi}\int_0^L f(x)e^{-ikx}dk$$

as we wanted. ■

4 A Curious Phenomenon

We have the following theorem about the Fourier Transform as in the previous section.

Theorem 4.1. *Let a function $f : \mathbb{R} \to \mathbb{R}$ be aperiodic. We can express this function as*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk$$

where

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

We now consider the following example.

Example. We find the Fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{if } -\pi < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

By the previous theorem, we have that

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}dx = \frac{1}{2\pi}[\frac{e^{-ikx}}{-ik}]_{-\pi}^{\pi} = \frac{1}{2\pi}\cdot\frac{1}{-ik}[-2i\sin(k\pi)] = \frac{\sin(k\pi)}{\pi k}$$

which implies that we have:

$$f(x) = \int_{-\infty}^{\infty} \frac{\sin(k\pi)}{k\pi}e^{ikx}dk$$

We now transform this continuous sum to a discrete sum over k. Notice then that $dk = 1$, so we have:

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{\sin(k\pi)}{\pi k}e^{ikx}$$

Therefore, our $c_k = \frac{\sin(k\pi)}{k\pi}$. But, since we take k over all the integers, we just get a bunch of 0s for the infinite series alongside $c_0 = 1$. Essentially we find that

$$f(x) = 1 + 0 + 0 + 0 + \cdots$$

However, it turns out that if we change the range of the function, we get a far more interesting series.

Example. We find the Fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

We repeat the following steps (which are quite similar to last time):

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-ikx} dx = \frac{\sin(\frac{k\pi}{2})}{k\pi}$$

after solving the integral. Therefore, our original function is

$$f(x) = \int_{-\infty}^{\infty} \frac{\sin(\frac{k\pi}{2})}{k\pi} e^{ikx} dx$$

Similarly, we turn this into a discrete summation:

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{\sin(\frac{k\pi}{2})}{k\pi} e^{ikx}$$

Now, we can note that c_0 is the average of the function or $\frac{1}{2}$. Consider summing up two "opposite" terms:

$$\frac{\sin(\frac{k\pi}{2})}{k\pi} e^{ikx} + \frac{\sin(\frac{-k\pi}{2})}{-k\pi} e^{-ikx} = \frac{\sin(\frac{k\pi}{2})}{k\pi}(\cos x + i \sin x) + \frac{\sin(\frac{k\pi}{2})}{k\pi}(\cos x - i \sin x) = \frac{2\sin(\frac{k\pi}{2})}{k\pi} \cos(x)$$

Thus, our infinite series is:

$$f(x) = c_0 + \frac{2}{\pi} \cos x - \frac{2}{3\pi} \cos x + \frac{2}{5x} \cos(x) - \cdots$$

after solving for the c_k terms. We substitute $x = 0$ and notice that $f(0) = 1$ by definition.

$$1 = \frac{1}{2} + \frac{2}{\pi} - \frac{2}{3\pi} + \frac{2}{5\pi} - \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

which is the famous Leibniz formula for $\frac{\pi}{4}$.

For curious readers such as myself, we may wonder as to why we actually have a value now? Why is it that changing the period, specifically making the pulse shorter, allowed us to get an infinite series instead of just 0s.

Consider the following Theorem as Poisson's Summation Formula (a useful trick with Fourier Transforms).

Theorem 4.2. *For a function $f(x)$ with period 2π and Fourier Transform $\hat{f}(k)$:*

$$\sum_{n=-\infty}^{\infty} f(x+2\pi n) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}$$

This is an equality between the integer values of f and the integer values of its Fourier Transform. Consider the first function we looked at where $\hat{f}(k) = 2\frac{\sin(k\pi)}{k}$ (taking constant factors out appropriately). Either $\hat{f}(k) = 2\pi$ for $k = 0$ or 0 elsewhere. Substituting this into Poisson's Summation Formula gives:

$$2\pi f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}$$

The only integer values for $\hat{f}(k)$ is at $k = 0$, so:

$$2\pi f(x) = 2\pi e^{i0x}$$

$$f(x) = 1$$

Every sampled Fourier value outside of the trivial one is 0. However, for the second function with a shorter pulse, its Fourier Transform is $\hat{f}(k) = 2\frac{\sin(\frac{k\pi}{2})}{k}$, which is not zero for all integers. There are actually infinitely many terms that contribute to the summation, producing a normal infinite series. This is entirely based upon the fact that sin has a period of 2π and we base the pulse of length 2π.

However, we can offer yet another explanation for what is happening. The Poisson Summation Formula lets us reconstruct the function by "adding" appropriate periodic portions of the "discrete" Fourier Transform. Sampling is the process of converting a continuous function into a sequence of values (discrete).

When we are sampling this function, we are doing so at intervals of $\frac{2\pi}{T} = 1$. Since the zeros of the sin function are 1 unit apart and we sample at integer k, every sample falls on a 0 (apart from the $k = 0$).

The following Theorem is called the Shannon-Nyquist Theorem, which details when a function can be reconstructed from its sampled values (which is exactly what we are doing when we find the function through its Fourier Transform).

Theorem 4.3. *If a function contains no frequencies higher than B hertz, then it can be completely determined from its sampled values at a sequence of points spaced less than $\frac{1}{2B}$ seconds across.*

For us, we are sampling in frequency to reconstruct a pulse. For the original example, the distance $\frac{1}{2B}$ is exactly 1, so the reconstruction "aliases" and misses each point to get 0s. However, for the new example, we "half" the pulse length, so the distance $\frac{1}{2B}$ doubles, and it is now 2. Since $1 < 2$, we can successfully reconstruct and catch the appropriate peaks.

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