Nowhere Differential functions

Abdul Rafay

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Abstract

Classical intuition suggests that continuous functions should be differentiable except at a few isolated points. However, this intuition was overturned by the construction of continuous functions that are nowhere differentiable. In this paper, we study such functions, focusing on the Weierstrass function, and explain how Fourier analysis provides a natural framework for understanding their behavior. In particular, we show how high-frequency oscillations lead to the failure of differentiability.

1 Introduction

In elementary calculus, most functions encountered are smooth and well-behaved. Even when a function is not differentiable, the failure typically occurs at isolated points. It is therefore surprising that there exist functions which are continuous everywhere but differentiable nowhere.

The first explicit example of such a function was given by Weierstrass. These examples reveal that continuity alone does not guarantee smoothness.

The goal of this paper is:

- To construct a nowhere differentiable function
- To explain its behavior using ideas from Fourier analysis

2 The Weierstrass Function

Consider the function

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$, b is a positive odd integer, and $ab > 1$.

2.1 Continuity

Theorem 1. *If a sequence of continuous functions converges uniformly on an interval, then the limit function is continuous.*

We first show that $W(x)$ is continuous. Since

$$|a^n \cos(b^n \pi x)| \leq a^n,$$

the series converges uniformly by comparison with the geometric series $\sum a^n$. Therefore, $W(x)$ is continuous.

3 Failure of Differentiability

We now explain why $W(x)$ is nowhere differentiable. The key idea is to study the difference quotient:

$$\frac{W(x+h) - W(x)}{h} = \sum_{n=0}^{\infty} a^n \frac{\cos(b^n \pi (x+h)) - \cos(b^n \pi x)}{h}.$$

3.1 Key Approximation

A crucial step is understanding the behavior of

$$\cos(b^n \pi (x+h)) - \cos(b^n \pi x).$$

Using the linear approximation for differentiable functions,

$$f(x+h) - f(x) \approx f'(x)h,$$

we apply this to

$$f(x) = \cos(b^n \pi x).$$

Then

$$f'(x) = -b^n \pi \sin(b^n \pi x),$$

so

$$\cos(b^n \pi (x+h)) - \cos(b^n \pi x) \approx -b^n \pi \sin(b^n \pi x)\, h.$$

Since $|\sin(\theta)| \leq 1$, we obtain the estimate

$$|\cos(b^n \pi (x+h)) - \cos(b^n \pi x)| \approx b^n |h|.$$

Dividing by h gives

$$\frac{\cos(b^n \pi(x+h)) - \cos(b^n \pi x)}{h} \approx b^n.$$

Multiplying by a^n, we see that each term contributes approximately

$$a^n b^n = (ab)^n.$$

3.2 Conclusion of Non-Differentiability

If $ab > 1$, then $(ab)^n \to \infty$. This means that higher-frequency terms contribute increasingly large values to the difference quotient.

As a result:

- The difference quotient does not converge
- The slope oscillates wildly at every scale

Thus, $W(x)$ is nowhere differentiable.

3.3 Remark (Mean Value Theorem)

Theorem 2 (Mean Value Theorem)**.** *Let f be differentiable on (a, b) and continuous on $[a, b]$. Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

A more rigorous argument uses the Mean Value Theorem. There exists c between x and $x + h$ such that

$$\frac{\cos(b^n \pi(x+h)) - \cos(b^n \pi x)}{h} = -b^n \pi \sin(b^n \pi c),$$

so

$$\left| \frac{\cos(b^n \pi(x+h)) - \cos(b^n \pi x)}{h} \right| \leq b^n \pi.$$

This confirms that the size of the difference quotient is controlled by b^n.

4 Connection to Fourier Analysis

The Weierstrass function can be viewed as a sum of oscillatory components:

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x).$$

Although this is not a classical Fourier series with linearly spaced frequencies, it still represents a decomposition into waves.

4.1 Frequency and Amplitude

Each term has:

- Amplitude: a^n
- Frequency: b^n

Thus:

- Amplitudes decrease
- Frequencies increase rapidly

4.2 Competition Between Decay and Oscillation

Differentiability depends on whether amplitudes decay fast enough to counteract oscillations.

In this case:

$$a^n b^n = (ab)^n,$$

which grows when $ab > 1$. Therefore, oscillations dominate.

4.3 General Principle

Theorem 3. *If a function f is k times continuously differentiable on $[0, 2\pi]$, then its Fourier coefficients c_n satisfy*

$$|c_n| \leq \frac{C}{|n|^k}$$

for some constant C.

Fourier analysis suggests the following.

- Smooth functions have rapidly decaying high-frequency components
- Rough functions retain significant high-frequency content

The Weierstrass function lies at the extreme end of this spectrum.

5 Conclusion

We have constructed a function that is continuous everywhere but differentiable nowhere. The key mechanism behind this phenomenon is the presence of oscillations at infinitely many scales.

Fourier analysis provides a natural framework for understanding this behavior. By decomposing functions into frequencies, it reveals how the interaction between amplitude and frequency determines smoothness.

This example illustrates a fundamental idea in analysis: local irregularity can arise from global structure.

References

- T. Tao, *Analysis I*, Hindustan Book Agency, 2006.
- E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, 2003.