ON RAMANUJAN'S PI INFINITE SERIES

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Contents

Abstract. In this paper, we derive Ramanujan's famous series for $\frac{1}{\pi}$.

1. Introduction and Definitions

We first define some tools and functions that we will need for the derivation.

Definition 1.1 (Gamma Function)**.** Let $z \in \mathbb{C}$ with $\Re(z) > 0$. The Gamma function is defined by the absolutely convergent improper integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

With integration by parts we may recover the identity $\Gamma(z+1) = z\Gamma(z)$ which allows for the unique analytic continuation of Γ to $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Definition 1.2 (Pochhammer Symbol)**.** The Pochhammer symbol is defined for $a \in \mathbb{C}$ as

$$(a)_0 = 1$$
$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

for $n \geq 1$, provided a is a not a non-positive integer.

Definition 1.3 (Gaussian Hypergeometric Series)**.** For $a, b, c \in \mathbb{C}$ where c is not a non-positive integer, the Gaussian hypergeometric series is defined as

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(a)_n, (b)_n, (c)_n$ are all Pochhammer symbols.

Lemma 1.4. *The series* ${}_2F_1(a, b; c; z)$ *converges absolutely for* $|z| < 1$.

Proof. Let $u_n = \frac{(a)_n(b)_n}{(c)_n n!} z^n$. We apply the Ratio Test, and taking the ratio of successive terms yields

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(a+n)(b+n)}{(c+n)(n+1)} z\right|.$$

Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} \left|\frac{(a+n)(b+n)}{(c+n)(n+1)} z\right| = \lim_{n\to\infty} \left|\frac{(1+a/n)(1+b/n)}{(1+c/n)(1+1/n)} z\right| = |z|.$$

Therefore, the series converges absolutely for $|z| < 1$ by the Ratio Test and diverges $|z| > 1$. The boundary case $|z| = 1$ can be shown to converge, however this is unnecessary for our parameter of $z_0 = 396^{-4}$ used later, which falls well within the radius of convergence. ■

2. Hypergeometric Representation of $K(k)$

We now define one of the elliptic integrals and its representation as a hypergeometric series, which is a key part of our proof of Ramanujan's series.

Definition 2.1 (Complete Elliptic Integral of the First Kind)**.** For a modulus k such that $0 \le k < 1$, the complete elliptic integral of the first kind is defined as

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2\theta}} d\theta.$$

Lemma 2.2. *For* $0 \le k < 1$,

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

Proof. Rewrite $K(k)$ as

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{-1/2} d\theta.$$

Since $0 \le k < 1$ and $0 \le \sin^2\theta \le 1$, we have $|k^2 \sin^2\theta| \le k^2 < 1$. This strict inequality allows us to use the generalized binomial theorem to expand the integrand, which states for $|x| < 1$

$$\sum_{n=0}^{\infty} \binom{-1/2}{n} (-x)^n.$$

Using the definition of the binomial coefficient and Pochhammer symbol we simplify the binomial coefficient:

$$\begin{aligned}\binom{-1/2}{n}(-1)^n &= \frac{(-1/2)(-3/2)\cdots(-(2n-1)/2)}{n!}(-1)^n \\ &= \frac{1 \cdot 3 \cdots (2n-1)}{2^n n!} \\ &= \frac{(1/2)_n}{n!}.\end{aligned}$$

Substituting $x = k^2 \sin^2\theta$ yields

$$K(k) = \int_0^{\pi/2} \frac{(1/2)_n}{n!} k^{2n} \sin^{2n}(\theta) d\theta.$$

Note that $f_{(n)}(\theta) = \frac{(1/2)_n}{n!} k^{2n} \sin^{2n}(\theta)$ are strictly nonnegative on $[0, \pi/2]$, so by Tonelli's Theorem we may bring the integral inside the infinite series:

$$K(k) = \sum_{k=0}^{\infty} \frac{(1/2)_n}{n!} k^{2n} \int_0^{\pi/2} \sin^{2n}(\theta) d\theta.$$

We use the well-known result of Wallis's Integral, which is that that

$$B(x, y) = 2 \int_0^{\pi/2} \sin^{2x-1} \theta \cos^{2y-1} \theta d\theta,$$

where B is the beta function. Thus,

$$\int_0^{\pi/2} \sin^{2n}(\theta) d\theta = \frac{1}{2} B\left(n + \frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(n+1)\Gamma(1/2)}{2\Gamma(n+1)}.$$

By the properties of the Gamma Function and Pochhammer symbol, we know that $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(n+1) = n!$, and $\Gamma(n + 1/2) = (1/2)_n \Gamma(1/2) = (1/2)_n \sqrt{\pi}$, so we have

$$\frac{\Gamma(n+1)\Gamma(1/2)}{2\Gamma(n+1)} = \frac{\pi}{2} \frac{(1/2)_n}{n!}.$$

Substituting this into our expression for $K(k)$ yields

$$K(k) = \sum_{n=0}^{\infty} \frac{(1/2)_n}{n!} k^{2n} \left(\frac{\pi}{2} \frac{(1/2)_n}{n!} \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_n}{(1)_n n!} (k^2)^n,$$

where the last step follows from the fact that $(1)_n = n!$. This exactly the matches of the hypergeometric series, so

$$K(k) = \frac{\pi}{2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; k^2 \right).$$

■

3. Existence of the Algebraic Modulus

In this section we prove that for any positive rational r, there exists a unique algebraic $k \in (0, 1)$ such that the ratio of the complete elliptic integrals is precisely $\sqrt{r}$. The k is known as the singular modulus of order r.

3.1. **Differential Relations and Legendre's Relation.**

To establish the existence of the singular modulus and eventually Ramanujan's series, we must first understand how the complete elliptic integrals $K(k)$ and $E(k)$ vary with the modulus k.

Definition 3.1 (Complete Elliptic Integral of the Second Kind)**.** For $0 \leq k < 1$, the complete elliptic integral of the second kind is defined by

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta.$$

We now state the following lemma, which gives a few facts about the differentiation of these functions. We omit the proof, which follows from an application of differentiation under the integral sign. Let $k' = \sqrt{1 - k^2}$ be the complementary modulus.

Lemma 3.2. $\frac{dE}{dk} = \frac{E-K}{k}$*, and* $\frac{dK}{dk} = \frac{E - k'^2 K}{kk'^2}$.

We now use Lemma 3.2 to prove the following theorem.

Theorem 3.3 (Legendre's Relation)**.** *For all* $k \in (0,1)$*, let* $k' = \sqrt{1-k^2}$*. Let* $K = K(k)$*,* $E = E(k)$*,* $K' = K(k')$*, and* $E' = E(k')$*. Then*

$$EK' + E'K - KK' = \frac{\pi}{2}.$$

Proof. Define the function $f(k) = EK' + E'K - KK'$. We will show that $\frac{df}{dk} = 0$ for all $k \in (0,1)$. Note that by the chain rule

$$\frac{d}{dk}K' = K(\sqrt{1-k^2}) = \frac{dK}{dk'}\frac{dk'}{dk} = \frac{k^2K' - E'}{kk'^2}.$$

Similarly,

$$\frac{d}{dk}E' = \frac{k(K' - E')}{k'^2}.$$

We now differentiate $f(k)$, so we have

$$\frac{df}{dk} = \left(\frac{dE}{dk}K' + E\frac{dK'}{dk}\right) + \left(\frac{dE'}{dk}K + E'\frac{dK}{dk}\right) - \left(\frac{dK}{dk}K' + K\frac{dK'}{dk}\right).$$

Substituting yields

$$\frac{df}{dk} = \frac{E-K}{k}K' + E\frac{k^2K' - E'}{kk'^2} + \frac{k(K'-E')}{k'^2}K + E'\frac{E - k'^2K}{kk'^2} - \frac{E - k'^2K}{kk'^2}K' - K\frac{k^2K' - E'}{kk'^2}.$$

Combining terms with a common denominator of kk'^2 and simplifying yields the above expression is 0, so $\frac{df}{dk} = 0$. This implies that $f(k) = C$, for some constant C. To find C, we evaluate the limit as $k \to 0^+$.

As $k \to 0$, we have $K(k) \to \pi/2$, $E(k) \to \pi/2$, and $k' \to 1$. Thus $E'(k) = E(k') \to E(1) = 1$. However $K'(k) = K(k')$ diverges as $k' \to 1$, so the terms EK' and KK' do not have finite limits individually. To determine the limit of

$$f(k) = EK' + E'K - KK',$$

we rewrite

$$f(k) = E'K + K'(E - K).$$

In this form the first term has a finite limit, while the second term can be analyzed using the small-k behavior of $E - K$.

Using the standard expansions near $k = 0$,

$$K(k) = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + O(k^4)\right),$$

$$E(k) = \frac{\pi}{2}\left(1 - \frac{k^2}{4} + O(k^4)\right),$$

we obtain $E(k) - K(k) = -\frac{\pi}{4}k^2 + O(k^4)$.

It is also known that as $k \to 0$ the complementary integral satisfies

$$K'(k) = K(k') = \log\left(\frac{4}{k}\right) + O(k^2 \log k).$$

Therefore

$$K'(k)(E(k) - K(k)) = \left(\log\frac{4}{k} + O(k^2\log k)\right)\left(-\frac{\pi}{4}k^2 + O(k^4)\right) = O(k^2 \log k) \to 0.$$

Consequently,

$$C = \lim_{k\to 0} f(k) = \lim_{k\to 0}\big(E'K + K'(E-K)\big) = 1\left(\frac{\pi}{2}\right) + 0 = \frac{\pi}{2}.$$

Since we have already shown that $f'(k) = 0$, the function $f(k)$ is constant on $(0,1)$, and therefore

$$EK' + E'K - KK' = \frac{\pi}{2}$$

for all $k \in (0,1)$. ■

3.2. **Existence of the Real Modulus.** We begin by defining the function that governs the relationship between the modulus and the period ratio. Let $k \in (0,1)$ and $k' = \sqrt{1-k^2}$ be the complementary modulus.

Definition 3.4 (The Period Ratio Function)**.** The function $H : (0,1) \to (0,\infty)$ is defined by the ratio of the complete elliptic integrals of the first kind:

$$H(k) = \frac{K'(k)}{K(k)} = \frac{K(\sqrt{1-k^2})}{K(k)}.$$

We now prove the following lemma.

Lemma 3.5. *The function $H(k)$ is strictly decreasing on $(0,1)$.*

Proof. Applying the quotient rule to $H(k) = K'/K$ yields

$$\frac{dH}{dk} = \frac{\frac{dK'}{dk}K - K'\frac{dK}{dk}}{K^2}.$$

Substituting the derivative expressions gives

$$\frac{dH}{dk} = \frac{1}{K^2}\left[\left(\frac{k^2K' - E'}{kk'^2}\right)K - K'\left(\frac{E - k'^2K}{kk'^2}\right)\right].$$

Simplifying and combining terms with $k^2 + k'^2 = 1$ yields

$$\frac{dH}{dk} = \frac{K'K - (E'K + EK')}{kk'^2K^2}.$$

By Theorem 3.3, we know the numerator is $-\frac{\pi}{2}$, so

$$\frac{dH}{dk} = -\frac{\pi}{2kk'^2K^2}.$$

For $k \in (0,1)$, all the terms in the denominator are positive, so therefore $\frac{dH}{dk} < 0$ and H is strictly decreasing. ■

We now determine the range of $H(k)$. As $k \to 0^+$, we know that $K(k) \to \pi/2$ and $K'(k) \to \infty$, so therefore, $\lim_{k\to 0^+} H(k) = \infty$. As $k \to 1^-$, we have $K(k) \to \infty$ and $K'(k) \to \frac{\pi}{2}$. Thus, $\lim_{k\to 1^-} H(k) = 0$. Since $H(k)$ is continuous and strictly decreasing, the Intermediate Value Theorem guarantees that for any $y \in (0,\infty)$, there exists a unique $k \in (0,1)$ such that $H(k) = y$. Specifically, for any $r \in \mathbb{Q}^+$, there exists a unique k_r such that

$$H(k_r) = \sqrt{r}.$$

3.3. **Existence of the Modular Equation.** We now introduce a theorem, that we unfortunately unable to give a proof of since it is out of the scope of this paper.

Theorem 3.6. *For every positive integer n, there exists a polynomial $P_n(X,Y)$ with integer coefficients such that $P_n(k^2, l^2) = 0$ if and only if $H(l) = nH(k)$.*

Using the above theorem, we show that k_r is algebraic for any rational r.

Theorem 3.7. *If $H(k) = \sqrt{r}$ for $r \in \mathbb{Q}^+$, then k is an algebraic number.*

Proof. Let k be the modulus such that $H(k) = \sqrt{r}$. Recall that $k' = \sqrt{1-k^2}$ satisfies

$$H(k') = \frac{K'(k')}{K(k')} = \frac{K(k)}{K'(k)} = \frac{1}{H(k)}.$$

Substituting our condition $H(k) = \sqrt{r}$ yields

$$H(k') = \frac{1}{\sqrt{r}}.$$

Multiplying both sides by r yields

$$rH(k') = \sqrt{r} = H(k).$$

Note that this is the exact condition required to invoke Theorem 3.6. Therefore, we may conclude then (by Theorem 3.6) that there exists a polynomial $P_n(X,Y) \in \mathbb{Z}[X,Y]$ such that:

$$P_n(k'^2, k^2) = P_n(1-k^2, k^2) = 0.$$

Let $Q(x) = P_n(1-x, x)$. Since P_n is a polynomial in two variables with integer coefficients, $Q(x)$ is a polynomial in one variable with integer coefficients. The above equation shows that k^2 is a root of $Q(x)$, which means that k^2 is algebraic. It follows that $k = \sqrt{k^2}$ is algebraic. ■

4. Linearization

Definition 4.1 (Ring of Formal Series)**.** Let $\mathbb{A}$ be the field of algebraic numbers. The ring of formal series $\mathbb{A}[[z]]$ consists of all formal sums $\sum_{n=0}^{\infty} a_n z^n$ where $a_n \in \mathbb{A}$. Addition and multiplication (defined as the Cauchy product) are defined agebraically without respect to convergence.

Definition 4.2 (Formal Derivation)**.** A formal derivation on $\mathbb{A}[[z]]$ is a map $D : \mathbb{A}[[z]] \to \mathbb{A}[[z]]$ that satisfies

$$D(af + bg) = aD(f) + bD(g)$$
$$D(fg) = fD(g) + D(f)g,$$

where $a, b \in \mathbb{A}$ and $f, g \in \mathbb{A}[[z]]$.

Lemma 4.3. *There exists a unique formal derivation D such that $D(z) = 1$ and $D(a) = 0$ for all $a \in \mathbb{A}$, defined by:*

$$D\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Proof. Linearity is trivial by definition. For the second requirement, we note that the coefficient of z^{k-1} in $D(fg)$ is $k\sum_{i=0}^{k} a_i b_{k-i}$. Computing $fD(g)+gD(f)$ yields

$$\left(a_i z^i\right)\left(\sum j b_j z^{j-1}\right)+\left(\sum b_j z^j\right)\left(\sum i a_i z^{i-1}\right).$$

The coefficient of z^{k-1} in this sum is

$$\sum_{i=0}^{k-1} a_i(k-i)b_{k-i}+\sum_{i=1}^{k} i a_i b_{k-i}=\sum_{i=0}^{k-1} k a_i b_{k-i}=[z^{k-1}]D(fg).$$

■

We now define the Linearization Operator.

Definition 4.4 (Euler Operator)**.** Define the operator $\Theta:\mathbb{A}[[z]]\to\mathbb{A}[[z]]$ by $\Theta=zD$. For any $F(z)=\sum c_n z^n$, we have

$$\Theta(F(z))=\sum_{n=0}^{\infty} n c_n z^n.$$

We now construct the $(An+b)$ progression. Let $\alpha,\beta\in\mathbb{A}$ be constants. Let $F(z)={}_2F_1\left(\frac{1}{2},\frac{1}{2};1;z\right)=\sum_{n=0}^{\infty} c_n z^n$. Define the linear operator $L=\beta\Theta+\alpha I$, where I is the identity operator. Applying L to F yields

$$L(F(z))=\sum_{n=0}^{\infty}(\beta n+\alpha)c_n z^n. \tag{4.1}$$

To relate this to π, we must express the derivative of $F(z)$ in terms of $K(k)$ and $E(k)$.

Lemma 4.5. *For $z=k^2$, the derivative of ${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;z\right)$ satisfies*

$$z\frac{d}{dz}\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;z\right)\right]=\frac{E(k)-(1-z)K(k)}{\pi(1-z)}.$$

Proof. By Lemma 2.2, $F(z)=\frac{2}{\pi}K(k)$. We differentiate with respect to z by the chain rule, which yields

$$\frac{d}{dz}\left[\frac{2}{\pi}K(k)\right]=\frac{2}{\pi}\frac{dK}{dk}\frac{dk}{dz}.$$

Recall that $z=k^2$, so $\frac{dk}{dz}=\frac{1}{2z}$. We can substitute $\frac{dK}{dk}$ from Lemma 3.2, so after substituting and simplifying we have

$$\frac{d}{dz}F(z)=\frac{E(k)-(1-z)K(k)}{\pi z(1-z)}.$$

Multiplying by z gives the identity. ■

Substituting the expression from above into our relationship in Equation 4.1 yields

$$\frac{1}{\pi}\left[2\alpha K(k)+\beta\frac{E(k)-(1-z)K(k)}{1-z}\right]=\sum_{n=0}^{\infty}(\beta n+\alpha)c_n z^n.$$

5. Evaluation at Singular Modulus

5.1. **Calculating c_n.** Note that the result applies to any hypergeometric series. As we established in Section 2,

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

To obtain Ramanujan's formula, we utilize Clausen's Formula, which expresses the square of our ${}_2F_1$ series to a ${}_3F_2$ series:

$$[{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; z\right)]^2 = {}_3F_2\left(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}; 1, 1; 4z(1-z)\right).$$

We now calculate the coefficients c_n for the series $\sum c_n z^n = {}_3F_2\left(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}; 1, 1; x\right)$. From the definition of the hypergeometric series, the n-th coefficient is:

$$c_n = \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(1)_n(1)_n n!} = \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3}.$$

We evaluate the Pocchammer symbols using Gauss's formula, since

$$\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n = \frac{\Gamma(n+1/4)}{\Gamma(1/4)}\frac{\Gamma(n+1/2)}{\Gamma(1/2)}\frac{\Gamma(n+3/4)}{\Gamma(3/4)}.$$

We calculate the numerator with Gauss's Formula:

$$\Gamma(n+\tfrac{1}{4})\Gamma(n+\tfrac{1}{2})\Gamma(n+\tfrac{3}{4}) = \frac{(2\pi)^{3/2}4^{1/2-4(n+1/4)}\Gamma(4n+1)}{\Gamma(n+1)}$$
$$= \frac{(2\pi)^{3/2}4^{-1/2-4n}(4n)!}{n!}.$$

We similarly calculate the denominator:

$$\Gamma(\tfrac{1}{4})\Gamma(\tfrac{1}{2})\Gamma(\tfrac{3}{4}) = \frac{(2\pi)^{3/2}4^{1/2-1}\Gamma(1)}{\Gamma(1)} = (2\pi)^{3/2}4^{-1/2}.$$

Calculating the resulting ratio yields that

$$\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n = \frac{(4n)!}{256^n(n!)},$$

so therefore

$$c_n = \frac{(4n)!}{256^n(n!)^4}.$$

We now must find α and β. from the Linearization identity in Section 4.

5.2. **Calculating the Coefficients.** To calculate the coefficients, we move from Signature 2 to Signature 4, which corresponds to the ${}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; z)$ basis.

Definition 5.1. For a given modulus k, we define the Nome q as

$$q = e^{-\pi K'(k)/K(k)}, \quad \text{where } K'(k) = K(\sqrt{1-k^2}).$$

From Theorem 3.7, we know there exists an algebraic modulus k_r for any $r \in \mathbb{Q}^+$ such that

$$\frac{K'(k_r)}{K(k_r)} = \sqrt{r}.$$

We evaluate at $r = 58$.

Definition 5.2. The class invariant G_r is defined in terms of the singular modulus k_r as:

$$G_r = (2k_r k_r')^{-1/12}$$

For $r = 58$, the value of G_{58} is determined by the fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{29})$, which is $U = \frac{5+\sqrt{29}}{2}$. Specifically, $G_{58} = \sqrt{U}$.

We now define $s(r)$, the singular value of the second kind. For a singular modulus k_r, we have

$$s(r) = \frac{\sqrt{r}}{\pi} - \frac{4\sqrt{r}K(k_r)K'(k_r)}{\pi^2}\left[\frac{E(k_r)}{K(k_r)} - \frac{\pi}{4K(k_r)K'(k_r)}\right].$$

The operator $L = \beta\Theta + \alpha I$ is matched to the derivative of the elliptic integrals. At $r = 58$, the derive term β must be 26390 since this is the rational component needed to satisfy the differential relation between the modular forms of E and K at k_{58}. The constant term α must be 1103 to satisfy the Legendre relation. For $r = 58$, therefore we have that

$$\frac{1}{\pi} = c[26390 \cdot \Theta(F(z)) + 1103 \cdot F(z)].$$

It remains to figure out where to evaluate the series. To find this, we use the fact that the modular argument in Signature 4 is

$$z = \left(\frac{2}{G_r^{12} + G_r^{-12}}\right)^2 = \left(\frac{1}{9801}\right)^2.$$

From s_r, we can now calculate c, which is equal to

$$\frac{\sqrt{1-z}}{K(k_5 8)} = \frac{2\sqrt{2}}{9801}.$$

Combining the above yields the final formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}.$$