THE ROGERS-RAMANUJAN IDENTITIES, THEIR COMBINATORIAL INTERPRETATION, AND GENERALIZATION

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Abstract. The Rogers-Ramanujan identities are proved analytically and their combinatorial interpretation in terms of congruences is subsequently given. Gordon's combinatorial generalization of these identities is then stated.

1. Introduction

The Rogers-Ramanujan identities are the following two relations between infinite products and infinite series:

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = 1 + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)\dots(1-q^k)}$$

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} = 1 + \sum_{k=1}^{\infty} \frac{q^{k(k+1)}}{(1-q)\dots(1-q^k)}.$$

An analytical proof due to Rogers and Ramanujan is given in [RR19]. The left sides have obvious combinatorial interpretations. For example, $\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$ is the generating function for the number of partitions into parts congruent to 1 or 4 $\mod 5$. The combinatorial meanings of the right sides are less obvious but it turns out that $1 + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)\dots(1-q^k)}$ is the generating function for the number of partitions with differences at least two. Similarly, $1 + \sum_{k=1}^{\infty} \frac{q^{k(k+1)}}{(1-q)\dots(1-q^k)}$ counts the number of partitions into parts with differences at least 2 and with no 1's. The Rogers-Ramanujan identities have the following combinatorial generalization due to Gordon [Gor61]: The number of partitions of an integer n into parts not congruent to $0, \pm t \mod 2d+1$, where $1 \leq t \leq d$, is equal to the number of partitions of

$$n = n_1 + n_2 + \dots + n_k$$

with $n_i \geq n_{i+1}$ and $n_i \geq n_{i+d-1} + 2$ and $n_{k-t+1} \geq 2$.

2. The Rogers-Ramanujan Identities

Definition 2.1 (Rogers-Ramanujan functions)**.**

$$G(q) := 1 + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)\dots(1-q^k)}$$

$$H(q) := 1 + \sum_{k=1}^{\infty} \frac{q^{k(k+1)}}{(1-q)\dots(1-q^k)}$$

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Of fundamental importance in both the analytical proof of the Rogers-Ramanujan identities and Gordon's combinatorial generalization is the Jacobi Triple Product Identity.

Theorem 2.2 (Jacobi Triple Product Identity). *For all $z \in \mathbb{C}$, and all $|q| < 1$,*

$$\sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz} = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1}e^{2iz})(1+q^{2n-1}e^{-2iz}).$$

Remark 2.3. If we denote $\Im(\tau) = t > 0$ and we suppose that $|z| \le M$ for arbitrary positive real numbers M, then

$$|\sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}| \le \sum_{n=-\infty}^{\infty} |e^{\pi i \tau n^2 + 2niz}| \le \sum_{n=-\infty}^{\infty} e^{-\pi t n^2 + 2nM}.$$

Since the last series converges, it follows that $\sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}$ is an entire function of z for every fixed $\Im(\tau) > 0$, and is a holomorphic function of q in the unit disk $|q| < 1$ for every fixed $z \in \mathbb{C}$.

We will give a short proof [And65] of Theorem 2.2 by first proving two lemmas due to Euler [Eul51] after discussing the holomorphic properties of an infinite product.

Lemma 2.4. *Let us denote the infinite product $\prod_{m=0}^{\infty}(1+y^m\omega)$ by $F(\omega)$ when $|y| < 1$. Then $F(\omega)$ is an entire function of ω for every fixed $|y| < 1$.*

Proof. Let $|\omega| < M$ for an arbitrary positive real number M. Since $|y| < 1$, there exists a minimum non-negative integer M_0 for which $M|y|^m < \frac{1}{2}$ whenever $m > M_0$. For every $m > M_0$, we have, if we make use of the principal branch of the logarithm,

$$|\log(1+\omega y^m)| \le M|y|^m \sum_{n=1}^{\infty} \frac{M^{n-1}|y|^{m(n-1)}}{n} \le M|y|^m \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^{n-1}} \le 2M|y|^m$$

and since $\sum_{m=M_0+1}^{\infty} |y|^m$ converges, it follows that $\sum_{m=M_0+1}^{\infty} \log(1+y^m\omega)$ is holomorphic on $|\omega| < M$ if $|y| < 1$ is fixed. From this it follows that

$$\exp\left(\sum_{m=M_0+1}^{\infty} \log(1+y^m\omega)\right) = \prod_{m=M_0+1}^{\infty} (1+y^m\omega),$$

and hence $\prod_{m=0}^{\infty}(1+y^m\omega)$, is an entire function of ω for every fixed $|y| < 1$. ■

Lemma 2.5. *For all $\omega \in \mathbb{C}$ and all y such that $|y| < 1$, we have*

$$\prod_{m=0}^{\infty}(1+y^m\omega) = \sum_{r=0}^{\infty} \frac{y^{\frac{r(r-1)}{2}}\omega^r}{(1-y)\dots(1-y^r)}.$$

Proof. We denote the Taylor expansion of the left side, which, by the previous lemma, is an entire function of ω provided that $|y| < 1$, by

$$F(\omega) = \sum_{n=0}^{\infty} a_n \omega^n. \tag{2.1}$$

Then it immediately follows that $a_0 = 1$. But now

$$F(\omega y) = \prod_{m=0}^{\infty}(1+\omega y^{m+1}),$$

so that

$$(1+\omega)F(\omega y) = F(\omega).$$

Therefore

$$\sum_{n=0}^{\infty} a_n \omega^n y^n + \sum_{n=1}^{\infty} a_{n-1}\omega^n y^{n-1} = \sum_{n=0}^{\infty} a_n \omega^n.$$

Equating coefficients of ω on both sides gives

$$a_n y^n + a_{n-1} y^{n-1} = a_n,$$

so that we have the recurrence relation

$$a_n = \frac{a_{n-1} y^{n-1}}{1-y^n}$$

for $n \geq 1$. Since $a_0 = 1$, it follows that

$$a_n = \frac{y^{0+1+\cdots+(n-1)}}{(1-y)\cdots(1-y^n)} = \frac{y^{n(n-1)/2}}{(1-y)\dots(1-y^n)}.$$

The lemma now follows by replacing this value of a_n in (2.1). ■

Lemma 2.6. *For all $|y| < 1$ and $|\omega| < 1$, we have*

$$\prod_{m=0}^{\infty} (1+y^m\omega)^{-1} = \sum_{r=0}^{\infty} \frac{(-1)^r \omega^r}{(1-y)\dots(1-y^r)}.$$

Proof. Inasmuch as the proof of this lemma is very similar to that of Lemma 2.5, we leave it as an exercise to the reader. ■

We can now prove Theorem 2.2.

Proof. In Lemma 2.5, put $y = q^2$ and $\omega = qe^{2iz}$ to find

$$\begin{aligned}\prod_{n=1}^{\infty}(1+q^{2n-1}e^{2iz}) &= \sum_{n=0}^{\infty} \frac{e^{2niz}q^{n^2}}{(1-q^2)\dots(1-q^{2n})} = \sum_{n=0}^{\infty} \frac{e^{2niz}q^{n^2}(1-q^{2n+2})(1-q^{2n+4})\dots}{(1-q^2)(1-q^4)\dots} \\ &= \left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{n=0}^{\infty} e^{2niz}q^{n^2}\prod_{m=0}^{\infty}(1-q^{2n+2+2m})\right] \\ &= \left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{n=-\infty}^{\infty} e^{2niz}q^{n^2}\prod_{m=0}^{\infty}(1-q^{2n+2+2m}),\right]\end{aligned}$$

because $\prod_{m=0}^{\infty}(1-q^{2n+2+2m})$ equals zero whenever n is a negative integer. By making use of Lemma 2.5 again but this time by putting $y = q^2$, and $\omega = -q^{2n+2}$, the last expression equals

$$\left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{n=-\infty}^{\infty} e^{2niz}q^{n^2}\sum_{r=0}^{\infty}\frac{(-1)^r q^{(2n+2)r+r^2-r}}{(1-q^2)\dots(1-q^{2r})}\right],$$

which can be written as

$$\left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{n=-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r e^{2niz}q^{(n+r)^2+r}}{(1-q^2)\dots(1-q^{2r})}.\right]$$

We can invert the double sum provided that it converges absolutely. For any fixed q such that $|q| = R$ where $R < 1$, we have

$$\left|\sum_{r=0}^{\infty} \frac{(-1)^r e^{2inz} q^{(n+r)^2+r}}{(1-q^2)\dots(1-q^{2r})}\right| \leq \sum_{r=0}^{\infty} \frac{|e^{2inz}| R^{(n+r)^2+r}}{(1-R^2)^r}$$

$$\leq |e^{2inz}| R^{n^2} \sum_{r=0}^{\infty} \frac{R^{r^2+r}}{(1-R^2)^r} = C|e^{2inz}| R^{n^2},$$

for some C independent of n. Since $\sum_{n=-\infty}^{\infty} |e^{2inz}| R^{n^2}$ is convergent, the inversion of the double sum is justified, so that

$$\prod_{n=0}^{\infty}(1+q^{2n+1}e^{2iz}) = \left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{r=0}^{\infty} \frac{(-1)^r e^{-2irz} q^r}{(1-q^2)\dots(1-q^{2r})} \sum_{n=-\infty}^{\infty} e^{2i(n+r)z} q^{(n+r)^2}\right].$$

The ultimate sum being absolutely convergent, we can re-index it by changing n into $n-r$ without altering the value and the right side can be factorised as

$$\left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\sum_{r=0}^{\infty} \frac{(-qe^{-2iz})^r}{(1-q^2)\dots(1-q^{2r})}\right]\left[\sum_{n=-\infty}^{\infty} e^{2inz} q^{n^2}\right].$$

But now if we restrict $|qe^{-2iz}| < 1$, we can make use of Lemma 2.6, with $y = q^2$ and $\omega = qe^{-2iz}$ to deduce that

$$\prod_{n=0}^{\infty}(1+q^{2n+1}e^{2iz}) = \left[\prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\right]\left[\prod_{k=0}^{\infty}(1+q^{2n+1}e^{-2iz})^{-1}\right]\left[\sum_{n=-\infty}^{\infty} e^{2inz} q^{n^2}\right],$$

i.e.

$$\sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz} = \prod_{n=0}^{\infty}(1+q^{2n+1}e^{2iz})(1+q^{2n+1}e^{-2iz})(1-q^{2n+2}),$$

when $|q| < 1$ and $|q| < |e^{2iz}|$. But both sides of the previous equality are entire functions of z for every fixed $|q| < 1$, by Lemma 2.4 and Remark 2.3. Since, moreover, both sides agree on a set of values of z, namely $|e^{2iz}| > |q|$, a subset of which is the open lower half z-plane, containing at least one limit point, therefore both sides agree on the entire complex z-plane for every fixed $|q| < 1$, by analytic continuation. ■

Two special cases of the Jacobi Triple Product Identity will be relevant to us.

Corollary 2.7. *For* $|x| < 1$,

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+1)}{2}} = \prod_{n=1}^{\infty}(1-x^{5n-3})(1-x^{5n-2})(1-x^{5n}).$$

Proof. Let $q = e^{5\pi iu}$ and $z = (1+u)\pi/2$, where $\Im(u) > 0$, in Theorem 2.2. This gives

$$\sum_{n=-\infty}^{\infty} e^{5\pi iun^2} e^{\pi in(1+u)} = \prod_{n=1}^{\infty}(1-e^{10\pi iun})(1+e^{5(2n-1)\pi iu+\pi i(1+u)})(1+e^{5(2n-1)\pi iun-\pi i(1+u)}),$$

or, letting $x = e^{2\pi iu}$,

$$\sum_{n=-\infty}^{\infty} e^{\pi in} x^{\frac{5n^2+n}{2}} = \prod_{n=1}^{\infty}(1-x^{5n})(1-x^{5n-2})(1-x^{5n-3}),$$

as desired.

■

Corollary 2.8. *For $|x|<1$,*

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+3)}{2}} = \prod_{n=1}^{\infty}(1-x^{5n-4})(1-x^{5n-1})(1-x^{5n}).$$

Proof. The proof proceeds similarly as in the last corollary, except that now we put $q = e^{5\pi iu}$ and $z = (1+3u)\pi/2$. ■

Remark 2.9. By absolute convergence, we find that

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+1)}{2}} = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n-1)}{2}},$$

by changing n into $-n$, when $|x|<1$. Similarly

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+3)}{2}} = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n-3)}{2}}.$$

We will use these representations interchangeably in the rest of the paper without further comment.

Theorem 2.10 (Rogers-Ramanujan Identities)**.**

$$G(q) = \prod_{n=1}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1}, \qquad H(q) = \prod_{n=1}^{\infty}(1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1}.$$

We give a sequence of lemmas, in the first of which we define a function $\mathcal{G}(x)$ which generalises the infinite sum of Corollary 2.7 which equals $\mathcal{G}(1)$. We will subsequently find a recurrence, of a similar type as that found in the proof of 2.5, for a closely related function which will then be expressible as a series which generalises both the Rogers-Ramanujan functions $G(q)$ and $H(q)$. This will enable us to prove Theorem 2.10.

Lemma 2.11. *If, for every $x \in \mathbb{C}$, we define*

$$\mathcal{G}(x) := 1 + \sum_{n=1}^{\infty}(-1)^n x^{2n} q^{\frac{n(5n-1)}{2}}(1-xq^{2n})\frac{(1-xq)(1-xq^2)\dots(1-xq^{n-1})}{(1-q)(1-q^2)\dots(1-q^n)},$$

then

$$\mathcal{G}(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n} q^{\frac{1}{2}n(5n+1)}(1-x^2q^{4n+2})\frac{(1-xq)\dots(1-xq^n)}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Remark 2.12. Although we have used a similar notation, the functions $\mathcal{G}$ and G are obviously different.

Proof.

$$\mathcal{G}(x) = 1 + \sum_{n=1}^{\infty}(-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} \frac{(1-xq)(1-xq^2)\dots(1-xq^{n-1})}{(1-q)(1-q^2)\dots(1-q^{n-1})} + \sum_{n=1}^{\infty}(-1)^n x^{2n} q^{\frac{1}{2}n(5n+1)} \frac{(1-xq)(1-xq^2)\dots(1-xq^{n})}{(1-q)(1-q^2)\dots(1-q^{n})}.$$

By taking away the first summand of the first infinite series on the right and re-indexing, we deduce that

$$\begin{aligned}\mathcal{G}(x) &= 1 - x^2q^2 + \sum_{n=1}^{\infty}(-1)^n \left(x^{2n} q^{\frac{n(5n+1)}{2}} - x^{2n+2} q^{\frac{(n+1)(5n+4)}{2}}\right) \frac{(1-xq)\dots(1-xq^n)}{(1-q)\dots(1-q^n)} \\ &= 1 - x^2q^2 + \sum_{n=1}^{\infty}(-1)^n x^{2n} q^{\frac{1}{2}n(5n+1)}(1-x^2q^{4n+2}) \frac{(1-xq)\dots(1-xq^n)}{(1-q)(1-q^2)\dots(1-q^n)}.\end{aligned}$$

■

Lemma 2.13. *For $\mathcal{G}$ defined as in the previous lemma,*

$$\frac{\mathcal{G}(x)}{1-xq} - \mathcal{G}(xq) = xq(1-xq^2)\mathcal{G}(xq^2).$$

Proof. If we denote the right side by $\mathcal{H}(x)$, then, using the previous lemma

$$\begin{aligned}\mathcal{H}(x) &= xq \\ &+ \frac{1}{1-xq}\sum_{n=1}^{\infty}\left\{\frac{(-1)^n(1-xq)\dots(1-xq^n)}{(1-q)(1-q^2)\dots(1-q^n)} x^{2n} q^{\frac{1}{2}n(5n+1)}\left[(1-x^2q^{4n+2}) - q^n(1-xq^{2n+1})\right]\right\} \\ &= xq + \frac{1}{1-xq}\sum_{n=1}^{\infty}(-1)^n \frac{(1-xq)\dots(1-xq^n)}{(1-q)\dots(1-q^{n-1})} x^{2n} q^{\frac{1}{2}n(5n+1)} \\ &\quad + \frac{1}{1-xq}\sum_{n=1}^{\infty}(-1)^n \frac{(1-xq)\dots(1-xq^{n+1})}{(1-q)\dots(1-q^n)} x^{2n+1} q^{\frac{1}{2}n(5n+1)+3n+1},\end{aligned}$$

where the last step has been achieved by rewriting

$$(1-x^2q^{4n+2}) - q^n(1-xq^{2n+1}) = (1-q^n) + xq^{3n+1}(1-xq^{n+1}).$$

Now we separate the first sum in the first infinite series on the right, re-index, and deduce that

$$\begin{aligned}\mathcal{H}(x) &= xq(1-xq^2) \\ &\quad + \frac{1}{1-xq}\sum_{n=1}^{\infty}\frac{(-1)^n(1-xq)\dots(1-xq^{n+1})}{(1-q)\dots(1-q^n)}\left[x^{2n+1}q^{\frac{n(5n+1)}{2}+3n+1} - x^{2n+2}q^{\frac{(n+1)(5n+6)}{2}}\right] \\ &\quad = xq(1-xq^2)\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n(1-xq^3)\dots(1-xq^{n+1})}{(1-q)\dots(1-q^n)} x^{2n} q^{\frac{n(5n+1)}{2}+3n}(1-xq^{2n+2})\right) \\ &\quad = xq(1-xq^2)\mathcal{G}(xq^2).\end{aligned}$$

■

Lemma 2.14. *For $\mathcal{G}(x)$ as in Lemma 2.11, if $|x| \leq 1$, we have*

$$\mathcal{G}(x) \prod_{n=1}^{\infty} (1 - xq^n)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{x^n q^{n^2}}{(1-q)\dots(1-q^n)}.$$

Proof. Let us denote the left side by $\mathcal{F}(x)$. Then Lemma 2.13 takes the form

$$\mathcal{F}(x) = \mathcal{F}(xq) + xq\mathcal{F}(xq^2).$$

Then we find that

$$\mathcal{F}(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n q^{n^2}}{(1-q)\dots(1-q^n)}.$$

We omit the last justification because the reader will have no difficulty in supplying it inasmuch as it is completely analogous to the proof of Lemma 2.5 detailed above. ∎

We can now prove Theorem 2.10.

Proof. By putting $x = 1$ and $x = q$ successively in Lemma 2.14, we find, using the Rogers-Ramanujan functions defined in Definition 2.1,

$$\mathcal{G}(1) \prod_{n=1}^{\infty} (1 - q^n)^{-1} = G(q), \qquad \mathcal{G}(q) \prod_{n=1}^{\infty} (1 - q^{n+1})^{-1} = H(q).$$

But now, using the definition of $\mathcal{G}$, given in Lemma 2.11, and Corollaries 2.7 and 2.8 we find that

$$\mathcal{G}(1) = 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(5n-1)}{2}} (1+q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(5n-1)}{2}} = \prod_{n=1}^{\infty} (1-q^{5n-3})(1-q^{5n-2})(1-q^{5n}),$$

and that

$$\begin{aligned}
(1-q)\mathcal{G}(q) &= 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(5n+3)}{2}} - q + \sum_{n=1}^{\infty} (-1)^{n+1} q^{\frac{n(5n+3)}{2}} q^{2n+1} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(5n+3)}{2}} + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(5n-3)}{2}} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(5n-3)}{2}} \\
&= \prod_{n=1}^{\infty} (1 - q^{5n-4})(1 - q^{5n-1})(1 - q^{5n}).
\end{aligned}$$

This means that

$$G(q) = \prod_{n=1}^{\infty} (1 - q^{5n-4})(1 - q^{5n-1}), \qquad H(q) = \prod_{n=1}^{\infty} (1 - q^{5n-2})(1 - q^{5n-3}).$$

∎

3. Combinatorial Interpretation of the Rogers-Ramanujan Identities

The two Rogers-Ramanujan Identities proved in the last section

$$\prod_{n=1}^{\infty}(1-q^{5n-4})(1-q^{5n-1}) = 1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)\dots(1-q^k)}$$
$$\prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3}) = 1+\sum_{k=1}^{\infty}\frac{q^{k(k+1)}}{(1-q)\dots(1-q^k)}$$

have interesting combinatorial interpretations.

Theorem 3.1. *The number of partitions of a positive integer N into distinct parts with differences at least 2 equals the number of partitions of N into parts congruent to either 1 or 4 modulo 5.*

Proof. In

$$\prod_{n=1}^{\infty}(1-q^{5n-4})(1-q^{5n-1}) = 1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)\dots(1-q^k)},$$

the left side is the generating function for the number of partitions of N into parts congruent to either 1 or 4 modulo 5. We must now show that the right side is the generating function for partitions of N into distinct parts with differences at least 2. The smallest such partition with k terms is

$$(2k-1)+(2k-3)+\dots+3+1 = k^2.$$

Therefore to obtain every partition of the form

$$n_1+n_2+\dots+n_k$$

where $n_i > 0$ and $n_i \geq n_{i+1}+2$ exactly once we must combine termwise[1] to $(2k-1)+(2k-3)+\dots+3+1$ every partition of the form

$$m_1+m_2+\dots+m_k$$

where $m_i \geq 0$ and $m_i \geq m_{i+1}$ exactly once. The generating function for all such partitions

$$m_1+m_2+\dots+m_k$$

is

$$\frac{1}{(1-q)(1-q^2)\dots(1-q^k)}$$

which can be seen by expanding each $(1-q^i)^{-1}$ into a geometric series as

$$\frac{1}{1-q^i} = 1+q^{1+1+\dots+1}+q^{2+2+\dots+2}+q^{3+3+\dots+3}+\dots$$

where we write the exponent of q^{ij} as the sum of i copies of j, for each positive integer j, and by combining terms from any two such geometric series by termwise addition of the partitions in their exponent e.g. in $(1-q)^{-1}(1-q^2)^{-1}$ we combine the term q^4, from the geometric expansion of $(1-q)^{-1}$, with the term q^{3+3}, from the geometric expansion of $(1-q^2)^{-1}$, by writing the resulting term as $q^{(4+3)+3} = q^{7+3}$. Therefore the generating function for the partitions

$$n_1+n_2+\dots+n_k$$

[1] by which is meant $(2k-1+m_1)+(2k-3+m_2)+\dots+(1+m_k)$.

is

$$\frac{q^{(2k-1)+(2k-3)+\cdots+3+1}}{(1-q)\ldots(1-q^k)} = \frac{q^{k^2}}{(1-q)\ldots(1-q^k)}.$$

Summing over all nonnegative k, it follows that the generating function for partitions into distinct parts with difference at least 2 is given by

$$1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)\ldots(1-q^k)}.$$

■

Example. The partitions of the positive integer 9 into distinct parts with differences at least 2 are 5 in number and they are: $9 = 8+1 = 7+2 = 6+3 = 5+3+1$. The partitions of 9 into parts $\equiv 1, 4 \mod 5$ are also 5 in number and they are: $9 = 6+1+1+1 = 4+4+1 = 4+1+1+1+1+1 = 1+1+\cdots+1$. As another example, the partitions of 10 into distinct parts with differences at least two are: $10 = 9+1 = 8+2 = 7+3 = 6+4 = 6+3+1$ and they are six in number, as are the partitions of 10 into parts $\equiv 1, 4 \mod 5$: $9+1 = 6+4 = 6+1+1+1+1 = 4+4+1+1 = 4+1+1+1+1+1+1 = 1+1+1+\cdots+1$.

Theorem 3.2. *The number of partitions of N into distinct parts with differences at least* 2 *and with no* 1*'s is equal to the number of partitions of N into parts congruent to either* 2 *or* 3 *modulo* 5.

Proof. The proof is similar to the previous one except that now we use the other Rogers-Ramanujan identity

$$\prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3}) = 1+\sum_{k=1}^{\infty}\frac{q^{k(k+1)}}{(1-q)\ldots(1-q^k)},$$

and we write

$$k(k+1) = 2+4+\cdots+2k.$$

■

4. Gordon's Combinatorial Generalization

A combinatorial generalization of the Rogers-Ramanujan identities has been obtained by Basil Gordon [Gor61].

Theorem 4.1. *The number of partitions of an integer n into parts not congruent to $0, \pm t$* $\mod 2d+1$, *where $1 \le t \le d$, is equal to the number of partitions of*

$$n = n_1 + n_2 + \cdots + n_k$$

with $n_i \ge n_{i+1}$ and $n_i \ge n_{i+d-1}+2$ and $n_{k-t+1} \ge 2$.

The Rogers-Ramanujan identities are the special cases $(d,t) = (2,1)$ and $(d,t) = (2,2)$ of the previous theorem; for when $d = 2$ and $t = 1$, it says that the number of partitions of an integer n into parts not congruent to $0, 1, 4 \mod 5$ is equal to the number of partitions of

$$n = n_1 + n_2 + \cdots + n_k$$

with $n_i \geq n_{i+1}, n_i \geq n_{i+1} + 2$, and $n_k \geq 2$, which is the combinatorial interpretation of the second Rogers-Ramanujan identity:

$$\prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3}) = 1 + \sum_{k=1}^{\infty} \frac{q^{k(k+1)}}{(1-q)\dots(1-q^k)}.$$

When $d = 2$ and $t = 2$ it says that the number of partitions of an integer n into parts not congruent to $0, 2, 3 \mod 5$ is equal to the number of partitions of

$$n = n_1 + n_2 + \cdots + n_k,$$

with $n_i \geq n_{i+1}, n_i \geq n_{i+1} + 2$ and $n_{k-1} \geq 2$, which is the combinatorial interpretation of the first Rogers-Ramanujan identity

$$\prod_{n=1}^{\infty}(1-q^{5n-4})(1-q^{5n-1}) = 1 + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)\dots(1-q^k)}.$$

Before proving the the theorem, let us illustrate some special cases of it.

Example. When $d = 3$ and $t = 1$ it says that the number of partitions of an integer n into parts not congruent to $0, 1, 6 \mod 7$ is equal to the number of partitions

$$n = n_1 + n_2 + \cdots + n_k$$

with $n_i \geq n_{i+1}, n_i \geq n_{i+2} + 2$ and $n_k \geq 2$. Thus, the partitions of 11 into parts congruent to $2, 3, 4, 5 \mod 7$ are $11 = 9 + 2 = 5 + 4 + 2 = 5 + 3 + 3 = 5 + 2 + 2 + 2 = 4 + 4 + 3 = 4 + 3 + 2 + 2 = 3 + 3 + 3 + 2 = 3 + 2 + 2 + 2 + 2$, which are 9 in number as are the partitions: $11 = 9 + 2 = 8 + 3 = 7 + 4 = 7 + 2 + 2 = 6 + 5 = 6 + 3 + 2 = 5 + 4 + 2 = 5 + 3 + 3$ which are those of the form $11 = n_1 + n_2 + \cdots + n_k$ where $n_i \geq n_{i+2} + 2$ and $n_k \geq 2$.

Example. When $d = 3$ and $t = 2$, the theorem says that the number of partitions of n into parts not congruent to $0, 2, 5 \mod 7$ is equal to the number of partitions of $n = n_1 + n_2 + \cdots + n_k$ where $n_i \geq n_{i+2} + 2$ and $n_{k-1} \geq 2$. Thus the partitions of 6 with parts $\equiv 1, 3, 4, 6 \mod 7$ are: $6 = 4 + 1 = 3 + 3 = 3 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1$ which are five in number, as are the partitions $6 = 5 + 1 = 4 + 2 = 3 + 3 = 3 + 2 + 1$ which are those of the form $9 = n_1 + n_2 + \cdots + n_k$ where $n_i \geq n_{i+2} + 2$ and $n_{k-1} \geq 2$.

Example. When $d = 3$ and $t = 3$, the theorem says that the number of partitions of n into parts not congruent to $0, 3, 4 \mod 7$ is equal to the number of partitions of $n = n_1 + n_2 + \cdots + n_k$ where $n_i \geq n_{i+2} + 2$ and $n_{k-2} \geq 2$. Thus the partitions of 6 with parts $\equiv 1, 2, 5, 6 \mod 7$ are: $6 = 5 + 1 = 2 + 2 + 2 = 2 + 2 + 1 + 1 = 2 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1$ which are six in number, as are the partitions $6 = 5 + 1 = 4 + 2 = 4 + 1 + 1 = 3 + 3 = 3 + 2 + 1$ which are those of the form $9 = n_1 + n_2 + \cdots + n_k$ where $n_i \geq n_{i+2} + 2$ and $n_{k-2} \geq 2$.

We will now prove the theorem.

Proof of Theorem. In Theorem 2.2, let us put $q = x^{d+1/2}$ and $e^{2iz} = -x^{d-t+1/2}$ so that we get

$$\prod_{n=1}^{\infty}(1-x^{(2d+1)n})(1-x^{(2d+1)n-t})(1-x^{(2d+1)n-2n-1+t}) = \sum_{m=-\infty}^{\infty}(-1)^m x^{(d+1/2)m^2+(d-t+1/2)m}.$$

But now

$$\prod_{n=1}^{\infty} \frac{(1-x^{(2d+1)n})(1-x^{(2d+1)n-t})(1-x^{(2d+1)n-2n-1+t})}{1-x^n}$$

is the generating function for the number of partitions of N into parts not congruent to $0, \pm t$ mod $2d-1$. It follows that, if we denote by $\phi_{dt}(x)$ the generating function for the number of partitions of N of the form $N = N_1 + \cdots + N_k$ where $N_i \geq N_{i+1}, N_i \geq N_{i+d-1} + 2$ and $N_k - t + 1 \geq 2$, then we must prove that

$$\phi_{dt}(x) \prod_{n=1}^{\infty} (1-x^n) = \sum_{m=-\infty}^{\infty} (-1)^m x^{(d+1/2)m^2 + (d-t+1/2)m}. \tag{4.1}$$

Now if we write

$$\phi_{dt}(x) = \sum_{n=0}^{\infty} F_{dt}(n) x^n,$$

so that $F_{dt}(n)$, for $n \geq 1$, is the number of partitions of N of the form $N = N_1 + \cdots + N_k$ where $N_i \geq N_{i+1}, N_i \geq N_{i+d-1} + 2$ and $N_k - t + 1 \geq 2$, and $F_{dt}(0) = 1$, then if we put

$$\phi_{dt}(x) \prod_{n=1}^{\infty} (1-x^n) = \sum_{n=0}^{\infty} c_{dt}(n) x^n,$$

we have that $c_{dt}(n) = \sum_{(a_1 \ldots a_k | b_1 \ldots b_l)} (-1)^k$ where the sum is over all partitions denoted by $(a_1 \ldots a_k | b_1 \ldots b_l)$ which is a partition of n such that

$$n = a_1 + \cdots + a_k + b_1 + \cdots + b_l$$

where

$$a_i \geq a_{i+1} + 1, \quad b_j \geq b_{j+1}, \quad b_j \geq b_{j+d-1} + 1, \quad b_{l-t+1} \geq 2.$$

We must show that there is a bijection between such partitions with k even and those with k odd except when n has the form

$$n = (d+1/2)m^2 + (d-t+1/2)m \quad \text{for } m = 0, \pm 1, \pm 2, \ldots$$

and in the case where n has the latter form we will show that the number of partitions where $k \equiv m \mod 2$ (i.e. k has the same parity as m) exceeds the number of partitions where $k \not\equiv m \mod 2$ by exactly 1. This will then establish Equation 4.1, and hence the theorem. For the bijection the reader is now referred to [Gor61]. ■

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