Pattern Avoidance in Permutations

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Abstract

A permutation $\sigma \in S_n$ *contains* a pattern $\pi \in S_k$ if some k-term subsequence of σ has the same relative order as π. If this isn't the case, then σ *avoids* π. After going through the basic definitions, we will prove all patterns of length 3 are Wilf-equivalent. And for our main result, we prove that $|\operatorname{Av}_n(123)| = C_n$ using a recursive argument, and hence all patterns π of length 3 follow $|\operatorname{Av}_n(\pi)| = C_n$.

1 Basic definitions and symmetries

A *permutation* of $[n] = \{1, 2, \ldots, n\}$ is a bijection $\sigma : [n] \to [n]$, written in one-line notation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$. Given a sequence of distinct numbers $w = (w_1, \ldots, w_k)$, its *standardization* $\operatorname{std}(w) \in S_k$ is the unique permutation with the same relative order as w.

Definition 1.1 (Pattern containment)**.** Fix $\pi \in S_k$. A permutation $\sigma \in S_n$ *contains* π if there exist indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $\operatorname{std}(\sigma_{i_1}, \ldots, \sigma_{i_k}) = \pi$. Otherwise, σ *avoids* π. We write $\operatorname{Av}_n(\pi) = \{\sigma \in S_n : \sigma \text{ avoids } \pi\}$, and $\operatorname{Av}(\pi) = \bigcup_{n \ge 0} \operatorname{Av}_n(\pi)$.

Definition 1.2 (Wilf equivalence)**.** Let π and σ be permutations (patterns). For each $n \ge 0$, let

$$\operatorname{Av}_n(\pi) = \{\tau \in S_n : \tau \text{ avoids the pattern } \pi\},$$

We say that π and σ are *Wilf equivalent* if

$$|\operatorname{Av}_n(\pi)| = |\operatorname{Av}_n(\sigma)| \quad \text{for all } n \ge 0.$$

Essentially, π and σ are Wilf equivalent if, for every n, the number of permutations of length n avoiding π and σ are the same.

Example 1.3. The permutation $4\,2\,5\,1\,3$ contains the pattern 213 via the subsequence $4, 2, 5$ (relative order $2, 1, 3$). It avoids 123 since there is no strictly increasing subsequence of length 3 read left-to-right.

We use three standard involutive symmetries on S_n:

- *Reverse* $r(\sigma)_i = \sigma_{n+1-i}$ (read right-to-left);
- *Complement* $c(\sigma)$ replaces each value x by $n + 1 - x$;
- *Inverse* $i(\sigma) = \sigma^{-1}$ (swap positions and values).

Each symmetry respects containment: σ avoids π iff $f(\sigma)$ avoids $f(\pi)$ for $f \in \{r, c, i\}$ and their compositions.

Lemma 1.4. *For any pattern π and f in the group generated by r, c, i, $|\operatorname{Av}_n(\pi)| = |\operatorname{Av}_n(f(\pi))|$ for all n.*

Remark 1.5. For length 3, the symmetries split S_3 into two orbits: $\{123, 321\}$ and $\{132, 213, 231, 312\}$. It is a classical theorem of Simion–Schmidt that all six length-3 patterns are nevertheless Wilf-equivalent (equinumerous avoidance classes). We prove this in the next section.

2 The Simion–Schmidt bijection

Theorem 2.1. *For every $n \geq 1$ there is a bijection*

$$\Phi : \mathrm{Av}_n(123) \longrightarrow \mathrm{Av}_n(132),$$

Proof. For $\sigma = a_1 a_2 \dots a_n \in S_n$, a position i is a *left-to-right minimum* (LR-minimum) if $a_i < a_j$ for all $j < i$. Let the LR-minima of σ be at positions

$$1 = i_1 < i_2 < \cdots < i_k, \qquad m_j := a_{i_j}.$$

Then $m_1 > m_2 > \cdots > m_k$, and we can write

$$\sigma = m_1\, w_1\, m_2\, w_2\, \dots\, m_k\, w_k,$$

where each w_j is the (possibly empty) block of entries strictly between m_j and m_{j+1} (or to the end).

Step 1: Structure of 123-avoiders.

Lemma 2.2. *If $\sigma \in \mathrm{Av}_n(123)$, then each block w_j is strictly decreasing from left to right.*

Proof. Fix j and consider positions $i_j + 1, \dots, i_{j+1} - 1$. No element smaller than m_j can appear there (otherwise m_j would not be an LR-minimum), so every element of w_j is $> m_j$. If w_j contained $p < q$ with $a_p < a_q$, then

$$a_{i_j} = m_j < a_p < a_q, \qquad i_j < p < q,$$

would be an occurrence of 123, contradicting $\sigma \in \mathrm{Av}_n(123)$. Thus w_j is strictly decreasing. □

Step 2: Our definition of Φ.

Let $\sigma = a_1 a_2 \dots a_n \in \mathrm{Av}_n(123)$. We construct $\tau = b_1 b_2 \dots b_n = \Phi(\sigma)$ as follows.

Maintain:

$$x = \text{current LR-minimum among } a_1, \dots, a_i, \qquad U = \{\text{values already used among } b_1, \dots, b_i\}.$$

Initialize:

$$b_1 := a_1, \quad x := a_1, \quad U := \{a_1\}.$$

For $i = 2, 3, \dots, n$:

- If $a_i < x$ (a new LR-minimum of σ), set

$$b_i = a_i, \quad x = a_i, \quad U = U \cup \{a_i\}.$$

- If $a_i > x$ (not an LR-minimum), set

$$b_i = \min\{t : t > x,\ t \notin U\}, \quad U = U \cup \{b_i\}.$$

Lemma 2.3. *The sequence τ is a permutation of $\{1, \dots, n\}$ and has the same LR-minima (positions and values) as σ.*

Proof. By construction each $b_i \in \{1, \dots, n\}$ and we never reuse a value, so τ is a permutation. We show by induction on i that after step i, the LR-minima among positions $1, \dots, i$ coincide for σ and τ, with the same values.

For $i = 1$ this is clear. Suppose it holds up to i.

If $a_{i+1} < x$, then a_{i+1} is the new minimum among $a_1, \dots, a_{i+1}$, so $i + 1$ is an LR-minimum of σ; we set $b_{i+1} = a_{i+1}$ and update $x := a_{i+1}$, so $i + 1$ is an LR-minimum of τ with the same value.

If $a_{i+1} > x$, then a_{i+1} is not an LR-minimum of σ, and we choose $b_{i+1} > x$, so $i + 1$ is not an LR-minimum of τ. Thus LR-minima positions and values match at every step. □

Hence, if

$$\sigma = m_1\, w_1\, m_2\, w_2\, \dots\, m_k\, w_k,$$

we can also write

$$\tau = m_1\, w_1'\, m_2\, w_2'\, \dots\, m_k\, w_k'$$

for some blocks w_j'.

Lemma 2.4. *For $\tau = \Phi(\sigma)$, each block w_j' is strictly increasing from left to right.*

Proof. Fix j. For positions in w_j' (between i_j and i_{j+1}), the current LR-minimum x equals m_j and never changes. Each time we are in this block we apply the second case and set

$$b_i = \min\{t : t > m_j,\ t \notin U\}.$$

Thus the values assigned in w_j' are taken in strictly increasing order, so w_j' is strictly increasing. □

Step 3: Characterization of 123- and 132-avoidance by blocks.

Lemma 2.5. *Let $\pi = m_1 v_1 m_2 v_2 \dots m_k v_k$ be a permutation written by LR-minima and blocks.*

1. *$\pi \in \mathrm{Av}_n(123)$ if and only if each v_j is strictly decreasing.*
2. *$\pi \in \mathrm{Av}_n(132)$ if and only if each v_j is strictly increasing.*

Proof. (1) The "only if" direction is Lemma 2.2. Conversely, if each v_j is decreasing, any triple entirely inside some v_j has pattern 321, 231, or 213, never 123. If a triple uses m_j and two elements of v_j, those two are decreasing, so again cannot form 123 with m_j as the smallest. If a triple uses entries from different blocks, the LR-minima strictly decrease ($m_1 > m_2 > \dots$), so the smallest element comes from the *latest* block, and the relative order of the other two cannot give 123 with indices $i < j < k$. Hence it follows that no 123-pattern appears.

(2) The argument is analogous. If each v_j is increasing, no three indices inside one block give 132 (they give 123 or 231). If a triple includes m_j and two elements of v_j, then m_j is the smallest, and the other two are increasing, so again one does not get 132. If indices lie in different blocks, the smallest value must come from the *latest* LR-minimum among them, and a case check shows this cannot yield $b_i < b_k < b_j$ with $i < j < k$. Conversely, if some v_j is not increasing, there exist $p < q$ in that block with $\pi_p > \pi_q$ and $\pi_{i_j} < \pi_q < \pi_p$, giving a 132-pattern $(\pi_{i_j}, \pi_p, \pi_q)$. □

Step 4: Bijectivity.

For $\sigma \in \mathrm{Av}_n(123)$ write

$$\sigma = m_1 w_1 m_2 w_2 \dots m_k w_k$$

by LR-minima and blocks; by Lemma 2.2 each w_j is strictly decreasing. Thus $\Phi(\sigma)$ has the same LR-minima and each block strictly increasing, so by Lemma 2.5(2) we have $\Phi(\sigma) \in \mathrm{Av}_n(132)$, i.e. $\Phi : \mathrm{Av}_n(123) \to \mathrm{Av}_n(132)$ is well-defined. Conversely, for $\tau \in \mathrm{Av}_n(132)$ with decomposition $\tau = m_1 v_1 m_2 v_2 \dots m_k v_k$ (each v_j increasing), define $\Psi(\tau)$ by keeping the same LR-minima and rewriting each block v_j in decreasing order; then by Lemma 2.2 we get $\Psi(\tau) \in \mathrm{Av}_n(123)$. We It is immediate that $\Psi(\Phi(\sigma)) = \sigma$ and $\Phi(\Psi(\tau)) = \tau$ for all σ and τ, so $\Psi = \Phi^{-1}$ and Φ is a bijection between $\mathrm{Av}_n(123)$ and $\mathrm{Av}_n(132)$.

□

3 Catalan Enumeration

Theorem 3.1. *For $n \geq 0$, the numbers $A_n = |\operatorname{Av}_n(132)|$ satisfy*

$$A_0 = 1$$

and, for every $n \geq 1$,

$$A_n = \sum_{k=1}^{n} A_{k-1} A_{n-k}. \tag{1}$$

Consequently, A_n is the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. We prove this by decomposing a 132-avoiding permutation according to the position of its largest element n, and inducting.

Step 1: The base case

For $n = 0$, there is exactly one permutation of the empty set, which vacuously avoids every pattern. Thus $A_0 = 1$.

Step 2: Decomposition by the position of n

Fix $n \geq 1$ and let $\pi \in \operatorname{Av}_n(132)$. Let k be the position of the largest element n in π. We write π in the form

$$\pi = L\, n\, R,$$

where

- L is the block of the first $k-1$ entries (to the left of n),
- R is the block of the last $n-k$ entries (to the right of n).

Step 3: Key lemma

Lemma 3.2. *Let $\pi \in \operatorname{Av}_n(132)$, and let $\pi = L\,n\,R$ as above. Then every entry in L is larger than every entry in R.*

Proof. Suppose, for contradiction, that there exist $x \in L$ and $y \in R$ with $x < y$.

Let $x = \pi_i$ and $y = \pi_j$, and let n be at position k. By construction we have $i < k < j$, so the positions of (x, n, y) are ordered as $i < k < j$. The values satisfy $x < y < n$ since n is the largest element. Therefore, the triple (x, n, y) has relative order $(1, 3, 2)$ giving us a contradiction.

Hence no such $x < y$ can exist, and every entry in L is larger than every entry in R. □

Step 4: Counting by the position of n

Fix $n \geq 1$ and fix a position $k \in \{1, \ldots, n\}$ for the entry n. By having every entry in L larger than every entry in R and n being the largest entry, the only way to introduce a 132 pattern into the permutation would for it to be completely contained in L or R. Hence:

- The block L is (after relabeling) a 132-avoiding permutation on $[k-1]$, so there are A_{k-1} possibilities for L.
- The block R is (after relabeling) a 132-avoiding permutation on $[n-k]$, so there are A_{n-k} possibilities for R.

Thus the number of 132-avoiding permutations $\pi \in S_n$ in which n occurs at position k is

$$A_{k-1}A_{n-k}.$$

Summing over all possible positions:

$$A_n = \sum_{k=1}^{n} A_{k-1}A_{n-k},$$

which is exactly the recurrence (1).

Together with the initial condition $A_0 = 1$, this recurrence uniquely characterizes the Catalan numbers, so $A_n = C_n$ for all $n \geq 0$. □

4 Conclusion

By constructing the Simion—Schmidt bijection, we showed that 123 and 132 are Wilf equivalent, and combining this with Lemma 1.4 it follows that all patterns of length 3 are Wilf equivalent. Hence, by Theorem 2.1 it follows that all patterns π of length 3, $|\operatorname{Av}_n(\pi)| = C_n$.

Although patterns of length 3 are well studied, there are many interesting problems to solve in the future about patterns of higher length. For example, it isn't true that all patterns of length 4 are Wilf equivalent, and we don't know the number of permutations avoiding higher length patterns in closed forms.

One natural direction is to study pattern avoidance for longer increasing patterns using the hook–length formula for standard Young tableaux. Since

$$|\operatorname{Av}_n(12\cdots k)| = \sum_{\lambda \vdash n,\ \lambda_1 \leq k-1} (f^\lambda)^2,$$

where f^λ is given by the hook–length formula, improving our understanding of these hook products could lead to sharper asymptotics or simpler expressions in special cases like the catalan numbers.

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