ANALYTIC COMBINATORICS

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1. Introduction

Generating functions provide one of the most powerful tools in enumerative combinatorics. A central theme of analytic combinatorics is that the asymptotic behavior of a sequence is governed not by its individual coefficients, but by the singularities of its generating function. The first part of this paper develops this point of view through *singularity analysis*. When a generating function has an isolated algebraic or logarithmic singularity, one can transfer the local expansion near that point directly into an asymptotic estimate for the coefficients. This explains, for example, why functions such as $G(z) = (1-z)^{-\alpha}$ have coefficients that grow like $n^{\alpha-1}$, and more generally why the qualitative nature of a singularity determines the polynomial factor in the asymptotic growth. Entire functions require a different approach. In these situations, we use the *saddle-point method*. The integral is turned into the integral of an exponential and then a local quadratic approximation then transforms the coefficient integral into a Gaussian integral. The majority of the exposition follows [FS09].

2. Preliminaries

We begin by recalling two basic tools from complex analysis that allow us to extract coefficients of analytic functions using contour integrals. Throughout, a function is called analytic on a domain $\Omega \subset \mathbb{C}$ if it admits a power series expansion that converges to the function on every compact subset of Ω.

Theorem 2.1 (Cauchy Residue Theorem)**.** *Let Γ be a closed contour and let $f(z)$ be analytic on and inside Γ, except for finitely many isolated singularities at $z_1, \ldots, z_m$. Then*

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \sum_{k=1}^{m} \operatorname{Res}(f)[z = z_k].$$

For analytic combinatorics, we apply this theorem to the integrand $f(z)/z^{n+1}$ and isolate the pole at $z = 0$, which leads to the coefficient extraction formula.

Corollary 2.1.1 (Coefficient formula)**.** Let f be analytic on a domain $\Omega \subset \mathbb{C}$ and assume $0 \in \Omega$. For any closed contour $\Gamma \subset \Omega$ that winds once counterclockwise around 0,

$$[z^n]\, f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z^{n+1}}\,dz.$$

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Let $f(z) = \sum_{n\geq 0} a_n z^n$ be analytic at 0 with radius of convergence $R > 0$. The circle $|z| = R$ is called the *circle of convergence*. A point z_0 with $|z_0| = R$ is a *singularity* of f if f fails to be analytic in every neighbourhood of z_0. All singularities of f lie on or outside the circle $|z| = R$, and none lie strictly inside.

Among all singularities of modulus R, those with minimal modulus determine the exponential growth of the coefficients. If f has exactly one such singularity of smallest modulus, say at $z = \rho$, we call ρ the *dominant singularity*. In this case the Cauchy integral representation shows that the contribution to $[z^n]f(z)$ obtained from a small neighbourhood of $z = \rho$ is of order ρ^{-n}, while the contribution from the remainder of the contour is exponentially smaller.

Thus,

(1) the *location* of the dominant singularity determines the exponential growth rate ρ^{-n}
(2) the *nature* of the singularity (for example algebraic or logarithmic behaviour) determines the subexponential factor, which is typically a power of n.

If f is analytic at $z = \rho$, then it admits a convergent Taylor series expansion in a neighbourhood of ρ, and therefore contributes only an exponentially small term to the coefficients $[z^n]f(z)$. In contrast, a singularity at $z = \rho$ produces polynomial factors such as $n^{\alpha-1}$ or $n^{\alpha-1}(\log n)^\beta$ in the asymptotic form of $[z^n]f(z)$. Consequently, when determining asymptotics one may ignore all parts of the generating function that are analytic at the dominant singularity and focus solely on the singular part of its local expansion.

Remark 2.1. Many generating functions arising in combinatorics have nonnegative coefficients. Pringsheim's Theorem states that in this situation the radius of convergence R is itself a singularity of f, so the dominant singularity lies on the positive real axis. We do not rely on this result in any proof, but it explains why the singularities in our examples always occur at positive real points.

The goal of the next section is to describe the singular behaviour of basic model functions such as $(1-z)^{-\alpha}$ and $(1-z)^{-\alpha}(\log \frac{1}{1-z})^\beta$, and to use these local expansions to obtain asymptotics for their coefficients via the coefficient formula.

3. Singularity Analysis

Before we can understand how singularities determine the growth of coefficients, it is useful to examine the simplest type of singular behaviour that appears in generating functions. Many functions admit a local expansion around $z = 1$ in powers of $(1-z)$, and the dominant term in such an expansion usually has the form $(1-z)^{-\alpha}$ for some complex α that is not a negative integer. Since the coefficients of $f(z)$ depend only on its behaviour near its dominant singularity, it becomes essential to know the asymptotics of

$$[z^n](1-z)^{-\alpha}.$$

The following theorem describes these coefficients in a complete way and serves as the basic scale from which all algebraic singular behaviours are derived.

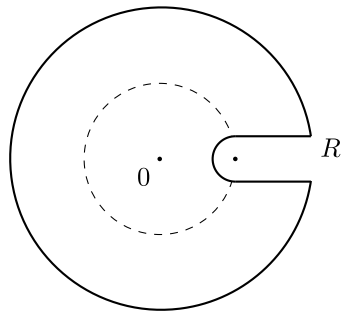

FIGURE 1. Contour $\mathcal{C}$

Theorem 3.1 (Standard function scale). *Let $\alpha \in \mathbb{C} \setminus (-\mathbb{N})$. The coefficient of z^n in $f(z) = (1-z)^{-\alpha}$ is asymptotically*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}.$$

Proof. We start with the Cauchy coefficient formula which is

$$f_n = \frac{1}{2\pi i}\int_\Gamma (1-z)^{-\alpha}\frac{\mathrm{d}z}{z^{n+1}}.$$

We want to define the contour that we take the integral on. We take a circular contour with some radius $R > 1$ with a notch so that it doesn't include $z = 1$. This is the contour $\mathcal{C}$ in Figure 1. Notice that the integral over the circular part of the contour will grow with $O(R^{-n})$. This means that we can let $R \to \infty$ and the integral of the circular part of the contour will go to 0. The remaining part is all we care about and this is called a *Hankel contour.*

We can formalize it as $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^- \cup \mathcal{H}^\circ$ where

$$\mathcal{H}^+ = \{z = \frac{i}{n} + t, \quad 1 \le t\}$$
$$\mathcal{H}^- = \{z = -\frac{i}{n} + t, \quad 1 \le t\}$$
$$\mathcal{H}^\circ = \{z = 1 - \frac{e^{it}}{n}, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

We can use the change of variables $z = 1 + \frac{t}{n}$ so $\mathrm{d}t = n\mathrm{d}z$. This gives

$$f_n = \frac{n^{\alpha-1}}{2\pi i}\int_{\mathcal{H}} (-t)^{-\alpha}\left(1+\frac{t}{n}\right)^{-n-1}\mathrm{d}t.$$

From here we have

$$\left(1+\frac{t}{n}\right)^{-n-1} = e^{-(n+1)\log(1+t/n)} = \exp\left(-(n+1)\left[\frac{t}{n} - \frac{t^2}{2n^2} + \cdots\right]\right).$$

Using the Taylor series for $\log(1+u)$,

$$-(n+1)\log(1+t/n) = -(n+1)\left[\frac{t}{n} - \frac{t^2}{2n^2} + \cdots\right] \approx -t.$$

This means that

$$f_n \approx \frac{n^{\alpha-1}}{2\pi i}\int_{\mathcal{H}} (-t)^{-\alpha} e^{-t} \mathrm{d}t.$$

We can apply Hankel's formula for the Gamma function (adjusted to go counterclockwise) [WW27] which says that

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\mathcal{H}} (-t)^{-z} e^{-t} \mathrm{d}t$$

where $\mathcal{H}$ is our same Hankel contour. Using this we get

$$f_n \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}.$$

□

In some situations the singular expansion of a generating function near $z = 1$ contains not only algebraic terms but also logarithmic factors. To handle such cases we need an extension of the algebraic scale that includes factors of the form $(\log \frac{1}{1-z})^{\beta}$. As in the purely algebraic setting, the coefficient asymptotics are determined entirely by the dominant term in the local expansion, so it suffices to understand

$$[z^n](1-z)^{-\alpha}\Big(\log \frac{1}{1-z}\Big)^{\beta}.$$

The next theorem provides the asymptotic behaviour for this family of functions and forms the standard scale for logarithmic singularities.

Theorem 3.2 (Standard function scale with logarithms)**.** *Let* $\alpha \in \mathbb{C} \setminus (-\mathbb{N})$ *and* $\beta \in \mathbb{C}$. *Then the coefficient of* z^n *in*

$$f(z) = (1-z)^{-\alpha}\left(\log \frac{1}{1-z}\right)^{\beta}$$

satisfies

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^{\beta}.$$

Proof. We start with the Cauchy coefficient formula

$$f_n = [z^n]f(z) = \frac{1}{2\pi i}\int_{\Gamma} (1-z)^{-\alpha}\left(\log \frac{1}{1-z}\right)^{\beta} \frac{dz}{z^{n+1}}.$$

The contour Γ is taken to be a circle of radius $R > 1$ with a notch at $z = 1$. The integral over the circular arc is $O(R^{-n})$, so letting $R \to \infty$ leaves the same Hankel contour $\mathcal{H}$, and hence

$$f_n \sim \frac{1}{2\pi i}\int_{\mathcal{H}} (1-z)^{-\alpha}\left(\log \frac{1}{1-z}\right)^{\beta} \frac{dz}{z^{n+1}}.$$

We now perform the same change of variables $z = 1 + \frac{t}{n}$ so $dz = n^{-1}dt$ and $(1-z) = -t/n$. We also have $(1+t/n)^{-n-1} = e^{-t}(1+O(n^{-1}))$ as $n \to \infty$. Furthermore

$$\log \frac{1}{1-z} = \log \frac{n}{t} = \log n - \log t.$$

Substituting into the integral gives

$$f_n = \frac{n^{\alpha-1}}{2\pi i} \int_{\mathcal{H}} (-t)^{-\alpha} e^{-t} \left(\log n - \log t\right)^{\beta} \left(1 + O(n^{-1})\right) dt.$$

We factor out the dominant term $\log n$:

$$(\log n - \log t)^{\beta} = (\log n)^{\beta} \left(1 - \frac{\log t}{\log n}\right)^{\beta} = (\log n)^{\beta} \left[1 - \frac{\beta \log t}{\log n} + O\!\left(\frac{1}{\log^2 n}\right)\right].$$

Plugging this into the integral yields

$$f_n = n^{\alpha-1} (\log n)^{\beta} \frac{1}{2\pi i} \int_{\mathcal{H}} (-t)^{-\alpha} e^{-t} \left[1 - \frac{\beta \log t}{\log n} + O\!\left(\frac{1}{\log^2 n}\right)\right] dt.$$

The leading term is

$$n^{\alpha-1} (\log n)^{\beta} \frac{1}{2\pi i} \int_{\mathcal{H}} (-t)^{-\alpha} e^{-t} dt = \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^{\beta}$$

by Hankel's formula for the Gamma function. This gives

$$[z^n] f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^{\beta}.$$

□

We have theorems called *transfer theorems* which will be stated without proof. These allow for bounding of later terms.

Theorem 3.3 (Transfer, Big-O and little-o)**.** *Let $\alpha, \beta \in \mathbb{R}$ and let $f(z)$ be analytic in a neighbourhood of the closed unit disc except for a singularity at $z = 1$.*

(i) Suppose that, as $z \to 1$,

$$f(z) = O\!\left((1-z)^{-\alpha} \left(\log \tfrac{1}{1-z}\right)^{\beta}\right).$$

Then

$$[z^n] f(z) = O\!\left(n^{\alpha-1} (\log n)^{\beta}\right).$$

(ii) Suppose that, as $z \to 1$,

$$f(z) = o\!\left((1-z)^{-\alpha} \left(\log \tfrac{1}{1-z}\right)^{\beta}\right).$$

Then

$$[z^n] f(z) = o\!\left(n^{\alpha-1} (\log n)^{\beta}\right).$$

Let's apply the method of singular analysis to an example like the *Catalan numbers*. In all examples considered here, the generating function may be written as the sum of an analytic part and a singular part. The analytic part is analytic in a disc of radius greater than 1, so its coefficients decay exponentially fast.

In contrast, algebraic or algebraic-logarithmic singularities at $z = 1$ produce coefficients that decay only polynomially. Since polynomial terms dominate exponentially small ones, the analytic portion contributes only negligible (exponentially small) terms to the coefficients, and the asymptotic behaviour of $[z^n]f(z)$ is completely determined by the singular part.

Example 3.1. The Catalan numbers have generating function $C(z) = \frac{1-\sqrt{1-4z}}{2z}$. This has a dominant singularity at $z = \frac{1}{4}$. To make this have a singularity at 1 let's create a new function $C_1(z) = C(z/4)$. We can see that $[z^n]C_1(z) = [z^n]C(z/4) = 4^{-n}[z^n]C(z)$. We have $C_1(z) = \frac{2(1-\sqrt{1-z})}{z} = \frac{2(1-\sqrt{1-z})}{1-(1-z)}$. This becomes

$$\begin{aligned} C_1(z) = \frac{2(1-\sqrt{1-z})}{1-(1-z)} &= 2(1-\sqrt{1-z})(1+(1-z)+(1-z)^2+\cdots) \\ &= \frac{2}{z} - 2\sqrt{1-z} - O((1-z)^{3/2}) \end{aligned}$$

As stated above we only care about the singular part which is $-2\sqrt{1-z} = -2(1-z)^{1/2}$. We know that it must grow like

$$[z^n]C_1(z) \sim (-2)\frac{n^{-3/2}}{\Gamma(-1/2)} = \frac{-2}{n^{3/2}(-2\sqrt{\pi})} = \frac{1}{n^{3/2}\sqrt{\pi}}.$$

This means that the Catalan numbers grow like

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

Example 3.2. Let's do another example, this time with an exponential generating function. This one will be the asymptotic for the number of permutations of a set with cycles of odd length. This is just

$$F(z) = \exp\left(\sum_{k \text{ odd}} \frac{z^k}{k}\right) = \exp\left(\sum_{m=0}^{\infty} \frac{z^{2m+1}}{2m+1}\right)$$

and this becomes

$$F(z) = \exp\left(\frac{1}{2}\log\frac{1+z}{1-z}\right) = \sqrt{\frac{1+z}{1-z}}.$$

We can see that there are 2 singularities, one at $z = 1$ and one at $z = -1$. Near $z = 1$ we have that

$$F(z) = (2-(1-z))^{1/2}(1-z)^{-1/2} = \sqrt{2}(1-z)^{-1/2}(1-O(1-z)).$$

This means that this will give a contribution proportionate to $n^{-3/2}$. Next to $z = -1$ we have

$$F(z) = (2-(1+z))^{-1/2}(1+z)^{1/2} = \sqrt{2}(1+z)^{1/2}(1+O(1+z)).$$

This gives a contribution proportionate to $n^{-1/2}$. This dominates the other one so we can only focus on this one. This will give us

$$[z^n]F(z) \sim \frac{\sqrt{2}n^{-1/2}}{\Gamma(1/2)} = \sqrt{\frac{2}{n\pi}}.$$

This means that since it is an EGF the number of permutations with cycles of odd length grows like

$$\sim n!\sqrt{\frac{2}{n\pi}}.$$

4. Saddle-Point Methods

In some cases, these generating functions won't have singularities and they will be analytic over all of $\mathbb{C}$. One example of this is the EGF for the Bell numbers, e^{e^z-1}, which is *entire*. Since this is the case, we are not able to use singularity analysis to find the asymptotic values of the Bell numbers. For this we have to turn to *saddle-point methods*.

To define saddle-points for complex valued functions we have to look at *modulus surfaces*, which are the three dimensional surfaces defined by $(x, y, |f(z)|)$ where $z = x + iy$. On these surfaces we have four kind of points:

(1) Ordinary point, where $f(z_0) \neq 0$ and $f'(z_0) \neq 0$. Locally, $f(z) \sim c$.
(2) Zereos of order p, where $f^{(k)}(z_0) = 0$ for $0 \leq k \leq p-1$ and $f^{(p)}(z_0) \neq 0$. Locally $f(z) \sim c(z - z_0)^p$.
(3) Pole of order p where locally $f(z) \sim c(z - z_0)^{-p}$.
(4) Saddle point, where $f(z_0) \neq 0$ but $f'(z_0) = 0$. Locally $f(z) \sim b + c(z - z_0)^2$.

Saddle points on the modulus surface resemble a saddle, having the property that the function follows a downward facing parabola in one direction (local maximum) and an upward facing parabola in the direction perpendicular to the first one (local minimum). We can utilize these points to create a bound and eventually an asymptotic for our coefficient integral.

Like with the singularities, since the combinatorial generating functions have nonnegative coefficients, the saddle point ζ will lie on the real line. Recall the coefficient formula

$$[z^n]G(z) = \frac{1}{2\pi i}\oint_C G(z)\frac{\mathrm{d}z}{z^{n+1}}.$$

We can take the contour to be the circle centered at 0 with the radius ζ. Notice that the modulus of the function $F(z) = \frac{G(z)}{z^{n+1}}$ is greatest at $z = \zeta$ or equivalently $\theta = 0$ where $z = \zeta e^{i\theta}$. We can see this with

$$\left|\frac{G(z)}{z^{n+1}}\right| = \zeta^{-(n+1)}\left|\sum_{k\geq 0} a_k \zeta^n e^{ik\theta}\right| \leq \zeta^{-(n+1)}\sum_{k\geq 0} a_k\zeta^n = F(\zeta).$$

Now we aim to solve for ζ. This is just when the derivative is 0 which means that

$$\frac{G'(z)z^{n+1} - G(z)(n+1)z^n}{(z^{n+1})^2} = 0 \implies \frac{zG'(z)}{G(z)} = n+1.$$

This is called the *saddle-point equation*. These together give us the saddle point bound.

Theorem 4.1 (Saddle-point bound)**.** *Let $G(z)$ be analytic at the origin with finite radius of convergence R. If G has nonnegative coefficients then*

$$[z^n]G(z) \leq \frac{G(\zeta)}{\zeta^n}$$

where ζ is a saddle point $z^{-(n+1)}G(z)$ being the unique real number that satisfies

$$\frac{\zeta G'(\zeta)}{G(\zeta)} = n+1.$$

Proof. Since the maximum modulus of the integrand is $G(\zeta)/\zeta^{n+1}$ and the perimeter of the circle is $2\pi\zeta$ we get

$$\left|\frac{1}{2\pi i}\oint_C G(z)\frac{\mathrm{d}z}{z^{n+1}}\right| \leq \frac{1}{2\pi}\cdot 2\pi\zeta\cdot\frac{G(\zeta)}{\zeta^{n+1}} = \frac{G(\zeta)}{\zeta^n}.$$

□

Example 4.1. For the function $G(z) = e^z$ where $[z^n]G(z) = 1/n!$, we can look at the nth coefficient. The saddle-point equation is $(\zeta e^z)/e^z = n+1$ so $\zeta = n+1$. This means that

$$\frac{1}{n!} \leq \frac{e^{n+1}}{(n+1)^n}.$$

As $n \to \infty$ this becomes

$$\frac{1}{n!} \leq \frac{e^{n+1}}{n^n(1+1/n)^n} \approx (e/n)^n$$

This is only off from Stirling's approximation for the factorial by a factor of $\sqrt{2\pi n}$.

To get this factor we take the process one step further looking at saddle-point asymptotics. Recall that near a saddle point ζ, we can write the second order Taylor expansion which is $f(z) \approx f(\zeta) + \frac{1}{2}f''(\zeta)(z-\zeta)^2$. We calculuate our coefficent integral using something called Laplace's method, which turns the integral into a simple Gaussian. Say we have $F(z) = e^{f(z)}$. The saddle point ζ of $f(z)$ is when $f'(z) = 0$. This is also satisfied when $F'(z) = 0$ as $F'(z) = f'(z)e^{f(z)}$. Near ζ we can write

$$F(z) = e^{f(z)} \approx e^{f(\zeta)+\frac{1}{2}f''(\zeta)(z-\zeta)^2}.$$

Taking the integral over the path of steepest descent, we have

$$\int_C F(z)\mathrm{d}z = e^{f(\zeta)}\int_{C_{\text{local}}} e^{\frac{1}{2}f''(\zeta)(z-\zeta)^2}\mathrm{d}z + \int_{C_{\text{far}}} F(z)\mathrm{d}z.$$

Far away from the saddle point, the real part of the exponent decreases strictly, so the integrand becomes exponentially small and contributes nothing to the asymptotic. Say that $f''(\zeta) = |f''(\zeta)|e^{-i\phi}$. Set $w = (z-\zeta)$. Since this is perpendicular to z we have $\mathrm{d}w = i\mathrm{d}z$. We can perform the change of variables $w = e^{-i\phi/2}x \implies \mathrm{d}w = e^{-i\phi/2}\mathrm{d}x$ so $\mathrm{d}z = ie^{-i\phi/2}\mathrm{d}x$. This shifts our integral to one over the real line. Since the tail of the Gaussian is so small we can just extend the bounds of our integral to ∞. This turns the integral into

$$\begin{aligned} &e^{f(\zeta)}ie^{-i\phi/2}\int_{-\infty}^{\infty} e^{-|f''(\zeta)|e^{i\phi}x^2e^{-i\phi}}\mathrm{d}x \\ &= e^{f(\zeta)}ie^{-i\phi/2}\int_{-\infty}^{\infty} e^{-|f''(\zeta)|x^2}\mathrm{d}x \\ &= e^{f(\zeta)}\cdot ie^{-i\phi/2}\sqrt{\frac{2\pi}{|f''(\zeta)|}}. \end{aligned}$$

Adding a familiar factor of $1/2\pi i$ we have

$$\frac{1}{2\pi i}\int_C F(z)\mathrm{d}z \approx \frac{1}{2\pi i}\cdot e^{f(\zeta)}\cdot ie^{-i\phi/2}\sqrt{\frac{2\pi}{|f''(\zeta)|}} = e^{-i\phi/2}\frac{e^{f(\zeta)}}{\sqrt{2\pi|f''(\zeta)|}}.$$

Since $\frac{1}{\sqrt{|f''(\zeta)|}} = \frac{1}{\sqrt{f''(\zeta)e^{-i\phi}}} = \frac{e^{i\phi/2}}{\sqrt{f''(\zeta)}}$ we just get

$$\frac{1}{2\pi i}\int_C F(z)\mathrm{d}z \approx \frac{e^{f(\zeta)}}{\sqrt{2\pi f''(\zeta)}}.$$

We can apply this to the coefficient integral where $F(z) = \frac{G(z)}{z^{n+1}}$ which gives us the final result.

Theorem 4.2 (Saddle-point transfer theorem)**.** *Let $G(z)$ be analytic at the origin with finite radius of convergence R. If G has nonnegative coefficients then*

$$[z^n]G(z) \sim \frac{e^{f(\zeta)}}{\sqrt{2\pi f''(\zeta)}}$$

where $f(z) = \log G(z) - (n+1)\log z$ and ζ is the positive real root of the saddle point equation $f'(z) = 0$ or $G'(z)/G(z) = (n+1)/z$.

Example 4.2 (Factorial)**.** We can return to our simple example of $G(z) = e^z$. We know that the saddle point is at $\zeta = n+1$. We have $f(z) = z - (n+1)\log z$. We also have $f''(z) = (n+1)/z^2$. This gives

$$[z^n]G(z) = \frac{e^{(n+1)-(n+1)\log(n+1)}}{\sqrt{2\pi(n+1)/(n+1)^2}} = \frac{e^{n+1}}{(n+1)^n\sqrt{2\pi(n+1)}}.$$

Writing $(n+1)^n$ as $(1+1/n)^n \cdot n^n$ makes it so that this becomes $e\cdot n^n$ as $n\to\infty$. Also $n+1$ becomes n as $n\to\infty$ so we get

$$\frac{1}{n!} \sim \left(\frac{e}{n}\right)^n \frac{1}{\sqrt{2\pi n}}$$

which matches Stirling's approximation for the factorial.

The motivating example was that of the Bell numbers so let us find the asymptotic for that.

Example 4.3 (Bell numbers)**.** We have that the EGF for the Bell numbers is $G(z) = e^{e^z-1}$. We get that $f(z) = e^z - 1 - (n+1)\log z$. For the saddle point equation we have

$$\frac{e^z e^{e^z-1}}{e^{e^z-1}} = \frac{n+1}{z} \implies e^z = \frac{n+1}{z} \implies \zeta = W(n+1).$$

We also get that $f''(z) = e^z + \frac{n+1}{z^2}$. This gives $f''(\zeta) = (n+1)(\frac{1}{\zeta}+\frac{1}{\zeta^2}) = (n+1)(\zeta+1)/\zeta^2 = e^\zeta(\zeta+1)/\zeta$. This gives us

$$[z^n]G(z) \sim \frac{e^{e^\zeta-1}}{\zeta^{n+1}\sqrt{2\pi e^\zeta(\zeta+1)\zeta^{-1}}}.$$

From here we get our result, that

$$B_n \sim \frac{n!e^{e^r-1}}{r^n\sqrt{2\pi r(r+1)e^r}}, \quad \text{where } re^r = n+1.$$

References

[FS09] Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009.

[WW27] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4 ed., Cambridge University Press, 1927.