ON q-ANALOGUES, GAUSSIAN COEFFICIENTS, AND THEIR APPLICATIONS

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ABSTRACT. A q-analogue is a way to generalize an expression, theorem, or idea to any real number q, such that when the limit is taken as $q \to 1$, the original object is recovered. This paper will explore common q-analogues in enumerative combinatorics, such as the q-factorial and binomial coefficient and the q-binomial theorem. It will also use q-analogues to derive a generating function for up-right paths by the area they enclose. Finally, it will discuss an application of q-analogues to vector spaces over finite fields with q elements.

1. Motivation, q-factorials, and Gaussian coefficients

One motivation for q-analogues is the geometric series identity

$$\frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}.$$

Thus taking the limit as $q \to 1$ implies $\lim_{q\to 1} \frac{1-q^n}{1-q} = n$. Thus we may define the q-analogue of a natural number n to be

$$[n]_q = \frac{1-q^n}{1-q}.$$

Having defined this, we can now define the q-factorial:

Definition 1.1. We define the *q-factorial* of n to be

$$[n]_q! = \prod_{i=1}^{n} \frac{1-q^i}{1-q} = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}.$$

Observe that as $q \to 1$, the ith term in the product converges to i, so the entire product converges to $n!$, making it a valid q-analogue for the factorial.

Definition 1.2. The *Gaussian coefficients*, or *q-binomial coefficients*, are defined as

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \prod_{i=1}^{k} \frac{1-q^{n-i+1}}{1-q^i}.$$

Again, since the q-factorial of n converges to $n!$ as $q \to 1$, the Gaussian coefficient $\binom{n}{k}_q$ converges to the normal binomial coefficient $\binom{n}{k}$ as $q \to 1$.

Date: December 8, 2025.

2. Identities regarding Gaussian coefficients

The binomial coefficients satisfy the following identity, known as **Pascal's Identity**:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Corresponding identities exist for the Gaussian coefficients:

Theorem 2.1 (q-Pascal's Identity)**.**

$$\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q.$$

Proof. We will first prove the first equality. To do so, we write

$$\begin{aligned} q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q &= q^k\prod_{i=1}^{k}\frac{1-q^{n-i}}{1-q^i} + \prod_{i=1}^{k-1}\frac{1-q^{n-i}}{1-q^i} \\ &= \prod_{i=1}^{k-1}\frac{1-q^{n-i}}{1-q^i}\left(1 + q^k\cdot\frac{1-q^{n-k}}{1-q^k}\right) \\ &= \prod_{i=1}^{k-1}\frac{1-q^{n-i}}{1-q^i}\left(\frac{1-q^n}{1-q^k}\right) \\ &= \prod_{i=1}^{k-1}\frac{1-q^{n-i}}{1-q^i}\cdot\frac{1-q^{n-k}}{1-q^k}\cdot\frac{1-q^n}{1-q^{n-k}} \\ &= \prod_{i=1}^{k}\frac{1-q^{n-i}}{1-q^i}\cdot\frac{1-q^n}{1-q^{n-k}} \\ &= \prod_{i=1}^{k}\frac{1-q^{n-i}}{1-q^i}\cdot\prod_{i=1}^{k}\frac{1-q^{n-i+1}}{1-q^{n-i}} \\ &= \prod_{i=1}^{k}\frac{1-q^{n-i+1}}{1-q^i} = \binom{n}{k}_q. \end{aligned}$$

Here, we take advantage of the product $\prod_{i=1}^{k}\frac{1-q^{n-i+1}}{1-q^{n-i}}$ telescoping to $\frac{1-q^n}{1-q^{n-k}}$.

To prove the second equality, we rewrite it as

$$(q^k-1)\binom{n-1}{k}_q = (q^{n-k}-1)\binom{n-1}{k-1}_q,$$

or

$$\binom{n-1}{k}_q = \binom{n-1}{k-1}_q\cdot\frac{q^{n-k}-1}{q^k-1}.$$

This follows by expanding both sides using the definition of Gaussian coefficients. Hence both equalities have been shown, completing the proof. ∎

Remark 2.2. Since $\binom{n}{0}_q = \binom{n}{n}_q$ for all $n \in \mathbb{N}$, one may use this theorem to show that $\binom{n}{k}_q$ is always a polynomial in q by induction on n.

The q-Pascal's Identity can be used to show the q-Binomial Theorem, a q-analog of the following theorem:

$$(x+1)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k.$$

The corresponding identity for the Gaussian coefficients is given below:

Theorem 2.3 (q-Binomial Theorem)**.**

$$\prod_{i=0}^{n-1}(1+q^i x) = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \binom{n}{k}_q x^k.$$

Proof. We will proceed by induction on n, with the base case $n = 0$ being clear. For the inductive step, assume that

$$\prod_{i=0}^{n-1}(1+q^i x) = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \binom{n}{k}_q x^k.$$

Multiplying both sides by $1 + q^n x$ gives

$$\prod_{i=0}^{n}(1+q^i x) = \left(\sum_{k=0}^{\infty} q^{\binom{k}{2}} \binom{n}{k}_q x^k\right)(1+q^n x).$$

Expanding the right-hand side, we see that the coefficient of x^m for $m \geq 1$ is

$$q^{\binom{m}{2}} \binom{n}{m}_q + q^n q^{\binom{m-1}{2}} \binom{n}{m-1}_q = q^{\binom{m}{2}} \left(\binom{n}{m}_q + q^{n-m+1} \binom{n}{m-1}_q \right).$$

By Theorem 2.1, the parenthesized expression is $\binom{n+1}{m}_q$. Therefore the right-hand side is

$$\sum_{k=0}^{\infty} q^{\binom{k}{2}} \binom{n+1}{k}_q x^k,$$

so the induction is complete. ∎

3. Classifying the area under a path

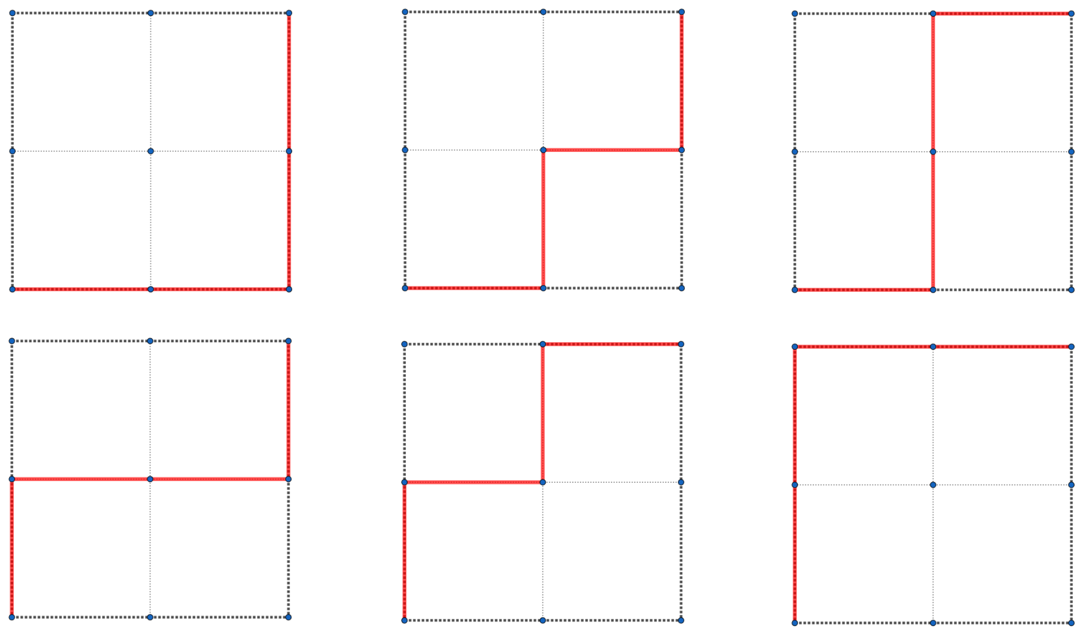
We will use the results in the previous discussion to prove the following theorem:

Theorem 3.1. *Let m and n be nonnegative integers. For each nonnegative integer k, let $F_{m,n}(k)$ be the number of up-right paths $\mathcal{P}$ from $(0,0)$ to (m,n) such that the area of the region bound by $\mathcal{P}$ and the lines $y = 0$ and $x = m$ is k. In particular, note that $F_{m,n}(k) = 0$ for $k > mn$. Then*

$$\sum_{k=0}^{\infty} F_{m,n}(k) q^k = \binom{m+n}{m}_q.$$

The case where $q = 1$ returns the formula for the number of up-right paths from $(0,0)$ to (m,n).

Example. Before we prove this remarkable theorem, let us look at a specific case, namely $m = n = 2$. All $\binom{4}{2} = 6$ up-right paths from $(0,0)$ to $(2,2)$ are shown below:



Thus $F_{2,2}(0) = F_{2,2}(1) = F_{2,2}(3) = F_{2,2}(4) = 1$ and $F_{2,2}(2) = 2$. This gives

$$\sum_{k=0}^{\infty} F_{2,2}(k)q^k = 1 + q + 2q^2 + q^3 + q^4.$$

Indeed,

$$\binom{4}{2}_q = \frac{1-q^4}{1-q}\cdot\frac{1-q^3}{1-q^2} = (1+q+q^2+q^3)\cdot\frac{q^2+q+1}{q+1} = (q^2+1)(q^2+q+1) = 1+q+2q^2+q^3+q^4.$$

Now we will prove the theorem. To do so, we write

$$\binom{m+n}{m}_k = \sum_{k=0}^{\infty} G_{m,n}(k)q^k.$$

We can do this since by Remark 2.2 $\binom{m+n}{m}_q$ is a polynomial in q. We aim to show that $F_{m,n}(k) = G_{m,n}(k)$. If we show that F and G satisfy the same recurrence and have the same base cases, then this will prove that they match for all m, n.

First, we tackle the base cases $m = 0$ or $n = 0$. In either case, we get $F_{m,0}(k) = F_{0,m}(k) = 1$ if $k = 0$ and 0 otherwise. Meanwhile, if $m = 0$, we get $\binom{n}{0}_q = 1$, and if $n = 0$, we get $\binom{m}{m}_q = 1$ again. Thus we have settled the base cases.

Now we will find a recurrence for F and G. First we deal with G. By Theorem 2.1, we know that

$$\binom{m+n}{m}_q = q^m\binom{m+n-1}{m}_q + \binom{m+n-1}{m-1}_q.$$

Expanding both, we get

$$\sum_{k=0}^{\infty} G_{m,n}(k)q^k = \sum_{k=0}^{\infty} G_{m,n-1}(k-m)q^k + \sum_{k=0}^{\infty} G_{m-1,n}(k)q^k,$$

working with the convention that $G_{m,n}(k) = 0$ if one of m, n is negative. Thus we get the recurrence

$$\boxed{G_{m,n}(k) = G_{m,n-1}(k-m) + G_{m-1,n}(k)}.$$

Hence it suffices to show that F satisfies the same recurrence, i.e.

$$F_{m,n}(k) = F_{m,n-1}(k-m) + F_{m-1,n}(k).$$

Indeed, consider an up-right path P from $(0,0)$ to (m,n) bounding an area of k. If the path visits $(1,0)$ to start with, then we get an area k path from $(1,0)$ to (m,n), giving $F_{m-1,n}(k)$ paths. Otherwise, the path visits $(0,1)$ to start with, so we get an area $k-m$ path from $(0,1)$ to (m,n), since we add m to this quantity to get the area under the path from $(0,0)$ to (m,n). This gives $F_{m,n-1}(k-m)$ paths. Summing up both cases, we conclude that F satisfies the same recurrence as G. Since they satisfy the same base cases, we are done. □

We conclude this section by describing what happens as $m, n \to \infty$, as well as an informal explanation for why this is so. Assume for our purposes that $|q| < 1$. By definition,

$$\binom{m+n}{m}_q = \frac{(1-q^{m+n})(1-q^{m+n-1})\cdots(1-q^{n+1})}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

Notice that the numerator is 1 plus q^{n+1} times a polynomial in q with nonzero constant coefficient. As $n \to \infty$, this second term approaches 0, so the numerator approaches 1. Meanwhile, the denominator is fixed. Hence

$$\lim_{n\to\infty} \binom{m+n}{m}_q = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

Taking the limit of this expression as $m \to \infty$ results in

$$\lim_{m\to\infty} \lim_{n\to\infty} \binom{m+n}{m}_r = \frac{1}{(1-q)(1-q^2)\cdots},$$

which is precisely the partition generating function. In other words, we must have that $\lim_{m\to\infty} \lim_{n\to\infty} F_{m,n}(k) = p(k)$, where $p(k)$ is the number of partitions of k. To explain this, rewrite $F_{m,n}(k)$ as the number of down-right paths from $(0,n)$ to $(m,0)$ bounding an area of k with the x and y-axis. By looking at each row, this is also the number of ways to write $k = a_1 + a_2 + \cdots + a_z$ for $1 \le z \le n$ and $1 \le a_1 \le m$, where order doesn't matter. As $m, n \to \infty$, the constraints $z \le n$ and $a_1 \le m$ are dropped, leaving behind the number of ways to write $k = a_1 + a_2 + \cdots + a_z$ for $z \ge 1$ and $a_1 \ge 1$, where order doesn't matter. By definition, this is $p(k)$.

4. Application to vector spaces and RREFs

In this section, we will compute the number of subspaces of a vector space over a finite field, and also the number of $k \times n$ matrices in reduced row echelon form (RREF) over a finite field, both using q-analogues. We solve the first problem below:

Theorem 4.1. *If V is an n-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements (so that q is a prime power), then the number of k-dimensional subspaces of V is $\binom{n}{k}_q$.*

Proof. It suffices to choose k linearly independent basis vectors from V, as those will determine our vector space. The first vector we choose has $q^n - 1$ possibilities (it can't be the zero vector). The second vector has $q^n - q$ possibilities, since it can't be a multiple of the first vector. In general the ith vector has $q^n - q^{i-1}$ possibilities since it can't be a linear combination of the first $i-1$ vectors. This gives a total of $(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})$ ways.

However, we need to divide out cases where two different sets of basis vectors determine the same subspace. Then this subspace W has dimension k, and we need to divide our count by the number of ways to select basis vectors from it. Similar work to what was shown above gives a total of $(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})$ here.

Therefore, after dividing our initial count by this factor, we end up with

$$\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1)})} = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} = \binom{n}{k}_q,$$

as desired. ■

Remark 4.2. To stay true to the nature of q-analogues to return the original object as $q \to 1$, let us look at what happens in the above theorem when $q \to 1$. Then V is an n-dimensional vector space over the field with one element, meaning that V contains just one element, namely $(0, 0, \ldots, 0)$, where there are n zeros. To form a k-dimensional subspace of V, we just need to select k of these n zeros, and the resulting point will be our entire subspace. This gives $\binom{n}{k} = \binom{n}{k}_1$ subspaces.

Now we will state our next result and prove it in two different ways. The first proof is just an application of Theorem 4.1, while the second proof demonstrates a neat bijection with up-right paths, giving a connection to Theorem 3.1.

Theorem 4.3. *The number of $k \times n$ matrices in RREF with no rows identically zero over the finite field $\mathbb{F}_q$ with q elements is $\binom{n}{k}_q$.*

We will first provide the first proof.

Proof. We establish a bijection between such matrices and k-dimensional subspaces of an n-dimensional vector space V over $\mathbb{F}_q$. For one direction, we simply take the k rows of the matrix to form the basis of our subspace. The matrix is of the form

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & - & \cdots & - & - & - & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & - & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $-$ denotes an entry with any value. Since the first 1 in each row has no other nonzero entries in its column by definition, we clearly see that these rows are linearly independent, so they indeed form a basis for a k-dimensional subspace of V.

For the other direction, suppose we have a k-dimensional space W, and suppose we have a basis $\beta = \{v_1, v_2, \ldots, v_k\}$ for W. Consider the matrix

$$\begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \end{pmatrix},$$

where v^T is the transpose of a vector. Since these vectors are linearly independent, we can apply Gaussian Elimination on this matrix to get an RREF matrix where no row is identically zero.

This establishes our bijection, so by Theorem 4.1, we are done. ■

Now we will look at the second proof.

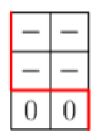
Proof. For simplicity, in an RREF matrix, we define a *free entry* as one which can be changed without breaking the RREF property. Also, we will define a *class* of RREF matrices of type i to be a set of RREF matrices such that the i free entries in each are fixed. For each i, we establish a bijection between every class of RREF matrices of type i and an up-right path from $(0, 0)$ to $(n - k, k)$ bounding an area of i. For one direction, we start with a class of RREF matrices of type i. We will represent this class as a single $k \times n$ matrix, where a $-$ is put in every free entry and the leading zeros and ones are left unchanged; note that there are i total $-$s. As an example, one may represent a class of 3×5 RREF matrices of type 4 as follows:

$$\begin{pmatrix} 1 & 0 & - & - & 0 \\ 0 & 1 & - & - & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

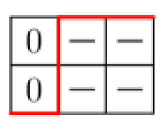
Now we will delete every column from this matrix with a leading 1. The result is a $k \times (n-k)$ matrix with only 0s and $-$s. In the matrix shown above, the resulting matrix is

$$\begin{pmatrix} - & - \\ - & - \\ 0 & 0 \end{pmatrix}.$$

Put these entries in a $k \times (n - k)$ grid in the natural way. By the form of RREF matrices, it is easy to see that there is a down-right path from the top left corner of the grid to the bottom right corner of the grid separating the zeros and the free entries. In the example shown above, the path is drawn below:



If we rotate this grid clockwise by 90°, the result is an up-right path in a $(n - k) \times k$ grid, with $-$s below and 0s above it:



Since there are i free entries, the path bounds an area of i underneath it. This establishes the first part of our bijection.

The second direction starts by undoing the steps shown above and getting a $k \times (n - k)$ matrix with only 0s and $-$s. The example from before is shown below:

$$\begin{pmatrix} - & - \\ - & - \\ 0 & 0 \end{pmatrix}$$

To turn this into an RREF matrix, for every row starting from the bottom, we add a column right before the last free entry of that row, where this column has zeros everywhere except for that row, in which it has a 1. For our example, the process is shown below:

$$\begin{pmatrix} - & - \\ - & - \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} - & - & 0 \\ - & - & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & - & - & 0 \\ 1 & - & - & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & - & - & 0 \\ 0 & 1 & - & - & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is not hard to verify that this is indeed RREF. Therefore, we have established our bijection between every class of RREF matrices of type i and an up-right path from $(0,0)$ to $(n-k,k)$ bounding an area of i.

To finish, we note that there are q^i matrices in a class of RREF matrices of type i. Therefore, summing over all classes of all types, we get that the total number of RREF matrices is

$$\sum_{\text{paths } P} q^{\text{area bounded by } P}.$$

By Theorem 3.1, this is equal to

$$\binom{(n-k)+k}{n-k}_q = \binom{n}{n-k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q!}{[k]_q![n-k]_q!} = \binom{n}{k}_q,$$

so we are done.

■

5. Acknowledgments

The author would like to thank Alex Cao and Simon Rubinstein-Salzedo for their valuable insights on this paper. The author would also like to thank Euler Circle for organizing the *Combinatorics* class and giving the opportunity to write this paper.

6. References

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