A Combinatorial and Geometric View of Rational Catalan Numbers

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1 Introduction

The Catalan are a useful sequence in combinatorics, appearing in many ways such as Dyck paths, polygon triangulations, and noncrossing partitions. A classical Catalan number is

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Fuss–Catalan numbers generalize this. For a positive integer m, one defines

$$\mathrm{Cat}^{(m)}(n) = \frac{1}{mn+1}\binom{(m+1)n}{n},$$

which count, among other things, m-divisible noncrossing partitions and $(m+2)$-angulations of convex polygons.

The rational Catalan number is

$$\mathrm{Cat}(a,b) = \frac{1}{a+b}\binom{a+b}{a,b} = \frac{1}{a+b}\binom{a+b}{a},$$

which connects to lattice path models (rational Dyck paths). Another version incorporates a q-parameter, giving a polynomial with non-negative integer coefficients:

$$\mathrm{Cat}_q(a,b) = \frac{\binom{a+b}{a}_q}{[a+b]_q}.$$

Before turning to the rational setting, we first review the classical and Fuss–Catalan numbers. After moving to rational Catalan numbers and their q-analogue, we then discuss the lattice-point interpretation.

2 Classical and Fuss–Catalan Numbers

We begin with the ordinary Catalan numbers, whose interpretations illustrate the branching and path-avoidance ideas.

2.1 Classical Catalan Numbers

Definition 2.1. The nth Catalan number is

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Three frequently used equivalent models that are enumerated by the Cataln numbers are

- Dyck paths: lattice paths from $(0,0)$ to $(2n,0)$ with steps $U=(1,1)$ and $D=(1,-1)$ that never go below the x-axis
- Plane (rooted ordered) trees: rooted trees in which the children of each vertex are linearly ordered, with n edges
- Polygon triangulations: ways to divide a convex $(n+2)$-gon into n triangles by noncrossing diagonals.

These models are connected by bijections. A preorder traversal of a rooted ordered tree produces a Dyck path (visit down-edge $\mapsto U$, return-edge $\mapsto D$), and taking the dual of a triangulation gives a rooted ordered tree. The common idea behind all these models is the "binary" decomposition, where a Catalan object is either empty or consists of a root together with an ordered pair of smaller Catalan objects. With this, we have the functional equation for the ordinary generating function $C(x) = \sum_{n\geq 0} \mathrm{Cat}(n)x^n$:

$$C(x) = 1 + x\,C(x)^2.$$

Solving this quadratic equation gives

$$C(x) = \frac{1-\sqrt{1-4x}}{2},$$

and taking coefficients by the binomial series, we have $\mathrm{Cat}(n) = \dfrac{1}{n+1}\binom{2n}{n}$.

A walks-on-trees view: Another interpretation views Catalan numbers as weighted counts of closed walks on (ordinary) unlabeled trees. For each unlabeled tree T on $n+1$ vertices and each vertex $v \in T$ one counts closed tours that traverse each edge twice (once away from v and once back), then sums these numbers over all vertices and divides by the order of the tree automorphism group. We write this as

$$\mathrm{Cat}(n) = \sum_{T \in \mathcal{T}_{n+1}} \sum_{v \in T} \frac{a_T(v)}{|\Gamma(T)|},$$

where $a_T(v)$ counts such tours and $\mathcal{T}_{n+1}$ is the set of unlabeled trees on $n+1$ vertices. Edges of size k naturally produce $(k-1)$-ary branching in associated expansion trees, and counting constrained tours on the associated bipartite graphs gives Fuss–Catalan-type numbers.

Now, increasing the branching factor from binary to $(r+1)$-ary leads to the Fuss–Catalan numbers.

2.2 Fuss–Catalan Numbers

Definition 2.2. For integers $r \geq 1$ and $n \geq 0$, the Fuss–Catalan number is

$$\mathrm{Cat}^{(r)}(n) = \frac{1}{rn+1}\binom{(r+1)n}{n}.$$

These numbers generalize Catalan numbers (which correspond to $r = 1$) and enumerate combinatorial families with $(r+1)$-fold branching. Some equivalent models are

(1) $(r+1)$-ary trees: rooted plane trees where each internal node has $r+1$ ordered children. Equivalently, (incomplete) $(r+1)$-ary trees with n internal nodes are counted by $\mathrm{Cat}^{(r)}(n)$.

(2) r-Dyck paths: lattice paths from $(0,0)$ to $((r+1)n, 0)$ with up-steps $U = (1,1)$ and down-steps $D = (1,-r)$ that never cross below the x-axis.

(3) Polygon $(r+2)$-angulations: dissections of a convex $(rn+2)$-gon into $(r+2)$-gons by noncrossing diagonals.

(4) r-ballot sequences: sequences with rn entries equal to 1 and n entries equal to $-r$ whose partial sums are nonnegative.

All these models share the same decomposition where an $(r+1)$-ary tree T is either empty or consists of a root together with an ordered $(r+1)$-tuple of $(r+1)$-ary trees. If $F(x) = \sum_{n\geq 0} \mathrm{Cat}^{(r)}(n)x^n$ is the ordinary generating function, the decomposition gives

$$F(x) = 1 + x\,F(x)^{r+1}.$$

Applying Lagrange inversion gives the formula

$$\mathrm{Cat}^{(r)}(n) = \frac{1}{rn+1}\binom{(r+1)n}{n}.$$

Proof via Lagrange inversion: Set $F(x) = 1 + xG(x)$ so that $G(x) = F(x) - 1$. The equation becomes $1+xG(x) = 1+x(1+xG(x))^{r+1}$, equivalently

$$G(x) = x(1+G(x))^{r+1}.$$

Lagrange inversion states that for $G(x) = x\phi(G(x))$ with ϕ analytic and $\phi(0) \neq 0$,

$$[x^n]G(x) = \frac{1}{n}[t^{n-1}]\phi(t)^n.$$

Here $\phi(t) = (1+t)^{r+1}$, hence

$$[x^n]G(x) = \frac{1}{n}\binom{n(r+1)}{n-1} = \frac{1}{rn+1}\binom{(r+1)n}{n},$$

and this is the formula for $\mathrm{Cat}^{(r)}(n)$.

Examples and small values

Here are the first few sequences (the $r = 1$ row is the classical Catalan sequence):

r	$n=0$	1	2	3	4	5
1	1	1	2	5	14	42
2	1	1	3	12	55	273
3	1	1	4	22	140	969

These numbers appear in many different ways. For instance $\mathrm{Cat}^{(2)}(3) = 12$ counts ternary trees with 3 internal nodes, triangulations of a certain 8-gon into pentagons, and 2-Dyck paths of semilength 3.

Unlike the classical and Fuss–Catalan numbers, the rational Catalan instead encode objects whose structure is governed by the slope a/b, which we will now cover.

2.3 Rational Catalan Numbers

Let (a, b) be a pair of positive coprime integers, $0 < a < b$.

Definition 2.3. The rational Catalan number $\mathrm{Cat}(a, b)$ is

$$\mathrm{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a, b}.$$

As Armstrong observes, if one writes $x = \frac{a}{b-a}$ then there is a symmetry

$$\mathrm{Cat}(a, b) = \mathrm{Cat}(b - a, b)$$

, called rational duality. This duality generalizes the fact that in the classical case $(n, n+1)$ one recovers the ordinary Catalan numbers.

There is also a symmetry under exchanging a and b: $\mathrm{Cat}(a, b) = \mathrm{Cat}(b, a)$. This implies the identity

$$\mathrm{Cat}\Big(1,\ x-1\Big) = \mathrm{Cat}\Big(x,\ 1\Big)$$

(where $x = a/(b-a)$).

The combinatorial definitions of rational Catalan numbers explain their symmetry, but they do not immediately show why the q-analogues should have nonnegative coefficients. Armstrong's interpretation resolves this by having $\mathrm{Cat}_q(a, b)$ be a lattice-point generating function inside a region defined by the Weyl group of type A.

3 Geometric Interpretation via Lattice Points

3.1 Weight Lattice of Type A_{a-1}

Fix a positive integer a. The type-A_{a-1} weight lattice is

$$\Lambda \;=\; \{(x_1, \ldots, x_a) \in \mathbb{Z}^a \;:\; x_1 + \cdots + x_a = 0\} \;\cong\; \mathbb{Z}^a/\mathbb{Z}(1, \ldots, 1).$$

Points of Λ are often represented by integer vectors modulo translation by the all-ones vector. The Weyl group $W = S_a$ acts on Λ by permuting coordinates, and the affine Weyl group

$$\widetilde{W} \;=\; S_a \ltimes a\mathbb{Z}^{a-1}$$

acts on $\Lambda \otimes \mathbb{R}$ by permuting and translating coordinates. The reflecting hyperplanes of $\widetilde{W}$ cut space into open simplices called alcoves. The fundamental alcove is

$$\mathcal{A} = \{x \in \mathbb{R}^a : \ x_1 < x_2 < \cdots < x_a < x_1 + 1\}\,.$$

All other alcoves are obtained from $\mathcal{A}$ by the action of $\widetilde{W}$.

For each coprime pair (a, b), Armstrong identifies a polyhedral region

$$\mathcal{R}(a, b) \;\subset\; \Lambda \otimes \mathbb{R},$$

cut out by affine-linear inequalities (involving the "slope" b/a), such that the lattice points of $\mathcal{R}(a, b)$ encode the combinatorics of rational Dyck paths.

Importantly, the height statistic on lattice points in $\mathcal{R}(a, b)$ is a natural geometric definition of the area statistic on rational Dyck paths. So,

$$\mathrm{Cat}_q(a, b) \;=\; \sum_{x \in \mathcal{R}(a,b) \cap \Lambda} q^{\mathrm{ht}(x)}.$$

3.2 Germs and the Residue Class of b Modulo a

Armstrong argues that the geometry of $\mathcal{R}(a, b)$ depends mostly on the residue class $b \pmod a$. For a fixed a, he constructs a finite family of polynomials

$$G_{a,r}(q) \qquad (r = 0, 1, \ldots, a - 1),$$

called the q-Catalan germs. Each germ $G_{a,r}(q)$ is defined as the generating function of lattice points in a certain canonical region of the weight lattice associated to residue class r. These regions are simpler than $\mathcal{R}(a, b)$ and depend only on r and a.

A theorem is that for any coprime pair (a, b),

$$\mathrm{Cat}_q(a, b) \;=\; \sum_{r=0}^{a-1} c_r\, G_{a,r}(q),$$

where each c_r is a nonnegative integer determined by the integer quotient and remainder of b modulo a.

Thus $\mathrm{Cat}_q(a, b)$ is written as a nonnegative integer combination of a small, fixed "basis" of lattice-point polynomials. This immediately implies that the coefficients of $\mathrm{Cat}_q(a, b)$ are nonnegative if each germ has nonnegative coefficients.

References

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